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PARABOLIC MARTINGALES AND NON-SYMMETRIC FOURIER MULTIPLIERS*

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Abstract. We give a class of Fourier multipliers with non-symmetric symbols and explicit norm bounds on L^p spaces by using the stochastic calculus of Lévy processes and Burkholder–Wang estimates for differentially subordinate martingales.

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1. INTRODUCTION AND MAIN RESULT

For each function $m : \mathbb{R}^d \to \mathbb{C}$ of absolute value bounded by one, there is a unique linear contraction M on $L^2(\mathbb{R}^d)$ defined in terms of the Fourier transform by

$$\widehat{Mf} = m\hat{f}\,, \tag{1.1}$$

or, in terms of bilinear forms and the Plancherel theorem, by

$$\Lambda(f,g) = \int_{\mathbb{R}^d} Mf(x)g(x)dx = (2\pi)^{-d}\int_{\mathbb{R}^d} m(\xi)\widehat{f}(\xi)\widehat{g}(-\xi)d\xi. \tag{1.2}$$

We are interested in *symbols* m for which the *Fourier multiplier* M has a finite operator norm $\|M\|_p$ on $L^p(\mathbb{R}^d)$ for all $p \in (1,\infty)$:

$$|\Lambda(f,g)| \leqslant \|M\|_p\|f\|_p\|g\|_q, \tag{1.3}$$

where $q = p/(p-1)$ and, say, $f, g \in C_c^\infty(\mathbb{R}^d)$. Motivated by [4] and [14], a wide class of multipliers was recently studied in [2] and [3] by transforming the so-

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called parabolic martingales of Lévy process. Burkholder–Wang inequalities for differentially subordinate martingales (see [15]) were used to bound their norms:

$$\|M\|_p \leqslant \max\left\{p-1, \frac{1}{p-1}\right\} =: p^* - 1. \tag{1.4}$$

Surprisingly, the symbols m obtained in [2] and [3] turned out to be symmetric, even when non-symmetric Lévy processes were used in the construction. In this paper we propose a new approach which leads to *non-symmetric* symbols. Namely, we use two different Lévy processes to drive the martingales defining the pairing Λ. Compared to [2] and [3] we also slightly modify the calculations of the Fourier symbol.

Let $d, n \in \mathbb{N}$ and consider the general Lévy–Khinchine exponent on $\mathbb{R}^n$,

$$\Psi(\zeta) = \int_{\mathbb{R}^n} \left(e^{i(\zeta,z)} - 1 - i(\zeta,z)1_{|z|\leqslant 1}\right)\nu(dz) - \frac{1}{2}\int_{\mathbb{S}} (\zeta,\theta)^2\,\mu(d\theta) + i(\zeta,\gamma), \tag{1.5}$$

where $\zeta, \gamma \in \mathbb{R}^n$, $\mu \geqslant 0$ is a (non-unique) finite measure on the unit sphere $\mathbb{S} \subset \mathbb{R}^n$, and $\nu \geqslant 0$ is a (unique) Lévy measure on $\mathbb{R}^n$: $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^n} \min(|z|^2, 1)\nu(dz) < \infty.$$

Here $(\xi, \eta) = \sum_k \xi_k \eta_k$ and $|\xi|^2 = \sum_k |\xi_k|^2 = (\xi, \bar{\xi})$ for $\xi, \eta \in \mathbb{R}^d$, $\mathbb{R}^n$, $\mathbb{C}^d$, $\mathbb{C}^n$. Consider complex-valued functions ϕ on $\mathbb{R}^n$ and φ on $\mathbb{S}$ such that $\|\phi\|_\infty \leqslant 1$ and $\|\varphi\|_\infty \leqslant 1$. For $\zeta \in \mathbb{R}^n$ we let

$$\widetilde{\Psi}(\zeta) = \int_{\mathbb{R}^n} \left(e^{i(\zeta,z)} - 1 - i(\zeta,z)1_{|z|\leqslant 1}\right)\phi(z)\nu(dz) - \frac{1}{2}\int_{\mathbb{S}} (\zeta,\theta)^2\varphi(\theta)\mu(d\theta). \tag{1.6}$$

Let $A, B \in \mathbb{R}^{d\times n}$. For $\xi \in \mathbb{R}^d$ we define

$$m(\xi) = [e^{\Psi(B^T\xi - A^T\xi)} - e^{\Psi(B^T\xi)+\Psi(-A^T\xi)}]\times \tag{1.7}$$
$$\frac{\int_{\mathbb{R}^d} (e^{i(B^T\xi,z)} - 1)(e^{i(-A^T\xi,z)} - 1)\phi\,(z)\,\nu(dz) - \int_{\mathbb{S}} (B^T\xi,\theta)(-A^T\xi,\theta)\varphi\,(\theta)\,\mu\,(d\theta)}{\int_{\mathbb{R}^d} (e^{i(B^T\xi,z)} - 1)(e^{i(-A^T\xi,z)} - 1)\nu(dz) - \int_{\mathbb{S}} (B^T\xi,\theta)\,(-A^T\xi,\theta)\,\mu\,(d\theta)},$$

with the convention that

$$m(\xi) = e^{\Psi(B^T\xi)+\Psi(-A^T\xi)}\times \tag{1.8}$$
$$\Big(\int_{\mathbb{R}^d}(e^{i(B^T\xi,z)} - 1)(e^{i(-A^T\xi,z)} - 1)\phi\,(z)\,\nu(dz) - \int_{\mathbb{S}}(B^T\xi,\theta)(-A^T\xi,\theta)\varphi\,(\theta)\,\mu(d\theta)\Big),$$

if the denominator in (1.7) is zero. To simplify (1.7) and (1.8), we note that

$$\int_{\mathbb{R}^n} (e^{i(\zeta_1,z)} - 1)(e^{i(\zeta_2,z)} - 1)\phi(z)\nu(dz) - \int_{\mathbb{S}} (\zeta_1,\theta)\,(\zeta_2,\theta)\,\varphi(\theta)\mu(d\theta) = \widetilde{\Psi}(\zeta_1+\zeta_2) - \widetilde{\Psi}(\zeta_1) - \widetilde{\Psi}(\zeta_2), \quad \zeta_1,\zeta_2 \in \mathbb{R}^n, \tag{1.9}$$

and a similar identity holds for the special case of Ψ. Thus, $m(\xi)$ equals

$$[e^{\Psi(B^T\xi - A^T\xi)} - e^{\Psi(B^T\xi)+\Psi(-A^T\xi)}]\frac{\widetilde{\Psi}(B^T\xi - A^T\xi) - \widetilde{\Psi}(B^T\xi) - \widetilde{\Psi}(-A^T\xi)}{\Psi(B^T\xi - A^T\xi) - \Psi(B^T\xi) - \Psi(-A^T\xi)} \tag{1.10}$$

with the convention that

$$m(\xi) = e^{\Psi(B^T\xi)+\Psi(-A^T\xi)}[\widetilde{\Psi}(B^T\xi - A^T\xi) - \widetilde{\Psi}(B^T\xi) - \widetilde{\Psi}(-A^T\xi)], \tag{1.11}$$

if the denominator in (1.10) is zero. In short,

$$m(\xi) = e^{\Psi(B^T\xi)+\Psi(-A^T\xi)}[\widetilde{\Psi}(B^T\xi - A^T\xi) - \widetilde{\Psi}(B^T\xi) - \widetilde{\Psi}(-A^T\xi)] \times q\left(\Psi(B^T\xi - A^T\xi) - \Psi(B^T\xi) - \Psi(-A^T\xi)\right), \tag{1.12}$$

where

$$q(z) = (e^z - 1)/z \text{ if } z \in \mathbb{C}\setminus\{0\}, \quad \text{and} \quad q(0) = 1\,.$$

We see that (1.7) with (1.8) are equivalent to (1.12). Here is our main result.

THEOREM 1.1. *If M satisfies* (1.1) *and* (1.12), *and* $1 < p < \infty$, *then* $\|M\|_p \leqslant p^* - 1$.

Theorem 1.1 is proved in Section 2 by using stochastic calculus of Lévy processes. In Section 3 we make some clarifying comments and point out a few symbols resulting from (1.12). An alternative approach for Gaussian Lévy processes is given in Section 4, where we use the familiar and more compact classical Itô calculus. This, however, boils down to taking $\nu = 0$ in (1.5), and yields only symmetric symbols. Details of the stochastic calculus needed in this note may be found in [2] and [3]. We refer to [6] and [12] for information on Lévy processes, including compound Poisson processes, and to [8], [9], and [11] for various expositions of stochastic calculus. Burkholder's method is discussed in depth in [1], and a classical treatment of Fourier multipliers may be found in [13]. A recent study of non-symmetric homogeneous symbols is given in [10]. As we already remarked, multipliers with symmetric symbols were obtained by similar methods in [2] and [3], and they include, e.g., Marcinkiewicz-type fractional multipliers, the Beurling–Ahlfors operator and the second order Riesz transforms. We also note that the bound (1.4) cannot in general be improved, because it is optimal for second order Riesz transforms (see [5]).

While we considerably extend the class of symbols manageable by our methods, we fall short of non-symmetric symbols homogeneous of degree zero. Specifically, homogeneous symbols may appear as the second factor (the ratio) in (1.7) or (1.10), but they are tempered at the origin and infinity by the first factor therein, which involves the Fourier transform of the semigroup. Replacing Ψ and $\widetilde{\Psi}$ by $u\Psi$ and $u\widetilde{\Psi}$ and letting $u \to \infty$ usually removes the first factor in (1.7) and (1.10) if $A = B$. The resulting symbols are given in (3.1) below, and include many symmetric symbols homogeneous of degree zero, see (3.2). We wonder if a different pairing or other modifications of our methods could produce symbols which are both discontinuous and non-symmetric.

Below we will often use the quadratic variation $[F, F]$ and covariation $[F, G]$ of square-integrable continuous-time *càdlàg* martingales F, G. Recall that $[F, F]$ is the unique adapted right-continuous non-decreasing process with jumps $[F, F]_t - [F, F]_{t-} = |F_t - F_{t-}|^2$, and such that $t \mapsto |F|_t^2 - [F, F]_t$ is a (continuous) martingale starting at zero ([8], VII.42). We say that F is *differentially subordinate* to G if $t \mapsto [G, G]_t - [F, F]_t$ is nonnegative and non-decreasing (see [15]). The covariation $[F, G]$ is defined by polarization, and we have $\mathbb{E}F_tG_t = \mathbb{E}[F, \overline{G}]_t$. All the functions and measures considered in this paper are assumed to be Borelian.

2. PROOF OF THEOREM 1.1

We will first prove the result for

$$\Psi(\zeta) = \int_{\mathbb{R}^d} (e^{i(\zeta,z)} - 1)\nu(dz), \qquad \zeta \in \mathbb{R}^n, \tag{2.1}$$

and

$$\widetilde{\Psi}(\zeta) = \int_{\mathbb{R}^n} (e^{i(\zeta,z)} - 1)\phi(z)\nu(dz), \qquad \zeta \in \mathbb{R}^n, \tag{2.2}$$

where ν is finite. To this end we only need to define Λ satisfying (1.2) and (1.3).

By f and g below we will denote complex-valued smooth compactly supported (i.e. C_c^∞) functions on $\mathbb{R}^d$ or $\mathbb{R}^n$. Let $(Y_t, t \geqslant 0)$ be a compound Poisson process on $\mathbb{R}^n$ with the Lévy measure ν, semigroup (P_t), expectation $\mathbb{E}$ and jumps $\Delta Y_t = Y_t - Y_{t-}$. Let $x \in \mathbb{R}^n$. Recall that

$$P_t f(x) = \mathbb{E}f(x + Y_t) = \int_{\mathbb{R}^d} f(x+y)p_t(dy), \quad \text{where } t \geqslant 0,$$

$$p_t = e^{-t|\nu|} \sum_{n=0}^{\infty} \frac{\nu^{*n}}{n!}, \quad \text{and} \quad \hat{p}_t(\zeta) = \mathbb{E}e^{i(\zeta,Y_t)} = e^{t\Psi(\zeta)} \text{ for } \zeta \in \mathbb{R}^n.$$

Let $A, B \in \mathbb{R}^{d\times n}$. The process $(AY_t, t \geqslant 0)$ is compound Poisson, too, with the Lévy measure equal to (the pushforward measure) $A\nu = \nu \circ A^{-1}$ on $\mathbb{R}^d \setminus \{0\}$

([12], Proposition 11.10). Indeed, for $\xi \in \mathbb{R}^d$,

$$\mathbb{E}e^{i(\xi, AY_t)} = e^{t\Psi(A^T\xi)} = \int_{\mathbb{R}^n} (e^{i(\xi, Az)} - 1)\nu(dz) = \int_{\mathbb{R}^d} (e^{i(\xi, z)} - 1)A\nu(dz).$$

We also have $\mathbb{E}f(x + AY_t) = P_t^A f(x)$, where

$$P_t^A f(x) = \int f(x + Ay)p_t(dy).$$

We proceed similarly for $(BY_t, t \geqslant 0)$. We remark that (AY_t) and (BY_t) have fairly general dependence structure, e.g. yield pairs of projections of Y.

We consider the filtration $\mathcal{F}_t = \sigma\{Y_s : 0 \leqslant s \leqslant t\}$. For $0 \leqslant t \leqslant 1$ we define the *parabolic* martingale $F_t = F_t(x; f, A)$, where

$$\begin{aligned} F_t(x; f, A) &= \mathbb{E}[f(x + AY_1)|\mathcal{F}_t] = \mathbb{E}\big[f\big(x + A(Y_1 - Y_t) + AY_t\big)|\mathcal{F}_t\big] \\ &= \int_{\mathbb{R}^d} f(x + Ay + AY_t)p_{1-t}(dy) = P_{1-t}^A f(x + AY_t). \end{aligned}$$

Thus F is of *function-type*, i.e. a composition of a (parabolic) function with a (space-time) stochastic process. By the Itô formula ([3], p. 17) for (AY_t),

$$\begin{aligned} F_t - F_0 = &\sum_{\substack{0<v\leqslant t \\ \Delta Y_v \neq 0}} [P_{1-v}^A f(x + AY_v) - P_{1-v}^A f(x + AY_{v-})] \\ &- \int_0^t \int_{\mathbb{R}^d} \big[P_{1-v}^A f\big(x + A(Y_v + z)\big) - P_{1-v}^A f(x + AY_v)\big]\nu(dz)dv. \end{aligned}$$

Following [2] and [3] we also define more *general* (i.e. non function-type) martingales

$$\begin{aligned} G_t(x; g, B, \phi) = &\sum_{\substack{0<v\leqslant t \\ \Delta Y_v \neq 0}} [P_{1-v}^B g(x + BY_v) - P_{1-v}^B g(x + BY_{v-})]\phi(\Delta Y_v) \\ &- \int_0^t \int_{\mathbb{R}^d} \big[P_{1-v}^B g\big(x + B(Y_v + z)\big) - P_{1-v}^B g(x + BY_v)\big]\phi(z)\nu(dz)dv \end{aligned}$$

driven by (BY_t). We see that $F_t(x; f, B) = G_t(x; f, B, 1)$. Let

$$\Lambda(f, g) = \int_{\mathbb{R}^d} \mathbb{E}F_1(x; f, A)G_1(x; g, B, \phi)dx. \tag{2.3}$$

By [3], p. 17, it follows that $G_t := G_t(x; g, B, \phi)$ has quadratic variation

$$[G, G]_t = \sum_{0<v\leqslant t} |P_{1-v}^B g(x + BY_v) - P_{1-v}^B g(x + BY_{v-})|^2 |\phi(\Delta Y_v)|^2.$$

The quadratic variation of F is

$$[F,F]_t = |F_0|^2 + \sum_{0<v\leqslant t} |P^A_{1-v} f(x+AY_v) - P^A_{1-v} f(x+AY_{v-})|^2.$$

Since $\|\phi\|_\infty \leqslant 1$, $G(x;g,B,\phi)$ is differentially subordinate to $F(x;g,B)$. Let $p,q\in(1,\infty)$ and $1/p+1/q=1$. By Fubini–Tonelli,

$$\begin{aligned}(2.4)\quad \int_{\mathbb{R}^d} \mathbb{E}|F_1(x;f,A)|^p dx &= \int_{\mathbb{R}^d} \mathbb{E}|f(x+AY_1)|^p dx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x+Ay)|^p p_1(dy)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x)|^p p_1(dy)dx = \|f\|_p^p.\end{aligned}$$

We then use Burkholder–Wang theory (see [15]) and the identity $p^*-1=q^*-1$:

$$\mathbb{E}|G_1|^q \leqslant (q^*-1)^q \mathbb{E}|g(x+BY_1)|^q = (p^*-1)^q \mathbb{E}|g(x+BY_1)|^q.$$

Following (2.4), we now obtain

$$\int_{\mathbb{R}^d} \mathbb{E}|G_1(x;g,B,\phi)|^q dx \leqslant (p^*-1)^q \int_{\mathbb{R}^d} |g(x)|^q dx.$$

By the Hölder inequality, $|\Lambda(f,g)| \leqslant (p^*-1)\|f\|_p\|g\|_q$, as required in (1.3). To obtain (1.2), we recall that $\mathbb{E}F_1G_1 = \mathbb{E}[F,\overline{G}]_1$. Furthermore,

$$\widehat{P^A_t f}(\xi) = \widehat{f}(\xi) e^{t\Psi(-A^T\xi)}.$$

By this, the Lévy system (see [3] and [8]) and the Plancherel theorem,

$$\begin{aligned}\Lambda(f,g) &= \int_{\mathbb{R}^d} \mathbb{E} \sum_{\substack{0<v\leqslant 1\\ \Delta Y_v\neq 0}} [P^A_{1-v} f(x+AY_v) - P^A_{1-v} f(x+AY_{v-})] \\ &\qquad\qquad \times [P^B_{1-v} g(x+BY_v) - P^B_{1-v} g(x+BY_{v-})]\phi(\Delta Y_v)dx \\ &= \int_{\mathbb{R}^d}\int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big[P^A_{1-v} f\big(x+A(y+z)\big) - P^A_{1-v} f(x+Ay)\big] \\ &\quad \times \big[P^B_{1-v} g\big(x+B(y+z)\big) - P^B_{1-v} g(x+By)\big]\phi(z)\nu(dz)p_v(dy)dvdx \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} m(\xi)\widehat{f}(\xi)\widehat{g}(-\xi)d\xi,\end{aligned}$$

where

$$(2.5)\quad m(\xi) = \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (e^{-i(\xi, A(y+z))} - e^{-i(\xi, Ay)})(e^{i(\xi, B(y+z))} - e^{i(\xi, By)}) \times e^{(1-v)\Psi(-A^T\xi)} e^{(1-v)\Psi(B^T\xi)} \phi(z)\nu(dz)p_v(dy)dv$$

$$= \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(B^T\xi - A^T\xi, y)} e^{(1-v)(\Psi(B^T\xi)+\Psi(-A^T\xi))} \times (e^{i(\xi,Bz)} - 1)(e^{-i(\xi,Az)} - 1)\phi(z)\nu(dz)p_v(dy)dv$$

$$= \int_0^1 \int_{\mathbb{R}^d} e^{v\Psi(B^T\xi - A^T\xi)} e^{(1-v)(\Psi(B^T\xi)+\Psi(-A^T\xi))} \times (e^{i(\xi,Bz)} - 1)(e^{-i(\xi,Az)} - 1)\phi(z)\nu(dz)dv.$$

We directly verify (cf. (1.9)) that

$$\int_{\mathbb{R}^d} (e^{i(\xi,Bz)} - 1)(e^{-i(\xi,Az)} - 1)\phi(z)\nu(dz) = \widetilde{\Psi}(B^T\xi - A^T\xi) - \widetilde{\Psi}(B^T\xi) - \widetilde{\Psi}(-A^T\xi).$$

We integrate (2.5) with respect to dv and obtain (1.12).

We shall next give an extension to compound Poisson processes with drift. We claim that the multiplier resulting from ϕ and the Lévy–Khinchine exponent

$$\int_{\mathbb{R}^d} \big(e^{i(\xi,z)} - 1 - i(\xi,z)1_{|z|\leqslant 1}\big)\nu(dz) + i(\xi,\gamma) = \int_{\mathbb{R}^d} (e^{i(\xi,z)} - 1)\nu(dz) + i(\xi,h),$$

where $h = \gamma - \int_{\mathbb{R}^d} z1_{|z|\leqslant 1}\nu(dz)$, has also the norm bounded by $p^* - 1$ on $L^p(\mathbb{R}^d)$. The operator $T_h f(x) = f(x-h)$ is an isometry of $L^p(\mathbb{R}^d)$, and also a Fourier multiplier with symbol $e^{i(\xi,h)}$. We can multiply $m(\xi)$ in (1.12) by $e^{i(B^T\xi - A^T\xi, h)}$, without changing the norm of the multiplier. The exponential function absorbs into the first factor on the right-hand side of (1.12), which grants the extension.

We will now pass to general Lévy processes, i.e. arbitrary Ψ and $\widetilde{\Psi}$ given by (1.5) and (1.6). We first note that the norm bound of our multipliers is preserved under pointwise convergence of the symbols, which follows from the Plancherel theorem and Fatou's lemma in the same way as in the proof of Theorem 1.1 in [3]. Then we remark that m in (1.12) depends continuously on Ψ and $\widetilde{\Psi}$. Finally, we recall the following approximation procedure: let $\varepsilon \to 0^+$,

$$\nu_\varepsilon = 1_{\{|z|>\varepsilon\}}\nu, \quad \text{and} \quad \mu_\varepsilon(drd\theta) = \varepsilon^{-2}\delta_\varepsilon(dr)\mu(d\theta).$$

Here $(r,\theta) \in (0,\infty) \times \mathbb{S}$ are the polar coordinates in $\mathbb{R}^n$, and δ_ε is the probability measure concentrated at ε. We consider

$$\Psi_\varepsilon(\xi) = \int_{\mathbb{R}^d} \big(e^{i(\xi,z)} - 1 - i(\xi,z)1_{|z|\leqslant 1}\big)(\nu_\varepsilon + \mu_\varepsilon)(dz) + i(\xi,\gamma),$$

and

$$\widetilde{\Psi}_\varepsilon(\xi) = \int_{\mathbb{R}^d} \left(e^{i(\xi,z)} - 1 - i(\xi,z)1_{|z|\leqslant 1}\right)\phi_\varepsilon(z)(\nu_\varepsilon + \mu_\varepsilon)(dz),$$

where $\phi_\varepsilon(z) = 1_{\{|z|>\varepsilon\}}\phi(z) + 1_{\{|z|=\varepsilon\}}\varphi(z/|z|)$. By dominated convergence, we have $\Psi_\varepsilon(\zeta) \to \Psi(\zeta)$ and $\widetilde{\Psi}_\varepsilon(\zeta) \to \widetilde{\Psi}(\zeta)$ (see [3], (3.3)), which yields the convergence of the resulting symbols (say, m_ε) to m in (1.12), and completes the proof. ■

3. COMMENTS AND EXAMPLES

Unless stated otherwise the multipliers discussed in this section have norms bounded by $p^* - 1$ on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, as results from the preceding discussion. We will focus on the symbols.

Note that $m(\xi)$ given by (1.12) is continuous in ξ, since so are $\Psi(\xi)$ and $\widetilde{\Psi}(\xi)$. By (1.1), the Plancherel theorem and (1.4) for $p = 2$ we also see that $|m(\xi)| \leqslant 1$.

Let $u > 0$. We may consider $u\Psi$ and $u\widetilde{\Psi}$ instead of Ψ and $\widetilde{\Psi}$ in (1.10). If $A = B$, $\Re\Psi(A\xi) < 0$ for $\xi \neq 0$, and $u \to \infty$, then we obtain the symbol

$$m(\xi) = \frac{\widetilde{\Psi}(A^T\xi) + \widetilde{\Psi}(-A^T\xi)}{\Psi(A^T\xi) + \Psi(-A^T\xi)}. \tag{3.1}$$

In fact, the assumption $A = B$ rules out non-symmetric symbols. On the other hand, if $A \neq B$, then the corresponding Lévy processes separate over time, and their parabolic martingales quickly decorrelate (cf. the proof of Theorem 1.1). We do not see a way to reproduce a nontrivial analogue of (3.1) in this situation. We also note that if $A = B = \mathbb{I}$ and $\Re\Psi(\xi) < 0$ for $\xi \neq 0$, then (3.1) is equivalent to (1.4) in [3]. Furthermore, if $A \in \mathbb{R}^{d\times d}$ and $\det A \neq 0$, then multipliers corresponding to symbols $m(\xi)$ and $m(A^T\xi)$ have equal norms on $L^p(\mathbb{R}^d)$. In such a case (3.1) is merely a trivial extension of (1.4) in [3]. If $\nu = 0$, then (3.1) yields, e.g., the symbols

$$m(\xi) = \frac{\int_{\mathbb{S}} (\xi,\theta)^2 \varphi(\theta)\mu(d\theta)}{\int_{\mathbb{S}} (\xi,\theta)^2 \mu(d\theta)}, \quad \xi \in \mathbb{R}^d. \tag{3.2}$$

Further discussion and examples related to (3.1) may be found in [3]. In particular, [3] compares (3.2) to the more usual matrix form. We also obtain the symbols

$$m(\xi) = \frac{\ln(1+\xi_j^2)}{\ln(1+\xi_1^2) + \ldots + \ln(1+\xi_d^2)} \tag{3.3}$$

(this corrects formulas (4.10) and (4.11) in [3]), and

$$m(\xi) = -2\xi_j\xi_k/|\xi|^2.$$

Here $\xi \in \mathbb{R}^d \setminus \{0\}$, $j,k = 1,\ldots,d$, and $j \neq k$.

To exhibit a non-symmetric symbol resulting from our construction, we let $n = d$, $\alpha \in (0,2)$ and $\Psi(\xi) = -|\xi|^\alpha$, so that $\mu = 0$, $\gamma = 0$, $\nu(dz) = c_\alpha |z|^{-d-\alpha} dz$, and $c_\alpha = \Gamma\big((d+\alpha)/2\big) 2^\alpha \pi^{-d/2} / |\Gamma(-\alpha/2)|$ in (1.5) (see [7]). These correspond to the isotropic α-stable Lévy process. If $\alpha \in (0,1)$ and $B = \mathbb{I} = -A$ in (1.12), then, by (1.7) and (1.9),

$$m(\xi) = \frac{e^{-|2\xi|^\alpha} - e^{-2|\xi|^\alpha}}{-|2\xi|^\alpha + 2|\xi|^\alpha} \int_{\mathbb{R}^d} (e^{i(\xi,z)} - 1)^2 \phi(z) \nu(dz).$$

Let $d = 1$ and $\phi(z) = \operatorname{sgn}(z)$. We have $(e^{i\xi z} - 1)^2 = (e^{2i\xi z} - 1) - 2(e^{i\xi z} - 1)$ and

$$\int_{\mathbb{R}} \frac{e^{i\xi z} - 1}{|z|^{1+\alpha}} \phi(z) dz = 2i \int_0^\infty \frac{\sin \xi z}{|z|^{1+\alpha}} dz = -2i\Gamma(-\alpha) \sin\frac{\pi\alpha}{2} \operatorname{sgn}(\xi) |\xi|^\alpha.$$

By this and the multiplication and reflection formulas for the gamma function,

$$\int_{\mathbb{R}} (e^{i\xi z} - 1)^2 \phi(z) \nu(dz) = -i \tan\frac{\pi\alpha}{2} \left[|2\xi|^\alpha - 2|\xi|^\alpha\right]. \tag{3.4}$$

Therefore,

$$m(\xi) = i \tan\frac{\pi\alpha}{2} \operatorname{sgn}(\xi) (e^{-|2\xi|^\alpha} - e^{-2|\xi|^\alpha}), \qquad \xi \in \mathbb{R}. \tag{3.5}$$

We may let $\alpha \to 1$ in (3.5), and use l'Hospital's rule to obtain

$$m(\xi) = \frac{4i \ln 2}{\pi} \, \xi e^{-2|\xi|}.$$

This agrees well with (1.8) and (1.11), see (3.4). By analytic continuation, (3.5) extends to $\alpha \in (1,2)$.

As seen in the proof of Theorem 1.1, the drift γ plays little role in our results, according with the conclusions of [3].

4. GAUSSIAN CASE

For multipliers resulting from the linear transformations of the Brownian motion there is an alternative direct approach based on the classical Itô calculus. The calculations are simpler and may shed some light on the procedures in Section 2.

THEOREM 4.1. *Let $d, n \in \mathbb{N}$ and $A, B \in \mathbb{R}^{d\times n}$. Let $K \in \mathbb{C}^{n\times n}$ satisfy*

$$|Kz| \leqslant |z| \quad \textit{for } z \in \mathbb{C}^n. \tag{4.1}$$

For each $p \in (1, \infty)$, *the Fourier multiplier* M *with the symbol*

$$m(\xi) = [e^{-|A^T\xi - B^T\xi|^2} - e^{-|A^T\xi|^2 - |B^T\xi|^2}] \frac{(A^T\xi, KB^T\xi)}{(A^T\xi, B^T\xi)} \tag{4.2}$$

is bounded in $L^p(\mathbb{R}^d)$. *In fact*, $\|Mf\|_p \leqslant (p^* - 1)\|f\|_p$ *for* $f \in L^p(\mathbb{R}^d)$, *where we assume* $m(\xi) = e^{-|A^T\xi|^2 - |B^T\xi|^2}(A^T\xi, KB^T\xi)$, *if the denominator in* (4.2) *is zero.*

Proof. Let $(W_t)_{t \geqslant 0}$ be the Brownian motion in $\mathbb{R}^n$. Let p_t denote the distribution of W_t. Thus, for $t > 0$ we have $p_t(dw) = p_t(w)dw$, where $p_t(w) = (2\pi t)^{-n/2} \exp\big(-|w|^2/(2t)\big)$. Let $f, g \in C_c^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. We consider the filtration

$$\mathcal{F}_t = \sigma\{W_s\,;\ 0 \leqslant s \leqslant t\}, \qquad t \geqslant 0,$$

and the parabolic martingale $F_t = F_t(x; f, A)$, where

$$\begin{aligned} F_t(x; f, A) &= \mathbb{E}[f(x + AW_1)|\mathcal{F}_t] = \mathbb{E}\big[f\big(x + AW_t + A(W_1 - W_t)\big)|\mathcal{F}_t\big] \\ &= \int_{\mathbb{R}^d} f(x + AW_t + Az)p_{1-t}(dz). \end{aligned}$$

Note that $F_1 = f(x + AW_1)$ and $F_0 = \mathbb{E}f(x + AW_1)$. Let $\tilde{f}(z) = f(Az)$. We have $\nabla\tilde{f}(y) = A^T\nabla f(Ay)$. For $0 \leqslant t \leqslant 1,\ w \in \mathbb{R}^d$, we define

$$h(t, w) = \int_{\mathbb{R}^d} f(x + Aw + Az)p_{1-t}(dz). \tag{4.3}$$

We observe that h is parabolic, i.e.

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \frac{1}{2}\Delta_w\right)h(t, w) &= \int_{\mathbb{R}^d} f(x + Aw + Az)\frac{\partial}{\partial t}\big[p_{1-t}(z)\big]dz \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^d} \Delta_z[f(x + Aw + Az)]p_{1-t}(z)dz = 0. \end{aligned} \tag{4.4}$$

Here $\Delta_w = \sum_{i=1}^n \partial^2/\partial w_i^2$ is the Laplacian, and the last inequality follows from integrating by parts and the heat equation

$$\frac{\partial}{\partial s}p_s(z) = \frac{1}{2}\Delta_z p_s(z), \qquad s > 0,\ z \in \mathbb{R}^n.$$

Let $p_t^A(dy)$ be the distribution of AW_t, i.e. $p_t^A = Ap_t$ (the pushforward measure). We have

$$\widehat{p_t^A}(\xi) = \exp(-t|A^T\xi|^2/2), \qquad \xi \in \mathbb{R}^d,$$

$h(t, w) = f * p_{1-t}^A(x + Aw)$, and $h(1, w) = f(x + Aw)$. Thus, $F_t(x; f, A) = h(t, W_t)$. By (4.4) and the Itô formula for h we obtain

$$F_t - F_0 = \int_0^t A^T(\nabla f) * p_{1-v}^A(x + AW_v)dW_v. \tag{4.5}$$

For $t \in [0, u]$ we define

$$G_t = G_t(x; g, B, K) = \int_0^t KB^T(\nabla g) * p_{1-v}^B(x + BW_v)dW_v,$$

where $p_t^B = Bp_t$. The quadratic variations of these martingales are:

$$[F, F]_t = |F_0|^2 + \int_0^t |A^T(\nabla f) * p_{1-v}^A(x + AW_v)|^2 dv, \tag{4.6}$$

$$[G, G]_t = \int_0^t |KB^T(\nabla g) * p_{1-v}^B(x + BW_v)|^2 dv. \tag{4.7}$$

By Burkholder–Wang theory of differentially subordinated martingales [15],

$$\mathbb{E}|G_t(x; g, B, K)|^p \leqslant (p^* - 1)^p \mathbb{E}|F_t(x; g, B)|^p. \tag{4.8}$$

Furthermore, we have

$$\begin{aligned} \int_{\mathbb{R}^d} |F_1(x; f)|^p dx &= \int_{\mathbb{R}^d} |f(x + AW_1)|^p dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x + Ay)|^p p_1(dy) dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x)|^p p_1(dy) dx = \|f\|_p^p. \end{aligned} \tag{4.9}$$

A similar identitity holds for g and $q = p/(p - 1)$. Therefore,

$$\int_{\mathbb{R}^d} \mathbb{E}|G_1(x; g, B, K)|^p dx \leqslant (p^* - 1)^p \|g\|_p^p. \tag{4.10}$$

We define

$$\Lambda(f, g) = \int_{\mathbb{R}^d} \mathbb{E}[F, \overline{G}]_1 dx.$$

By (4.9), (4.10), and the Hölder inequality for the measure $P \otimes dx$, we have

$$\Lambda(f, g) \leqslant (p^* - 1)\|f\|_q \|g\|_p. \tag{4.11}$$

By the Plancherel theorem,

$$\Lambda(f,g) = \int_0^1 \int_{\mathbb{R}^d} (2\pi)^{-d} \int_{\mathbb{R}^d} (A^T\xi, KB^T\xi) e^{-(1-t)|A^T\xi|^2/2}$$
$$\times e^{-(1-t)|B^T\xi|^2/2} e^{-i(A^T\xi,y)} e^{i(B^T\xi,y)} p_t(y) \widehat{f}(\xi)\widehat{g}(-\xi)\, d\xi dy dt$$
$$= \int_0^1 \int_{\mathbb{R}^d} (2\pi)^{-d} (A^T\xi, KB^T\xi) e^{-(1-t)(|A^T\xi|^2+|B^T\xi|^2)/2} e^{-t|B^T\xi - A^T\xi|^2/2}$$
$$\times \widehat{f}(\xi)\widehat{g}(-\xi)\, d\xi dt$$
$$= \int_{\mathbb{R}^d} (2\pi)^{-d} \widehat{f}(\xi)\widehat{g}(-\xi)(A^T\xi, KB^T\xi) e^{-(|A^T\xi|^2+|B^T\xi|^2)/2}$$
$$\times \int_0^1 e^{-t\left[|B^T\xi - A^T\xi|^2 - |A^T\xi|^2 - |B^T\xi|^2\right]/2} dt d\xi \tag{4.12}$$
$$= \int_{\mathbb{R}^d} (2\pi)^{-d} \widehat{f}(\xi)\widehat{g}(-\xi)(A^T\xi, KB^T\xi) e^{-(|A^T\xi|^2+|B^T\xi|^2)/2} \frac{e^{(A^T\xi, B^T\xi)} - 1}{(A^T\xi, B^T\xi)} d\xi.$$

Here we used the identity $|A^T\xi|^2 + |B^T\xi|^2 - 2(A^T\xi, B^T\xi) = |B^T\xi - A^T\xi|^2$ (if $(A^T\xi, B^T\xi) = 0$, then the inner integral in (4.12) equals one). This yields the symbol. The multiplier's norm bound follows from (4.11), as in the proof of Theorem 1.1. ■

If $A\xi = B\xi \neq 0$ for all $\xi \neq 0$, and we multiply the matrices by $u \to \infty$, then

$$m(\xi) = \frac{(A^T\xi, KA^T\xi)}{(A^T\xi, A^T\xi)}, \tag{4.13}$$

and the corresponding multiplier has the same norm bound $p^* - 1$ (see the remarks following Theorem 1.1). Such symbols were discussed in some detail in [2] and [3].

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