PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 32, Fasc. 2 (2012), pp. 301–322

DATA DRIVEN SCORE TESTS FOR UNIVARIATE SYMMETRY BASED ON NON-SMOOTH FUNCTIONS

BY

JADWIGA JÓZEFCZYK (WROCŁAW)

Abstract. We propose data driven score rank tests for univariate symmetry around a known center based on non-smooth functions. A choice of non-smooth functions is motivated by very special properties of a certain function on $[0, 1]$ determined by a distribution which is responsible for its asymmetry. We modify recently introduced data driven penalty selection rules and apply Schwarz-type penalty as well. We prove basic asymptotic results for the test statistics. In a simulation study we compare the empirical behavior of the new tests with the data driven tests based on the Legendre basis and with the so-called hybrid test. We show good power behavior of the new tests often overcoming their competitors.

2000 AMS Mathematics Subject Classification: Primary: 62G10; Secondary: 65C05, 62G99.

Key words and phrases: Testing symmetry, data driven score test, rank test, selection rule, hybrid test, Monte Carlo study.

1. INTRODUCTION

Testing for symmetry is one of the oldest classical nonparametric problems and has an extensive literature. A lot of variables which are used in the statistical modeling have antisymmetric behavior. An answer to the question about symmetry is usually essential for many problems in econometrics, computer science, engineering and social sciences. Similarly, the symmetry of a distribution is sometimes crucial for a correct application of statistical procedures (see, e.g., Zheng and Gastwirth [23]).

The sign test, which is the oldest symmetry test, has already been used by Arbuthnot in 1710. In the paper "An argument for divine providence, taken from the constant regularity observed in the births of both sexes" [1] he analyzed data on birth process and showed that men are born more often than women.

In the last two decades, the problem of testing symmetry has paid renew attention of many statisticians. We can mention, among others, McWilliams [15], Modarres and Gastwirth [17], Cheng and Balakrishnan [5], Thas et al. [22], and

Bakshaev [3]. They considered different types of symmetry tests. Cheng and Balakrishnan [5] modified the sign test. The Wilcoxon sign test was at the beginning of the investigation of Tajuddin [21] and Thas et al. [22]. Modarres and Gastwirth [16] proposed a modified runs test. The last two authors considered also a very interesting two-stage procedure called the hybrid test which will be taken into account in our study. Bai and Ng [2] and also Premaratne and Bera [19] studied tests based on moments. Rank tests were throughly developed by Hájek et al. [8]. A more detailed study of the literature on testing symmetry can be found, e.g., in Inglot et al. [11].

In the present paper we propose and study some new data driven tests. Janic-Wróblewska [12] introduced and investigated a data driven test for symmetry based on the Legendre polynomials and a Schwarz-type selection rule. Her study was continued and extended by Inglot et al. in [10] and [11], where the authors added new selection rules, provided new theoretical results, and presented a simulation study. In the present paper we focus on score statistics based on non-smooth systems of orthogonal functions. Such a choice is motivated by very special properties of a function on $[0, 1]$ which contains all information about an asymmetry of the distribution under consideration (see Section 2).

The paper is organized as follows. In Section 2 we construct the new test statistics and give assumptions and notation. In Section 3 basic asymptotic results for the new tests are stated. Section 4 is devoted to presenting the simulation study. In Section 5 we formulate some conclusions. Section 6 contains proofs of theorems stated in Section 3.

In the sequel we shall use c, C to denote positive constants, possibly different in each case.

2. CONSTRUCTION OF THE TEST STATISTICS

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with a continuous distribution function $F(x)$. We are going to test

$$\mathcal{H}_0 : F(m + x) = 1 - F(m - x), \quad x \in \mathbb{R}, \tag{2.1}$$

where m is known. Without loss of generality we may assume m to be equal to zero.

Denote by $F_s(x) = \big(F(x) + 1 - F(-x)\big)/2$ the distribution function of the symmetric part of F and put $F_a = F - F_s$. So, $\mathcal{H}_0$ is equivalent to testing whether $F_a = 0$. Using F_s, we transform the data into the unit interval and obtain $U_i = F_s(X_i)$, $i = 1, 2, \ldots, n$. It implies that the distribution of U_i is absolutely continuous with respect to the uniform distribution on $(0, 1)$ and its distribution function has the form

$$F \circ F_s^{-1}(t) = t + A(t), \quad t \in (0, 1),$$

where $A = F_a \circ F_s^{-1}$ is an absolutely continuous function, symmetric with respect to $1/2$. It means that U_i has the density

$$p(t) = 1 + a(t), \tag{2.2}$$

where $a(t) = A'(t)$ a.s. A function a is antisymmetric with respect to $1/2$ and is responsible for asymmetry of data distribution. So, testing $\mathcal{H}_0$ is equivalent to testing whether $a = 0$ with F_s being a nuisance parameter.

It is interesting to realize how asymmetry of a distribution F is described by the function a defined in (2.2). Observe that when F has a density f, the function a satisfies the following relation:

$$\frac{f(x)}{f_s(x)} - 1 = a\big(F_s(x)\big), \qquad x \in \mathbb{R},$$

where f_s is the density of F_s. Hence the function a takes values in $[-1, 1]$. The median of F is equal to zero if and only if $\int_0^{1/2} a(t)dt = 0$. If $a(t) = 0$ in some neighborhood of zero, then tails of F are identical beginning from some point. Moreover, if F has an asymmetric support, the function a takes values -1 and 1 at some intervals near zero and one. Typically, $a(0.5) = 0$, $a(t)$ has one zero in the interval $(0, 1/2)$ and attains the largest values near zero.

Denote by $\mathcal{F}_n$ the empirical distribution function of $X_1, X_2, \dots, X_n$. Then as an estimator of an unspecified, symmetric part $F_s(x)$ one may take

$$\mathcal{F}_{ns}(x) = \frac{1}{2}\big(\mathcal{F}_n(x) + 1 - \mathcal{F}_n(-x)\big).$$

Let $d(n) \geqslant 1$ be a (bounded or not) sequence of natural numbers. For every $n \geqslant 1$ consider a triangular array

$$g^k = g^k(n) = (g_{k1}^{(n)}, g_{k2}^{(n)}, \dots, g_{kk}^{(n)}) = (g_{k1}, g_{k2}, \dots, g_{kk}), \tag{2.3}$$

$k = 1, 2, \dots, d(n)$, of bounded, rowwise orthonormal functions in $L_2[0, 1]$ which are antisymmetric with respect to $1/2$. In our notation below we shall omit n and write simply g_{kj} and g^k. In particular, if $g_1, g_2, \dots$ is an infinite orthonormal system on $[0, 1]$, one may, for every n, take $g_{kj} = g_j$ for $j = 1, 2, \dots, k,\ k = 1, 2, \dots, d(n)$. In Inglot et al. [11], odd Legendre polynomials were considered. Here, we extend that construction by removing the smoothness assumption and by considering triangular arrays instead of common orthonormal systems. We briefly repeat a construction of test statistics for the reader's convenience.

Going back to the idea of Neyman [18] consider for each fixed $k = 1, \dots, d(n)$ the exponential family of densities

$$p_k(t, \vartheta) = c_k(\vartheta) \exp\Big\{ \sum_{j=1}^{k} \vartheta_j g_{kj}(t)\Big\}, \tag{2.4}$$

where $\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_k) \in \mathbb{R}^k$, while $c_k(\vartheta)$ is the normalizing constant. Suppose that $p(t) = 1 + a(t)$ can be treated approximately as a member of the family (2.4). Then the hypothesis $\mathcal{H}_0$ is equivalent to the parametric one, i.e., $\vartheta = 0$. By the orthonormality of the system g^k, the normalized score statistic for such a parametric problem takes the form

$$\sum_{j=1}^{k} \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_{kj}\big(F_s(X_i)\big) \right\}^2.$$

Since the distribution function F_s is unknown, we replace F_s with its natural estimator $\mathcal{F}_{ns}$. Then we have

$$\mathcal{F}_{ns}(X_i) = \frac{R_i}{2n} = \frac{n + 0.5 + \operatorname{sign}(X_i)(R_i^+ - 0.5)}{2n},$$

where R_i is the rank of X_i in the pooled sample $X_1, \ldots, X_n; -X_1, \ldots, -X_n$, R_i^+ is the rank of the absolute value $|X_i|$ among $|X_1|, |X_2|, \ldots, |X_n|$, and $\operatorname{sign}(X_i)$ denotes the sign of X_i (cf. Janic-Wróblewska [12]). Taking into account a usual continuity correction, we obtain the score statistic for testing $\mathcal{H}_0$ in the form

$$N_k = \sum_{j=1}^{k} \widehat{g}_{kj}^2, \tag{2.5}$$

where

$$\begin{aligned} \widehat{g}_{kj} &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_{kj} \left(\mathcal{F}_{ns}(X_i) - \frac{1}{4n} \right) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \operatorname{sign}(X_i) g_{kj} \left(\frac{n + R_i^+ - 0.5}{2n} \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_{kj} \left(\frac{R_i - 0.5}{2n} \right). \end{aligned}$$

In the sequel, $\widehat{g}_{kj}$ will be called *empirical Fourier coefficients*.

To get a flexible test we choose the dimension k in (2.5) using the data at hand. Basing on the idea of Kallenberg and Ledwina [14] and following Inglot et al. [11] we consider two types of selection rules. The first one is a score based Schwarz selection rule (cf. Schwarz [20])

$$S = \min\{k : 1 \leqslant k \leqslant d(n),\ N_k - k \log n = \max_{1 \leqslant j \leqslant d(n)} (N_j - j \log n)\}. \tag{2.6}$$

The second one is a less conservative class of selection rules defined by

$$L = \min\big\{k : 1 \leqslant k \leqslant d(n),\ N_k - \Pi(k,n) = \max_{1 \leqslant j \leqslant d(n)} \big(N_j - \Pi(j,n)\big)\big\}, \tag{2.7}$$

where

$$\Pi(j,n) = (j \log n) \mathbb{1}_{W_n^c} + (2j) \mathbb{1}_{W_n},$$

while W_n is an event which gives the information how many empirical Fourier coefficients among $\widehat{g}_{d(n)1}, \ldots, \widehat{g}_{d(n)d(n)}$ overcome some thresholds. More precisely, choose $1 \leqslant K_n \leqslant d(n)$, $1 \leqslant D_n \leqslant K_n$, $\lambda \geqslant 0$, and $\delta_n \to 0$. For $j = 1, \ldots, K_n$ define thresholds c_{jn} by the formula

$$(2.8) \qquad \big(1 - \mathbf{\Phi}(c_{jn})\big)\big(2\mathbf{\Phi}(c_{jn}) - 1\big)^{\lambda} = \frac{1}{2}\left(\frac{\delta_n}{D_n}\binom{d(n)}{j}^{-1}\right)^{1/j}$$

with $\mathbf{\Phi}(c_{jn}) > 1 - (2\lambda + 2)^{-1}$, where $\mathbf{\Phi}$ denotes the standard normal distribution function. Next, order $\widehat{g}^2_{d(n)1}, \ldots, \widehat{g}^2_{d(n)d(n)}$ from the largest to the smallest, obtaining $\widehat{g}^2_{d(n)(1)} \geqslant \widehat{g}^2_{d(n)(2)} \geqslant \ldots \geqslant \widehat{g}^2_{d(n)(d(n))}$. Then choose a D_n-element subset $J_n \subseteq \{1, \ldots, K_n\}$ and define

$$(2.9) \qquad W_n = \bigcup_{j \in J_n} \{\widehat{g}^2_{d(n)(j)} \geqslant c^2_{jn}\}.$$

In Inglot and Janic [9] and Inglot et al. [11] only the case $K_n = D_n$ and $\lambda = 0$ was considered. Here, it is suitable to extend that construction. For more comments concerning the idea of the construction of L see Inglot and Janic [9].

Finally, we propose for testing $\mathcal{H}_0$ the test statistics N_S and N_L with N_k given by (2.5). In the present paper we focus on triangular arrays g^k of non-smooth functions, which better detect possibly a nonzero function $a(t)$. In particular, in Section 3 below, we propose to take the Haar System and the so-called Complemented Haar System.

3. TEST STATISTICS ASYMPTOTICS

A detailed study of asymptotic behavior of the statistics N_S and N_L based on the systems of orthonormal, absolutely continuous functions was presented in Inglot et al. [11]. Here, we extend some of those results when the orthonormal systems consist of non-smooth functions that are sufficiently regular.

Let $g^k = (g_{k1}, g_{k2}, \ldots, g_{kk})$, $k = 1, 2, \ldots, d(n)$, be a triangular array of orthonormal and antisymmetric functions as in (2.3) such that each g_{kj} is a linear combination of $l_{kj} \geqslant 1$ indicators of disjoint intervals in $[0, 1]$. Assume

$$(3.1) \qquad \max_{1 \leqslant k \leqslant d(n)} \max_{1 \leqslant j \leqslant k} \sup_{t \in [0,1]} |g_{kj}(t)| \leqslant c[d(n)]^{\eta}$$

for some $\eta \geqslant 0$ and

$$(3.2) \qquad \max_{1 \leqslant k \leqslant d(n)} \max_{1 \leqslant j \leqslant k} l_{kj} \leqslant c[d(n)]^{\zeta} = o(n)$$

for some $\zeta \geqslant 0$.

Our idea is to approximate functions g_{kj} by absolutely continuous functions and to reduce the problem to the case which has been already considered in Inglot et al. [11].

We shall replace each indicator with a trapezoid. More precisely, the indicator $\mathbb{1}_{[\alpha,\beta]}(t)$ will be approximated by the function

$$\begin{cases} 1 & \text{if } \alpha + 1/(2n) \leqslant t \leqslant \beta - 1/(2n), \\ 2n(t-\alpha) & \text{if } \alpha \leqslant t < \alpha + 1/(2n), \\ -2n(t-\beta) & \text{if } \beta - 1/(2n) < t \leqslant \beta, \\ 0 & \text{otherwise.} \end{cases}$$

In this way each function g_{kj} is approximated by the function, say

$$\varphi_{kj}^{(n)} = \varphi_{kj},$$

which is nonconstant on intervals of joint length l_{kj}/n. Obviously, φ_{kj}'s are absolutely continuous. Note also that φ_{kj}'s are antisymmetric with respect to $1/2$. However, in general, functions $\varphi_{k1}, \varphi_{k2}, \ldots, \varphi_{kk}$ are not necessarily orthogonal. We shall assume that they are orthogonal, i.e.

(3.3) $\qquad \varphi_{k1}, \varphi_{k2}, \ldots, \varphi_{kk}$ are orthogonal for each $k = 2, \ldots, d(n)$.

The empirical Fourier coefficients with respect to g_{kj} and φ_{kj} have the form (cf. (2.5))

$$\widehat{g}_{kj} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \operatorname{sign}(X_i) g_{kj}\left(\frac{n + R_i^+ - 0.5}{2n}\right)$$

and

$$\widehat{\varphi}_{kj} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \operatorname{sign}(X_i) \varphi_{kj}^{(n)}\left(\frac{n + R_i^+ - 0.5}{2n}\right).$$

We have the following lemma.

LEMMA 3.1. *If the conditions* (3.1) *and* (3.2) *are satisfied, then*

(3.4) $$\max_{1 \leqslant k \leqslant d(n)} |\widehat{g}^k - \widehat{\varphi}^k|_k^2 \leqslant c \frac{[d(n)]^{2\eta + 2\zeta + 1}}{n} \quad \text{a.s.},$$

where $|v|_k = (v_1^2 + v_2^2 + \ldots + v_k^2)^{1/2}$ *is the k-dimensional Euclidean norm of a vector* $v = (v_1, v_2, \ldots)$ *and* $\widehat{\varphi}^k = (\widehat{\varphi}_{k1}, \ldots, \widehat{\varphi}_{kk})$.

The proof of Lemma 3.1 is given in Section 6.

The following theorem establishes an asymptotic behavior of N_S and N_L under $\mathcal{H}_0$.

THEOREM 3.1. *Suppose that $\mathcal{H}_0$ is true and the conditions* (3.1), (3.2), *and* (3.3) *are satisfied.*

(1) *If $d(n) = O(n^\tau)$ for some $\tau < 1/(4\eta + 4\zeta + 3)$, then*

$$S \xrightarrow{P} 1 \quad \text{and} \quad N_S \xrightarrow{\mathcal{D}} \chi_1^2 \quad \text{as } n \to \infty,$$

where χ_k^2 denotes a random variable with the central chi-square distribution with k degrees of freedom.

(2) *If $D_n = D \geqslant 1$ and $K_n = K \geqslant 1$ are fixed natural numbers such that $D \leqslant K < d(n)$, $d(n) = O(n^\tau)$ for some $\tau < 1/(4\eta + 4\zeta + 3)$, and $\delta_n > 0$* (cf. (2.8)) *is such that $\delta_n \to 0$, $\log(1/\delta_n) = o(n^{-1/(4\eta+4\zeta+3)})$, and $\log(1/\delta_n)/d(n) \to \infty$, then*

$$P(L = S) \to 1 \quad \text{and} \quad N_L \xrightarrow{\mathcal{D}} \chi_1^2 \quad \text{as } n \to \infty.$$

The proof of Theorem 3.1 is given in Section 6.

The next theorem concerns the asymptotic behavior of upper tail tests based on N_S and N_L under alternatives.

THEOREM 3.2. *Suppose* (3.1), (3.2), *and* (3.3) *are satisfied, $d(n) \to \infty$, $d(n) = O(n^\tau)$ with $\tau < 1/(4\eta + 4\zeta + 3)$, and F is a fixed antisymmetric distribution function such that*

$$\frac{n}{[d(n)]^{2\eta+2\zeta+1}\log^4 n}\Big|\int_0^1 [g(t)]^{d(n)} a(t)dt\Big|_{d(n)}^2 \to \infty \quad \text{as } n \to \infty, \tag{3.5}$$

where a is defined in (2.2). *Then $N_S \to \infty$ in probability and $N_L \to \infty$ in probability. Consequently, for D_n, K_n, and δ_n as in Theorem* 3.1 *the upper-tail tests based on N_S and N_L are consistent in the family of alternatives satisfying* (3.5).

Below, we specify two triangular arrays of functions g^k which will be applied in the sequel. The first one is based on the following system of orthogonal functions:

$$h_j(t) = \sqrt{d(n)}\text{sign}(t - 0.5)\mathbb{1}_{A_j}(t), \quad j = 1, \ldots, d(n),$$

where

$$A_j = \left(\frac{j-1}{2d(n)}, \frac{j}{2d(n)}\right) \cup \left(1 - \frac{j}{2d(n)}, 1 - \frac{j-1}{2d(n)}\right).$$

Since the function $a(t)$ defined in (2.2) has specific properties (see Section 2, comments after the definition of a), an ordering of h_j's strongly influences the power of the corresponding test. Let $\pi_n = \pi$ be a permutation of the set $\{1, 2, \ldots, d(n)\}$. Then we define the triangular array (h_{kj}) as follows: $h_{kj} = h_{\pi(j)}$, $j = 1, \ldots, k$, $k = 1, \ldots, d(n)$, and we shall refer to it as to the *Haar System*. A particular choice of π will be proposed in the next section.

The *Complemented Haar System* is also based on h_j's and is defined as follows:

$$c_{11}(t) = \mathrm{sign}(t-0.5),$$
$$c_{21}(t) = h_{\pi(1)}(t), \quad c_{22} = \sqrt{d(n)/\big(d(n)-1\big)}\mathrm{sign}(t-0.5)\mathbb{1}_{\{t:c_{21}(t)=0\}}(t),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$c_{k1}(t) = h_{\pi(1)}(t), \quad \ldots, \quad c_{k,k-1}(t) = h_{\pi(k-1)}(t),$$
$$c_{kk}(t) = \sqrt{d(n)/\big(d(n)-k+1\big)}\mathrm{sign}(t-0.5)\mathbb{1}_{C_k}(t), \quad k = 3,\ldots,d(n),$$

with $C_k = \bigcap_{j=1}^{k-1}\{t : c_{kj}(t) = 0\}$, and again $\pi_n = \pi$, a permutation of the set $\{1, 2, \ldots, d(n)\}$.

Note that for both systems the assumptions (3.1), (3.2), and (3.3) are fulfilled. In particular, for the Haar System we have $\eta = 1/2$, $l_{kj} = 2$, and hence $\zeta = 0$, while for the Complemented Haar System we get $\eta = 1/2$, $l_{kj} = 2$ for $j < k$ and $l_{kk} \leqslant 2k$, which results in $\zeta = 1$. Moreover, both systems satisfy (3.5) for every function $a \neq 0$. Indeed, if $a \neq 0$, then $a > 0$ on some interval which for n sufficiently large contains an interval of the form $\big((j-1)/\big(2d(n)\big), j/\big(2d(n)\big)\big)$ due to the assumption $d(n) \to \infty$.

4. SIMULATION STUDY

In this section we present results of a simulation study for the new tests in comparison to the hybrid test proposed by Modarres and Gastwirth in [16], denoted below by H, and two data driven tests investigated in Inglot et al. [11], denoted below by NS and NL ($NL3$ in their notation). In our computations we used R. In all cases we took $10{,}000$ Monte Carlo runs. We restricted our attention to the sample size $n = 100$ and significance level $\alpha = 0.05$. As it was suggested in Inglot and Janic [9] and Inglot et al. [11] we took $d(n) = 12$.

We restrict our presentation to the tests based on two triangular arrays described in the previous section – the Haar System and the Complemented Haar System. The following permutation is applied in both cases:

$$\pi_{12} = \pi = (1, 3, 9, 5, 7, 11, 2, 4, 10, 6, 8, 12).$$

Some other choices of g^k and π were also considered, but resulted in weaker tests.

For the Haar System the test statistics under the null hypothesis take a small number of values which have significant probabilities. It causes a difficulty in determining exact critical values. To remove this difficulty, we modify the first function $h_{(12)1} = h_1$ as follows:

$$\widetilde{h}_{(12)1}(t) = \frac{144}{\sqrt{61}}\left[\left(t-\frac{5}{24}\right)\mathbb{1}_{[0,1/24]}(t) + \left(t-\frac{19}{24}\right)\mathbb{1}_{[23/24,1]}(t)\right].$$

As in Inglot et al. [11] we consider tests applying the Schwarz-type rule S (see (2.6)) and the rule L (see (2.7)) determined by the parameters $\delta_n = 0.05$, $K_n = D_n = 3$, and $\lambda = 0$. It can be observed that for systems (h_{kj}) and (c_{kj}) there is usually no empirical Fourier coefficient which dominates and they often have rather equalized values. It results in too conservative behavior of both rules. To make the test more sensitive we propose one more choice of parameters, i.e. $\delta_n = 0.05$, $K_n = 5$, $D_n = 3$, $A_n = \{1, 3, 5\}$, and $\lambda = 1$. The corresponding selection rule will be denoted by L^*.

In effect, we are going to study empirical behavior of five new data driven tests. We will use the following notation:

Notation	**Selection rule**	**System**
NSH	S	Haar System,
NLH	L	Haar System,
NL^*H	L^*	Haar System,
NSC	S	Complemented Haar System,
NLC	L	Complemented Haar System.

For more reliable comparison, we repeat simulations presented in Inglot et al. [10], [11] for the data driven tests NS and NL based on the Legendre System. We decided to include into our study also the hybrid test H since it proved to be a very sensitive symmetry test (cf. Inglot et al. [10], [11]). It occurs that for typical alternatives the hybrid test attains high power in comparison to most classical or recently proposed procedures. For the reader's convenience, we briefly present the construction of this test. This is a two-stage test procedure. Adopting the notation from Section 2, at the first stage, the sign test given by the statistic $Z = \sum_{i=1}^{n} \operatorname{sign}(X_i)$ at the significance level $\alpha_1 < \alpha$ is applied. If $\mathcal{H}_0$ is accepted at the first stage, then a modified Tajuddin's procedure (cf. Tajuddin [21]) is applied at level α_2. The test statistic takes the form

$$H = \frac{W_p - E(W_p|Z)}{[\operatorname{Var}(W_p|Z)]^{1/2}},$$

where

$$W_p = \sum_{k=1}^{n} (R_i^+ - np)^+ \mathbb{1}_{[0,+\infty)}\big(\operatorname{sign}(X_i)\big),$$

while $p \in (0, 1)$ is a trimming proportion and $u^+ = \max(0, u)$. Modarres and Gastwirth recommended to choose $p = 0.8$ for sample sizes $n \geqslant 50$ and $\alpha_1 = 0.01$, $\alpha_2 = 0.0404$. We accepted these recommendations in our simulations. For detailed description we refer to Modarres and Gastwirth [17].

4.1. Critical values. As usual for data driven tests, the slow convergence of the rules S, L, and L^* to one in probability under the null hypothesis implies that exact critical values are far from the asymptotic ones for moderate sample sizes like we just have taken. That is the reason why we prefer to take the empirical critical values in our simulation study. We present them in Table 1.

TABLE 1. Empirical critical values, $\alpha = 0.05$, $n = 100$; $10,000$ MC runs

NS	NL	NSH	NLH	NL^*H	NSC	NLC
5.355	6.121	4.332	4.694	4.560	4.843	6.614

Observe that for the Haar System we get significantly lower critical values. Note that it is possible to find out some approximation formulas for the null distribution of the test statistics and in consequence for critical values. However, in a first order approximation formula it is difficult to take into account several parameters defining the current rule L. Anyway, in the range approximation formula works well, empirical critical values are quite close to that calculated from this formula. For more comments see Inglot et al. [11].

4.2. Alternatives. For power comparisons we have considered a broad spectrum of alternatives. Most of them were described in Inglot et al. [10], [11]. All alternatives which we take here into consideration have median zero. Below, we list several alternatives we applied in our study. The first one is from Inglot et al. [10], the second from Fan [6], the rest are new. In each case, the letter m denotes the median of the underlying distribution.

Notation	**Density**
MixBeta($a, b; \theta$)	$f(x) = 0.1\big(\beta_{(a,b)}(x-1) + \beta_{(b,a)}(x)\big)$ $+0.8\beta_{(a,\theta)}(x+m), \quad x \in [0,2], a, b, \theta > 0;$
Sin(θ, j)	$f(x) = 0.5 + \theta \sin(\pi j x), \quad \lvert x\rvert \leqslant 1, \lvert\theta\rvert \leqslant 0.5;$
ShiftBeta(θ)	$f(x) = \beta_{(2,\theta)}(x+m), \quad x \in [0,2], \theta > 0;$
ShiftChiSq(θ)	$f(x) = \chi^2_\theta(x+m), \quad x \geqslant 0, \theta = 1, 2, \ldots;$
BiBeta(θ)	$f(x) = 0.5\big(\beta_{(2,\theta)}(x-1) + \beta_{(2,2)}(x)\big), \quad x \in [0,2], \theta > 0;$
BiChiSq(θ)	$f(x) = 0.5\big(\chi^2_\theta(-x) + \chi^2_6(x)\big), \quad x \geqslant 0, \theta = 1, 2, \ldots;$
IG(μ, λ)	$f(x) = \sqrt{\lambda/(2\mu x^3)}\exp\big(-\lambda(x-\mu)^2/(2\mu^2 x)\big),$ $x > 0, \lambda > 0, \mu > 0;$
Ra(θ)	$f(x) = \theta^{-2} x \exp(-x^2/2\theta^2), \quad x \geqslant 0, \theta > 0;$
Lehm(θ)	$f(x) = \theta 0.5^\theta (x+1)^{\theta-1}\mathbb{1}_{[-1,1]}(x), \quad x \in [-1,1], \theta > 1.$

In the above formulas, $\beta_{(a,b)}(x)$ denotes the density of the beta distribution with parameters a and b. Apart from the alternatives described above, we also consider alternatives belonging to the families well known from the literature. We take into account the Generalized Lambda Family denoted here by Lambda (see, e.g., Modarres and Gastwirth [17]). We choose three the most interesting cases among nine frequently appearing in the literature. We also take the Generalized Tukey-Lambda Family denoted by Tukey (see Freimer et al. [7]), two particular cases of the Fechner Family denoted by N-Fechner and C-Fechner (see Cassart et al. [4]), and recently introduced Sinh-arcsinh Family, denoted here by Sh-Ash (see Jones and Pewsey [13]). Most of these families were taken into account by Inglot et al. in [10], [11], where one can also find their more precise description.

4.3. Power comparisons. In Tables 2 and 3 we present powers of our five new tests as well as NS, NL, and H for the selected alternatives.

TABLE 2. Empirical powers in per cent, $\alpha = 0.05$, $n = 100$; $10,000$ MC runs

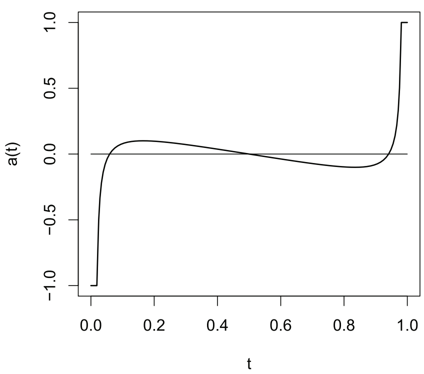
alternative	NS	NL	NSH	NLH	NL^*H	NSC	NLC	H
MixBeta$(1, 2; 3)$	61	69	31	63	60	17	59	8
MixBeta$(2, 3; 16)$	35	47	32	51	49	18	44	6
Sin$(0.5; 8)$	43	98	61	98	98	60	98	43
Sin$(0.3; 3.5)$	54	72	34	65	65	32	66	43
ShiftBeta(4)	60	57	86	69	79	78	73	77
ShiftChiSq(10)	70	67	88	70	79	79	74	85
BiBeta(5)	83	85	41	87	84	40	88	70
BiChiSq(3)	91	91	61	90	89	59	91	87
IG$(0.1, 1)$	69	66	87	67	76	77	72	85
IG$(15, 10)$	52	48	71	45	55	57	50	68
Ra(2)	56	53	83	64	75	73	68	74
Lehm(0.2)	52	50	87	74	83	80	77	71

TABLE 3. Empirical powers in per cent, $\alpha = 0.05$, $n = 100$; $10,000$ MC runs

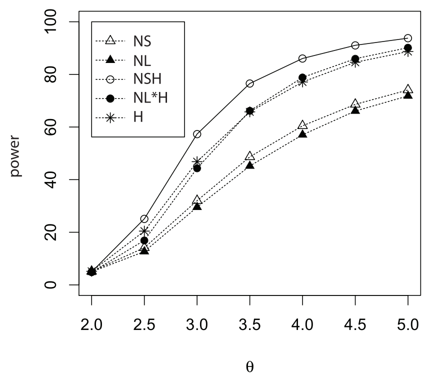
alternative	NS	NL	NSH	NLH	NL^*H	NSC	NLC	H
Lambda(Case 4)	74	71	81	60	69	71	66	86
Lambda(Case 5)	89	88	92	80	86	87	84	96
Lambda(Case 7)	39	36	41	23	29	32	28	50
Tukey$(0.1; 0.5)$	67	64	91	78	86	84	81	83
Tukey$(10; 0.9)$	49	48	85	72	82	78	75	67
Tukey$(1.1; 1.6)$	56	54	88	77	86	83	80	74
Tukey$(7; 1.5)$	56	53	81	63	74	71	66	72
Tukey$(1.1; 6.5)$	70	66	86	71	80	78	74	82
N-Fechner(0.5)	73	70	83	62	71	72	67	86
C-Fechner(0.4)	81	79	43	45	48	41	49	74
Sh-Ash$(+\infty, 4)$	59	55	78	55	65	66	61	75
Sh-Ash$(0.4, 1)$	60	56	68	45	54	55	50	74

The following observations can be drawn from the results in Tables 2 and 3. The powers of tests applying the selection rule L are more stable. What is more, the rule L^* for the Haar System gives a significant gain in power in comparison to the rule L (ca. 6% on average). The test NSH behaves very well, but it is not so stable as NL^*H. In most cases the tests based on the Haar System and the Complemented Haar System give better power in comparison with tests based on the Legendre polynomials. The behavior of the tests with the Complemented Haar System changes with the applied rule. The test NLC usually attains high power while NSC has some weak points. The hybrid test H is very good, but fails completely when asymmetry is beyond distribution's tails (see the first two rows in Table 2). Note that for the Generalized Lambda Family the highest power is achieved by the hybrid test, while for the Generalized Tukey-Lambda Family by the tests NL^*H and NSH.

For further illustration of the behavior of the compared tests, in Figures 1, 2, and 3 we present power curves for three alternatives when changing the parameter θ and the corresponding plots of the functions a. For readability we omit the tests based on the Complemented Haar System. In Figure 4 we show other six alternatives.



(a) Graph of the function a for $\theta = 3$



(b) Power curves, $\alpha = 0.05$, $n = 100$; $10{,}000$ MC runs

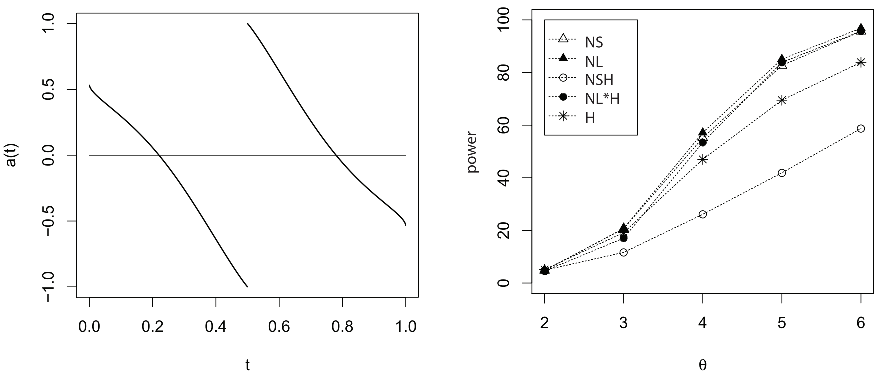
FIGURE 1. Empirical behavior of selected tests for the alternative ShiftBeta(θ). Absolute values of the three biggest empirical Fourier coefficients $\widehat{g}_{kj}$ (number): Legendre – 1.48 (2); 1.05 (3); 0.62 (4); Haar – 1.71 (1); 0.29 (4); 0.27 (8)

One can observe that for alternatives ShiftBeta (Fig. 1), IG (Fig. 4a) and Tukey (Fig. 4b) the tests with the Haar System attain higher power than their competitors. In these cases the hybrid test behaves quite well.

For C-Fechner (Fig. 4c) the highest power is attained by the tests NS and NL, while H also behaves very well.

For alternatives N-Fechner and Sh-Ash (Figs. 4e, 4f) the tests with the Schwarz type rule S perform better behavior. Here the hybrid test H attains the highest power.

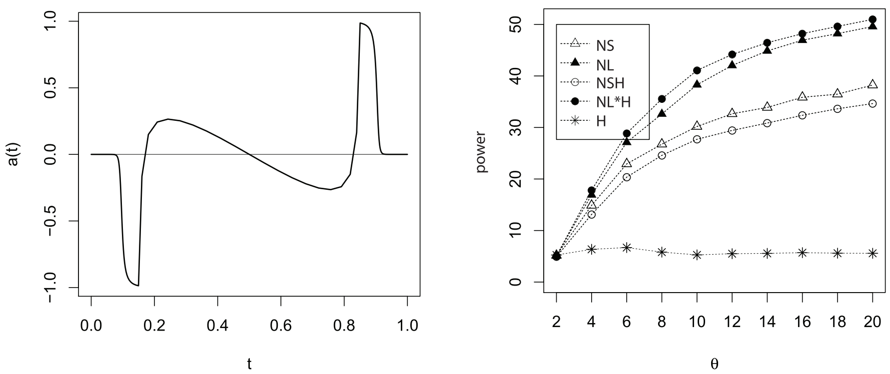
Finally, for alternatives BiBeta, MixBeta, and Sin (Figs. 2, 3, 4d) the tests with the rules L and L^* give better results. Here H is much worse and for MixBeta breaks down.



(a) Graph of the function a for $\theta = 4$

(b) Power curves, $\alpha = 0.05$, $n = 100$; $10,000$ MC runs

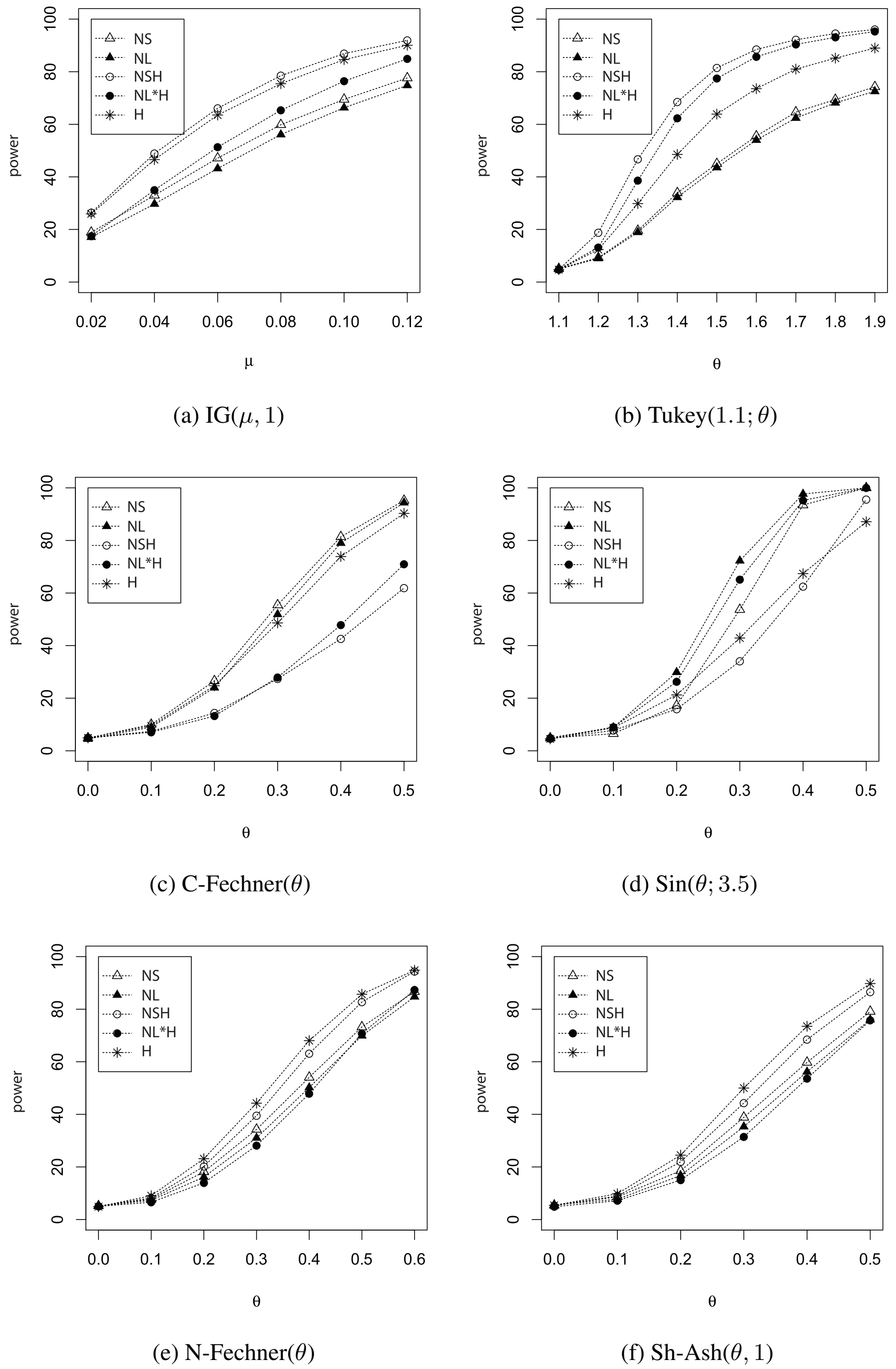
FIGURE 2. Empirical behavior of selected tests for the alternative BiBeta(θ). Absolute values of the three biggest empirical Fourier coefficients $\widehat{g}_{kj}$ (number): Legendre – 1.95 (12); 1.93 (1); 1.68 (2); Haar – 2.31 (12); 1.69 (6); 1.31 (1)



(a) Graph of the function a for $\theta = 16$

(b) Power curves, $\alpha = 0.05$, $n = 100$; $10,000$ MC runs

FIGURE 3. Empirical behavior of selected tests for the alternative MixBeta($2, 3; \theta$). Absolute values of the three biggest empirical Fourier coefficients $\widehat{g}_{kj}$ (number): Legendre – 2.09 (3); 1.20 (7); 1.16 (12); Haar – 1.68 (2); 1.53 (8); 0.71 (5)



(a) IG($\mu, 1$) (b) Tukey($1.1; \theta$)

(c) C-Fechner(θ) (d) Sin($\theta; 3.5$)

(e) N-Fechner(θ) (f) Sh-Ash($\theta, 1$)

FIGURE 4. Empirical power curves of selected tests and alternatives, $\alpha = 0.05, n = 100$; $10,000$ MC runs

5. CONCLUSIONS

The presented simulation study shows that the new data driven tests can really compete with the best tests and in many cases overcome them. Simultaneously, they have essentially wider sensitivity. The tests with the rule L behave definitely better. One can also observe that applying the rule L^* gives in most cases a gain in power, even about 10%, and sometimes loses about 3%. The tests based on the Haar System attain the highest or close to the highest power for all considered families of alternatives and behave stably. That is why we recommend NL^*H as the best symmetry test.

Some insight into a compared tests ability of detection various types of alternatives may give an average power. We calculated it for all 24 alternatives presented in Tables 2 and 3. We obtained (in $\%$): 71.8 for NL^*H, 70.0 for NSH, 68.4 for NLC, 68.2 for H, 65.6 for NLH, 64.3 for NL, 62.5 for NS, and 62.0 for NSC. It is seen that the test NL^*H has on average 6.2% higher power in comparison to NLH and 7.5% to NL. It means that, in fact, L^* increases sensitivity of the test. The surplus of NL^*H to H of 3.6% on average proves its essentially better performance. It is worth recalling that, as it was shown in Inglot et al. [11], NL loses, on average, to the optimal Bayes test approximately 16.7% (for a specially designed comparison). It means that there is not much space for improving it. Therefore, NL^*H seems to be much closer to the optimal test than NL and its other competitors.

6. PROOFS

In this section we provide proofs of theorems stated in Section 3.

Proof of Lemma 3.1. For each $j = 1, \ldots, k$ and $k = 1, \ldots, d(n)$ let us put

$$\Delta_{kj}(r) = g_{kj}\left(\frac{r-0.5}{2n}\right) - \varphi_{kj}\left(\frac{r-0.5}{2n}\right), \qquad r = 1, 2, \ldots, 2n.$$

By the construction of φ_{kj}, the number of nonzero Δ_{kj}'s equals $2l_{kj}$ and for nonzero Δ_{kj}'s we have from (3.1) an estimate $|\Delta_{kj}(r)| \leqslant c[d(n)]^{\eta}$. These facts and (3.1) together with (3.2) imply that for each k and j we have the following estimate:

$$|\widehat{g}_{kj} - \widehat{\varphi}_{kj}| = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} |\Delta_{kj}(R_i)| \leqslant \frac{c}{\sqrt{n}} l_{kj}[d(n)]^{\eta} \leqslant \frac{c}{\sqrt{n}}[d(n)]^{\eta+\zeta}.$$

Now, (3.4) follows immediately. ∎

P r o o f o f T h e o r e m 3.1. By (3.3) for each $k = 1, 2, \ldots, d(n)$ the functions

$$\widetilde{\varphi}_{kj} = \frac{\varphi_{kj}}{\|\varphi_{kj}\|}, \quad j = 1, 2, \ldots, k,$$

where $\|v\|$ denotes the L_2-norm of a function v, are orthonormal and absolutely continuous. By the construction of φ_{kj} it follows that $|g_{kj}(t)| \geqslant |\varphi_{kj}(t)|$ for all t and $\varphi_{kj} = g_{kj}$ outside a set of the Lebesgue measure l_{kj}/n. Hence, from (3.1) and (3.2) we have for each k and j

$$0 < \int_0^1 \left(g_{kj}^2(t) - \varphi_{kj}^2(t)\right) dt \leqslant \frac{c[d(n)]^{2\eta+\zeta}}{n}. \tag{6.1}$$

By the assumption on $d(n)$ the right-hand side of (6.1) tends to zero. Since g_{kj} are normalized, this gives for sufficiently large n

$$1 < \frac{1}{\|\varphi_{kj}\|} \leqslant \left(1 - \frac{c[d(n)]^{2\eta+\zeta}}{n}\right)^{-1/2} \leqslant 1 + \frac{c[d(n)]^{2\eta+\zeta}}{n}. \tag{6.2}$$

Now, we need to apply Theorems A.3 and A.5 from Inglot et al. [11]. For our present purpose these theorems can be stated as follows.

Let $\{\widetilde{\varphi}_{kj},\ j=1,\ldots,k,\ k=1,\ldots,d(n)\}$ be a triangular array of orthonormal absolutely continuous functions (possibly changing with n). For $k = 1, \ldots, d(n)$ let us set

$$\psi^2(k) = \sum_{j=1}^{k} \Big(\int_0^1 |\widetilde{\varphi}'_{kj}(t)| dt\Big)^2$$

which, in general, may depend on n.

THEOREM A (Inglot et al. [11]). *Suppose $\mathcal{H}_0$ is true.*

1. *For each fixed k, $1 \leqslant k \leqslant d(n)$,*

$$|\widehat{\widetilde{\varphi}}^{k}|_k^2 \xrightarrow{\mathcal{D}} \chi_k^2 \quad \text{as } n \to \infty$$

provided $\left(\psi^4(k)\log^6 n\right)/n \to 0$ *as* $n \to \infty$.

2. *For any sequence $k(n)$ of natural numbers, $k(n) \leqslant d(n)$, any $\nu \in (0, 1/2)$, and every sequence x_n of positive numbers such that*

$$x_n^{1-2\nu}\psi^2\big(k(n)\big) \to 0, \quad \frac{nx_n^2}{k(n)} \to \infty, \quad \frac{n^3 x_n^{4(1+\nu)}}{\psi^4\big(k(n)\big)\log^6 n} \to \infty \tag{6.3}$$

it follows that

$$P(|\widehat{\widetilde{\varphi}}^{k(n)}|_{k(n)}^2 \geqslant nx_n^2) = \exp\left\{-\frac{nx_n^2}{2} + O(nx_n^{2+\nu}) + O\left(k(n)\log nx_n^2\right)\right\}.$$

3. *For any sequence $k(n)$ of natural numbers, $k(n) \leqslant d(n)$, any $\nu \in (0, 1/2)$, and every sequence x_n of positive numbers such that*

$$x_n^{1-2\nu}\psi^2\big(d(n)\big) \to 0, \quad \frac{nx_n^2}{d(n)} \to \infty, \quad \frac{n^3 x_n^{4(1+\nu)}}{\psi^4\big(d(n)\big)\log^6 n} \to \infty$$

it follows that

$$P(|\widehat{\widetilde{\varphi}}^{d(n)}|_E^2 \geqslant nx_n^2) = \exp\left\{-\frac{nx_n^2}{2} + O(nx_n^{2+\nu}) + O\left(k(n)\log nx_n^2\right)\right\}$$

for any $k(n)$-element subset $E \subset \{1, \dots, d(n)\}$, where $|v|_E^2 = \sum_{i\in E} v_i^2$.

We shall apply Theorem A to particular triangular arrays $\widetilde{\varphi}_{kj}$ defined in Section 3.

Since $\widetilde{\varphi}'_{kj}(t) = 0$ outside the set of Lebesgue measure l_{kj}/n and for every t $|\widetilde{\varphi}'_{kj}(t)| \leqslant cn[d(n)]^\eta$, by (3.1) and (3.2) we have $\psi^2(k) \leqslant ck[d(n)]^{2\eta+2\zeta}$ uniformly in n.

From part 1 of Theorem A and the assumption on $d(n)$ we immediately obtain $|\widehat{\widetilde{\varphi}}|_1^2 = \widehat{\widetilde{\varphi}}_{11}^2 \xrightarrow{\mathcal{D}} \chi_1^2$, which by (6.2) and the assumption on $d(n)$ implies $\widehat{\varphi}_{11}^2 \xrightarrow{\mathcal{D}} \chi_1^2$. From Lemma 3.1 we have $\widehat{g}_{11} - \widehat{\varphi}_{11} \to 0$ a.s. and, consequently,

$$N_1 = \widehat{g}_{11}^2 \xrightarrow{\mathcal{D}} \chi_1^2.$$

From the definition of S (see (2.6)) we have

$$\begin{aligned} P(S \geqslant 2) &= \sum_{k=2}^{d(n)} P(S = k) \leqslant \sum_{k=2}^{d(n)} P\big(N_k \geqslant (k-1)\log n\big) \\ &= \sum_{k=2}^{d(n)} P\big(|\widehat{g}^k|_k^2 \geqslant (k-1)\log n\big). \end{aligned} \tag{6.4}$$

Applying part 2 of Theorem A for $2 \leqslant k = k(n) \leqslant d(n)$, $x_n^2 = \big(k(n)-1\big)(\log n)/n$ and $0 < \nu < [1 - \tau(4\eta + 4\zeta + 3)]/2(1-\tau)$ we see that (6.3) is fulfilled. So, for sufficiently large n we get

$$P\big(|\widehat{\widetilde{\varphi}}^k|_k^2 \geqslant (k-1)\log n\big) = \exp\left\{-\frac{1}{2}(k-1)(\log n)\big(1+o(1)\big)\right\} \leqslant n^{-\frac{1}{2}(1+o(1))}.$$

Hence and from Lemma 3.1, (6.2) and the assumptions on $d(n)$ we have for n sufficiently large

(6.5)

$$\begin{aligned} P\big(|\widehat{g}^k|_k^2 \geqslant (k-1)\log n\big) &\leqslant P\left(|\widehat{\varphi}^k|_k + c\frac{d(n)^{\eta+\zeta+1/2}}{\sqrt{n}} \geqslant \sqrt{(k-1)\log n}\right) \\ &\leqslant P\left(|\widehat{\widetilde{\varphi}}^k|_k^2 \geqslant \left(1 - \frac{1}{n^{1/4}}\right)(k-1)\log n\right) \leqslant n^{-\frac{1}{2}(1+o(1))}. \end{aligned}$$

Combining (6.4) and (6.5) we finally get for n sufficiently large

$$P(S \geqslant 2) \leqslant d(n) n^{-\frac{1}{2}(1+o(1))}$$

which tends to zero, and hence proves the first part of the theorem.

To prove the second part, consider the family $\mathcal{E}_j$ of all j-element subsets of the set $\{1, 2, \ldots, d(n)\}$. Then from the definition of the event W_n and the corresponding set J_n (see (2.9)), Lemma 3.1, and (6.2) we get similarly as above

$$\begin{aligned} P(W_n) &\leqslant \sum_{j=1}^{D} \sum_{E \in \mathcal{E}_j} P(|\widehat{g}^{d(n)}|_E^2 \geqslant j c_{jn}^2) \\ &\leqslant \sum_{j=1}^{K} \sum_{E \in \mathcal{E}_j} P\left(|\widehat{\widehat{\varphi}}^{d(n)}|_E^2 \geqslant \left(1 - \frac{1}{n^{1/4}}\right) j c_{jn}^2\right). \end{aligned} \tag{6.6}$$

Proceeding as in the proof of Theorem 3.1 in Inglot et al. [11] we prove that

$$\log \frac{c_{jn}^2}{2} \leqslant \log\log \frac{2Dd(n)}{\delta_n}, \qquad j c_{jn}^2 \geqslant \log \frac{1}{\delta_n} + 2 \log \binom{d(n)}{j}. \tag{6.7}$$

Applying part 3 of Theorem A to $n x_n^2 = (1 - n^{-1/4})^2 (j c_{jn}^2)$ and $0 < \nu < \big(1 - \tau(4\eta + 4\zeta + 3)\big)/2$ we see the assumption of part 3 is fulfilled and the right-hand side of (6.6) is estimated by

$$\sum_{j=1}^{K} \sum_{E \in \mathcal{E}_j} \exp\left\{ -\frac{j c_{jn}^2}{2}\left(1 - \frac{1}{n^{1/4}}\right) + O\big(n^{-\nu/2} (j c_{jn}^2)^{1+\nu/2}\big) + O(\log c_{jn}^2)\right\}.$$

By (6.7) this can be further estimated by the expression

$$\sum_{j=1}^{K} \exp\left\{ -\frac{1}{2} \log \frac{1}{\delta_n} + \frac{j c_{jn}^2}{n^{1/4}} + O\big(n^{-\nu/2} (j c_{jn}^2)^{1+\nu/2}\big) + O(\log c_{jn}^2)\right\}.$$

Using again (6.7) and the assumptions on δ_n we see that the above expression also tends to zero (cf. the proof of Theorem 3.1 in Inglot et al. [11]). Hence $P(W_n) \to 0$ and consequently $P(L = S) \to 1$. ∎

It remains to prove Theorem 3.2.

Proof of Theorem 3.2. We have $N_S - S \log n \geqslant N_{d(n)} - d(n) \log n$ a.s. for each n, which means that

$$N_S \geqslant |\widehat{g}^{d(n)}|_{d(n)}^2 - d(n) \log n \ \text{ a.s.}$$

So, to prove that $N_S \to \infty$ in probability it is enough to show

$$P\big(|\widehat{g}^{d(n)}|_{d(n)}^2 \geqslant 2 d(n) \log n\big) \to 1 \quad \text{as } n \to \infty. \tag{6.8}$$

By Lemma 3.1 and the assumption on $d(n)$ we have

$$|\widehat{g}^{d(n)} - \widehat{\varphi}^{d(n)}|^2_{d(n)} \leqslant c[d(n)]^{2\eta+2\zeta+1}/n \to 0 \ \text{a.s.}$$

Thus, by (6.2) and the triangle inequality, (6.8) can be reduced to

$$P\big(|\widehat{\widehat{\varphi}}^{d(n)}|^2_{d(n)} \geqslant 4d(n)\log n\big) \to 1$$

or to the stronger condition

$$P\big(|\widehat{\widehat{\varphi}}^{d(n)}|^2_{d(n)} \geqslant [d(n)]^{2\eta+2\zeta+1}\log^4 n\big) \to 1. \tag{6.9}$$

Now, Proposition A.8 in Inglot et al. [11] can be applied to $\widehat{\widehat{\varphi}}^{d(n)}$. We have (cf. (A.3) and (A.5) in Inglot et al. [11])

$$\widehat{\widehat{\varphi}}^{d(n)} =$$

$$= \Big[\int_0^1 \big[\widetilde{\varphi}\big(\mathcal{H}_{ns}(t)\big)\big]^{d(n)} d\varepsilon_n(t) - \widehat{\widehat{\varphi}}_0^{d(n)}\Big] + \widehat{\widehat{\varphi}}_0^{d(n)} + \sqrt{n}\int_0^1 \big[\widetilde{\varphi}\big(\mathcal{H}_{ns}(t)\big)\big]^{d(n)} dA(t),$$

where $\widehat{\widehat{\varphi}}_0^{d(n)} = \int_0^1 [\widetilde{\varphi}(t)]^{d(n)} d\varepsilon_n(t)$. Hence, putting $\kappa_n = [d(n)]^{\eta+\zeta+1/2}\log^2 n$, we get

$$P(|\widehat{\widehat{\varphi}}^{d(n)}|_{d(n)} \geqslant \kappa_n) \geqslant P\Big(\big|\int_0^1 \big[\widetilde{\varphi}\big(\mathcal{H}_{ns}(t)\big)\big]^{d(n)} dA(t)\big|_{d(n)} \geqslant 3\frac{\kappa_n}{\sqrt{n}}\Big)$$

$$- P\Big(\big|\int_0^1 \big[\widetilde{\varphi}\big(\mathcal{H}_{ns}(t)\big)\big]^{d(n)} d\varepsilon_n(t) - \widehat{\widehat{\varphi}}_0^{d(n)}\big|_{d(n)} \geqslant \kappa_n\Big) - P(|\widehat{\widehat{\varphi}}_0^{d(n)}|_{d(n)} \geqslant \kappa_n).$$

So, to prove (6.9) it is enough to check

$$P\Big(\big|\int_0^1 \big[\widetilde{\varphi}\big(\mathcal{H}_{ns}(t)\big)\big]^{d(n)} dA(t)\big|_{d(n)} \geqslant 3\frac{\kappa_n}{\sqrt{n}}\Big) \to 1, \tag{i}$$

$$P\Big(\big|\int_0^1 \big[\widetilde{\varphi}\big(\mathcal{H}_{ns}(t)\big)\big]^{d(n)} d\varepsilon_n(t) - \widehat{\widehat{\varphi}}_0^{d(n)}\big|_{d(n)} \geqslant \kappa_n\Big) \to 0, \tag{ii}$$

and

$$P(|\widehat{\widehat{\varphi}}_0^{d(n)}|_{d(n)} \geqslant \kappa_n) \to 0. \tag{iii}$$

To prove (i) observe that the construction of φ_{kj}, (3.1), (3.2), (6.2), boundedness of a, and (3.5) imply $\big|\int_0^1 [\widetilde{\varphi}(t)]^{d(n)} dA(t)\big|_{d(n)} \geqslant 2\xi_n\kappa_n/\sqrt{n}$ for n sufficiently large, where ξ_n is some sequence tending to infinity. By the triangle inequality, to get (i) it is enough to prove

$$P\Big(\big|\int_0^1 \big[\widetilde{\varphi}\big(\mathcal{H}_{ns}(t)\big)\big]^{d(n)} dA(t) - \int_0^1 [\widetilde{\varphi}(t)]^{d(n)} dA(t)\big|_{d(n)} \geqslant \xi_n \frac{\kappa_n}{\sqrt{n}}\Big) \to 0. \tag{6.10}$$

Applying (A.22) from Inglot et al. [11] to $x_n^2 = (\log^4 n)/n$ we get

$$\begin{aligned} P\Big(\big|\int_0^1 \big[\widetilde{\varphi}\big(\mathcal{H}_{ns}(t)\big)\big]^{d(n)} dA(t) - \int_0^1 [\widetilde{\varphi}(t)]^{d(n)} dA(t)\big|_{d(n)} \geqslant 3\psi\big(d(n)\big)\frac{\log^2 n}{\sqrt{n}}\Big) \\ \leqslant C\exp\{-c\log^4 n\} \to 0. \end{aligned}$$

Since

$$\frac{\xi_n\kappa_n}{\sqrt{n}} > \frac{3\psi\big(d(n)\big)(\log^2 n)}{\sqrt{n}}$$

for n sufficiently large, the relation (6.10) follows, thus proving (i).

To prove (ii) we apply (A.21) from Inglot et al. [11] to $x_n = 1/(\log n)$ and $\sigma = 1/4$ and obtain

$$\begin{aligned} &P\Big(\big|\int_0^1 \big[\widetilde{\varphi}\big(\mathcal{H}_{ns}(t)\big)\big]^{d(n)} d\varepsilon_n(t) - \widehat{\widetilde{\varphi}}_0^{d(n)}\big|_{d(n)} \geqslant \Delta(n)\Big) \\ &\qquad \leqslant P\Big(\big|\int_0^1 \big[\widetilde{\varphi}\big(\mathcal{H}_{ns}(t)\big)\big]^{d(n)} d\varepsilon_n(t) - \widehat{\widetilde{\varphi}}_0^{d(n)}\big|_{d(n)} \geqslant \psi\big(d(n)\big)x_n^\sigma\Big) \\ &\qquad \leqslant C\exp\{-cnx_n^2\} + C\exp\{-cx_n^{2\sigma-1}\} \to 0. \end{aligned}$$

Finally, to prove (iii) observe that

$$\begin{aligned} |\widehat{\widetilde{\varphi}}_0^{d(n)}|_{d(n)} = \big|\int_0^1 (\widetilde{\varphi}^{d(n)})'(t)\varepsilon_n(t)dt\big|_{d(n)} &\leqslant \psi\big(d(n)\big)\sup_t|\varepsilon_n(t)| \\ &\leqslant c[d(n)]^{\eta+\zeta+1/2}\sup_t|\varepsilon_n(t)|. \end{aligned}$$

So, (iii) reduces to

$$P\big(\sup_t|\varepsilon_n(t)| \geqslant \log^{3/2} n\big) \to 0,$$

which follows immediately from the Komlós, Major, and Tusnády inequality and the estimate $P\big(\sup_t|B(t)| \geqslant x\big) \leqslant 2\exp\{-2x^2\}$, where $B(t)$ denotes the Brownian bridge.

Since $N_L \geqslant N_{d(n)} - d(n)\log n$ a.s., the above proof also remains valid for the statistics N_L. This completes the proof of Theorem 3.2. ■

Acknowledgments. We are very grateful to the referee for detailed remarks improving the presentation.

REFERENCES

[1] J. Arbuthnot, *An argument for divine providence, taken from the constant regularity observed in the births of both sexes*, Philos. Trans. R. Soc. Lond. 27 (1710), pp. 186–190.

[2] J. Bai and S. Ng, *Tests for skewness, kurtosis, and normality for time series data*, J. Bus. Econom. Statist. 23 (2005), pp. 49–60.

[3] A. Bakshaev, *Nonparametric tests based on N-distances*, Lituanian Math. J. 48 (2008), pp. 368–379.

[4] D. Cassart, M. Hallin, and D. Paindaveine, *Optimal detection on Fechner-asymmetry*, J. Statist. Plann. Inference 138 (2008), pp. 2499–2525.

[5] W.-H. Cheng and N. Balakrishnan, *A modified sign test for symmetry*, Comm. Statist. Simulation Comput. 33 (2004), pp. 703–709.

[6] J. Fan, *Test of significance based on wavelet thresholding and Neyman's truncation*, J. Amer. Statist. Assoc. 91 (1996), pp. 674–688.

[7] M. Freimer, G. Kollia, G. S. Mudholkar, and C. T. Lin, *A study of the generalized Tukey lambda family*, Comm. Statist. Theory Methods 17 (1988), pp. 3547–3567.

[8] J. Hájek, Z. Šidák, and P. K. Sen, *Theory of Rank Tests*, Academic Press, 1999.

[9] T. Inglot and A. Janic, *How powerful are data driven score tests for uniformity*, Appl. Math. (Warsaw) 36 (2009), pp. 375–395.

[10] T. Inglot, A. Janic, and J. Józefczyk, *Data driven rank test for univariate symmetry. Simulation results*, Technical Report No. 30/2010, Institute of Mathematics and Computer Science, Wrocław University of Technology, 2010.

[11] T. Inglot, A. Janic, and J. Józefczyk, *Data driven tests for univariate symmetry*, this volume, pp. 323–358.

[12] A. Janic-Wróblewska, *Data driven rank test for univariate symmetry*, Technical Report No. I18/98/P-020, Institute of Mathematics, Wrocław University of Technology, 1998.

[13] M. C. Jones and A. Pewsey, *Sinh-arcsinh distributions*, Biometrika 96 (2009), pp. 761–780.

[14] W. C. M. Kallenberg and T. Ledwina, *Data driven smooth tests when the hypothesis is composite*, J. Amer. Statist. Assoc. 92 (1997), pp. 1094–1104.

[15] T. P. McWilliams, *A distribution-free test for symmetry based on a runs statistics*, J. Amer. Statist. Assoc. 85 (1990), pp. 1130–1133.

[16] R. Modarres and J. L. Gastwirth, *A modified runs test for symmetry*, Statist. Probab. Lett. 31 (1996), pp. 107–112.

[17] R. Modarres and J. L. Gastwirth, *Hybrid test for the hypothesis of symmetry*, J. Appl. Stat. 25 (1998), pp. 777–783.

[18] J. Neyman, *'Smooth test' for goodness of fit*, Skand. Aktuarietidskr. 20 (1937), pp. 149–199.

[19] G. Premaratne and A. Bera, *A test for symmetry with leptokurtic financial data*, J. Financial Econometrics 3 (2005), pp. 169–187.

[20] G. Schwarz, *Estimating the dimension of a model*, Ann. Statist. 6 (1978), pp. 461–464.

[21] I. Tajuddin, *Distribution-free test for symmetry based on Wilcoxon two-sample test*, J. Appl. Stat. 21 (1994), pp. 409–415.

[22] O. Thas, J. C. W. Rayner, and D. J. Best, *Tests for symmetry based on the one-sample Wilcoxon signed rank statistic*, Comm. Statist. Simulation Comput. 34 (2005), pp. 957–973.

[23] T. Zheng and J. Gastwirth, *On bootstrap tests of symmetry about an unknown median*, J. Data Science 8 (2010), pp. 397–412.

Institute of Mathematics and Computer Science
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: jadwiga.jozefczyk@pwr.wroc.pl

Received on 28.11.2011;
revised version on 29.11.2012