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A SIMPLE PROOF OF THE CLASSIFICATION THEOREM FOR POSITIVE NATURAL PRODUCTS

BY

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Abstract. A simplification of the proof of the classification theorem for natural notions of stochastic independence is given. This simplification is made possible after adding the positivity condition to the algebraic axioms for a (non-symmetric) universal product (i.e. a natural product). Indeed, this simplification is nothing but a simplification, under the positivity, of the proof of the claim that, for any natural product, the 'wrong-ordered' coefficients all vanish in the expansion form. The known proof of this claim involves a cumbersome process of solving a system of quadratic equations in 102 unknowns, but in our new proof under the positivity we can avoid such a process.

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1. INTRODUCTION

In non-commutative probability theory, there exist several different notions of stochastic independence, for example, tensor independence (i.e. classical independence), free independence, Boolean independence, and monotone independence (see the references in [1] and [4] for the detailed explanations).

This phenomenon is very specific to the non-commutative situation because in the commutative case there exists only one notion of independence, i.e. tensor independence. We expect that based on the various notions of independence one can develop various probability theories in some way parallel to classical probability theory.

A classification program for universal notions of stochastic independence was carried out in a series of papers (see [1] and [3]–[6]). In [5] Schürmann proposed that universal notions of stochastic independence should be formulated as universal products among non-commutative probability spaces. Speicher [6] (in the expansion form), and Ben Ghorbal and Schürmann [1] (in the form of canonical axioms) proved that the only possible ‘symmetric’ (or ‘commutative’) universal products are three products: tensor, free, and Boolean products. Extending this result to the non-symmetric case, we proved in [3] (in the expansion form) and in [4] (in the form of canonical axioms) that the only possible ‘non-symmetric’ universal products (i.e. natural products) are five products: tensor, free, Boolean, monotone, and anti-monotone products.

However, the proof of the classification theorem for natural products (Theorem 2.2 in [4]) contains a cumbersome step with complicated calculations, unfortunately. The claim of the step is that the coefficient $t(\pi, \lambda; \sigma)$, which is associated with a given natural product, vanishes whenever the partition σ is ‘wrong-ordered’. In the present paper, the explanations of the coefficient $t(\pi, \lambda; \sigma)$ and ‘wrong-orderedness’ will be given in Section 3. In [4], to complete this step, a system of quadratic equations in 102 unknowns was solved by hands. This is a heavy calculation that consists of 22 pages in [4] in its compact form.

The aim of this paper is to improve this cumbersome proof of the claim (V), i.e. the vanishment of wrong coefficients $\{t(\pi, \lambda; \sigma)\}$, to obtain a more clear proof, without using such a big system of equations. However, this simplification of the proof of the claim (V) is made possible after adding the condition of positivity (P) to the algebraic axioms for a natural product. Up to now we do not know if such a simplification of the proof of the claim (V) is possible or not, without using the positivity (P).

This paper consists of the following sections. In Section 2 we prepare some notation concerning partitions of a finite linearly ordered set. In Section 3, after introducing some conditions on a product among algebraic probability spaces (universality, associativity, positivity, etc.), we explain the relation beteween the classification theorem for natural products and the vanishment result (V). In Section 4 we give a simple proof of the vanishment result (V) under the positivity assumption (P), which avoids using a big system of equations.

Throughout the paper, $\mathbb{C}$ is the field of complex numbers, $\mathbb{N}^*$ is the set of all natural numbers ($\neq 0$), and $\#A$ or $|A|$ denotes the cardinality of a finite set A.

2. NOTATION FOR PARTITIONS

Here we describe some notation for partitions used in this paper (see [4]).

Let S be a finite linearly ordered set. A collection $\pi = \{U_1, U_2, \dots, U_p\}$ of subsets of S is called a *partition* of S if $S = \bigcup_{i=1}^{p} U_i,\ U_i \neq \emptyset$, and $U_i \cap U_j = \emptyset$ for all $i, j \in \{1, 2, \dots, p\}$ with $i \neq j$. A pair $(\pi, \lambda) = \{U_1 \prec U_2 \prec \dots \prec U_p\}$ of a partition π and a linear ordering λ among blocks in π is called a *linearly ordered partition* of S (see [3]). A collection $\pi = \{U_1, U_2, \dots, U_p\}$ of finite sequences U of elements from S is called a *BGS-partition* of S if $\#\{i_1, i_2, \dots, i_k\} = k$ for each $U = (i_1 i_2 \dots i_k) \in \pi$ and if $\overline{\pi} := \{\overline{U_1}, \overline{U_2}, \dots, \overline{U_p}\}$ is a (usual) partition of S (see [1]). Here we put $\overline{U} := \{i_1, i_2, \dots, i_k\}$ for $U = (i_1 i_2 \dots i_k) \in \pi$. For each block U in a BGS-partition π, we put $lg(U) := \#(\overline{U})$ the length of U.

Denote by $\mathcal{P}(S)$, $\mathcal{LP}(S)$, and $\vec{\mathcal{P}}(S)$ the set of all partitions, linearly ordered partitions, and BGS-partitions of S, respectively. For each BGS-partition $\sigma = \{U_1, U_2, \ldots, U_p\}$ in $\vec{\mathcal{P}}(S)$, there exists naturally the associated usual partition in $\mathcal{P}(S)$ given by $\overline{\sigma} := \{\overline{U_1}, \overline{U_2}, \ldots, \overline{U_p}\}$. Conversely, we identify $\mathcal{P}(S)$ as the subset of $\vec{\mathcal{P}}(S)$ through the natural correspondence $\pi \mapsto \pi'$ given by

$$\pi \ni U_q = \{i_1 < i_2 < \ldots < i_k\} \mapsto U' = (i_1 i_2 \ldots i_k) \in \pi'.$$

For usual partitions $\pi, \sigma \in \mathcal{P}(S)$, we write $\sigma \leqslant \pi$ when σ is a refinement of π, i.e. when for each $U \in \sigma$ there exists $V \in \pi$ such that $U \subset V$. When the ordered set S is given by $\{1, 2, \ldots, n\}$ with the natural order, we write $\mathcal{P}(n)$, $\mathcal{LP}(n)$, $\vec{\mathcal{P}}(n)$ instead of $\mathcal{P}(S)$, $\mathcal{LP}(S)$, $\vec{\mathcal{P}}(S)$, respectively.

3. CLASSIFICATION THEOREM AND VANISHMENT RESULT

In this section we describe the relation between the classfication theorem and the vanishment result (V). For details see [4].

An *algebraic probability space* $(\varphi, \mathcal{A})$ is a pair of an associative $\mathbb{C}$-algebra $\mathcal{A}$ and a $\mathbb{C}$-linear functional φ over $\mathcal{A}$. Denote by $\mathcal{K}$ the class of all algebraic probability spaces $(\varphi, \mathcal{A})$. We do not assume the existence of unit elements for these algebras $\mathcal{A}$. Denote by $\mathcal{A}'$ the set of all $\mathbb{C}$-linear finctionals φ over $\mathcal{A}$, and by $\mathcal{A}_1 \sqcup \mathcal{A}_2$ the free product of algebras $\mathcal{A}_1$ and $\mathcal{A}_2$. Then for any algebra homomorphisms $j_l : \mathcal{B}_l \to \mathcal{A}_l$ $(l = 1, 2)$, there exists a unique algebra homomorphism $j_1 \amalg j_2 : \mathcal{B}_1 \sqcup \mathcal{B}_2 \to \mathcal{A}_1 \sqcup \mathcal{A}_2$ such that $i_l \circ j_l = (j_1 \amalg j_2) \circ \iota_l$ for all $l = 1, 2$. Here $i_l : \mathcal{A}_l \to \mathcal{A}_1 \sqcup \mathcal{A}_2$ $(l = 1, 2)$ and $\iota_l : \mathcal{B}_l \to \mathcal{B}_1 \sqcup \mathcal{B}_2$ $(l = 1, 2)$ are the natural embeddings. We denote the 'expectation' of $a \in \mathcal{A}$ with respect to φ by $\varphi[a]$ instead of $\varphi(a)$.

Any map $\Box : \mathcal{K} \times \mathcal{K} \to \mathcal{K} : \big((\varphi_1, \mathcal{A}_1), (\varphi_2, \mathcal{A}_2)\big) \mapsto (\varphi_1 \Box \varphi_2, \mathcal{A}_1 \sqcup \mathcal{A}_2)$ is called a *product* over $\mathcal{K}$. For simplicity we used here the same symbol $\Box$ to denote two different levels of operations.

A *natural product* $\Box$ is a product over $\mathcal{K}$ satisfying the following four conditions.

(N1) *Universality.* For any algebra homomorphisms $j_l : \mathcal{B}_l \to \mathcal{A}_l$ and any $\varphi_l \in \mathcal{A}_l{}'$ $(l = 1, 2)$, we have

$$(\varphi_1 \Box \varphi_2) \circ (j_1 \amalg j_2) = (\varphi_1 \circ j_1) \Box (\varphi_2 \circ j_2).$$

(N2) *Associativity.* For all $(\varphi_l, \mathcal{A}_l) \in \mathcal{K}$ $(l = 1, 2, 3)$, we have

$$(\varphi_1 \Box \varphi_2) \Box \varphi_3 = \varphi_1 \Box (\varphi_2 \Box \varphi_3)$$

under the identification $(\mathcal{A}_1 \sqcup \mathcal{A}_2) \sqcup \mathcal{A}_3 = \mathcal{A}_1 \sqcup (\mathcal{A}_2 \sqcup \mathcal{A}_3)$.

(N3) *Extension.* For all $(\varphi_l, \mathcal{A}_l) \in \mathcal{K}$ $(l = 1, 2)$, we have

$$(\varphi_1 \Box \varphi_2) \circ i_l = \varphi_l \quad (l = 1, 2),$$

where i_l are the natural embeddings of $\mathcal{A}_l$ in the free product $\mathcal{A}_1 \sqcup \mathcal{A}_2$.

(N4) *Factorization.* For all $(\varphi_l, \mathcal{A}_l) \in \mathcal{K}$ $(l = 1, 2)$, we have

$$(\varphi_1 \Box \varphi_2)[i_1(a) i_2(b)] = (\varphi_1 \Box \varphi_2)[i_2(b) i_1(a)] = \varphi_1[a]\varphi_2[b]$$

for all $a \in \mathcal{A}_1$ and $b \in \mathcal{A}_2$.

The notion of natural product is nothing but a modification, to the non-symmetric case, of the notion of universal product of Schürmann in [5]. In [4] we proved the following classification theorem for natural products.

THEOREM 3.1. *There exist exactly five natural products*: *tensor*, *free*, *Boolean*, *monotone*, *and anti-monotone products.*

Here we omit the definitions of these five products (tensor, free, Boolean, monotone, and anti-monotone), because in our discussions in the present paper, we have no need to know them. For the detailed explanation of these products and independences, see the references in [1] and [4].

The strategy to prove Theorem 3.1 is to reduce this theorem to the next Theorem 3.2 through Theorems 3.3 and 3.4. It is the same strategy as that in the case of 'symmetric' products in [1] but in our non-symmetric case we must take care of some order structures on partitions.

Let us prepare some notation for the description of these theorems. Let $(\pi, \lambda) = \{V_1 \prec V_2 \prec \ldots \prec V_p\} \in \mathcal{LP}(n)$ be a linearly ordered partition, and $(\varphi_l, \mathcal{A}_l)_{l=1}^p$ be a family of algebraic probability spaces. Then we simply write $a_1 a_2 \ldots a_n \in \mathcal{A}_{(\pi,\lambda)}$ to describe the situation in which $a_1 \in \mathcal{A}_{l_1}$, $a_2 \in \mathcal{A}_{l_2}$, ..., $a_n \in \mathcal{A}_{l_n}$, and $l_k = q$ if and only if $k \in V_q$. We always identify $a \in \mathcal{A}_l$ with its natural image $i_l(a) \in \sqcup_{l=1}^p \mathcal{A}_l$. Let $a_1 a_2 \ldots a_n \in \mathcal{A}_{(\pi,\lambda)}$ and $\sigma \in \vec{\mathcal{P}}(n)$ with $\overline{\sigma} \leqslant \pi$ be fixed. Then we put for each $U = (i_1 i_2 \ldots i_k) \in \sigma$

$$\varphi_U[a_1, a_2, \ldots, a_n] := \varphi_{l(U)}[a_U] := \varphi_{l(U)}[a_{i_1} a_{i_2} \ldots a_{i_k}].$$

Here $l(U)$ is the label $l \in \{1, 2, \ldots, p\}$ such that $l_{i_s} = l$ for all $i_s \in \overline{U}$.

A *quasi-universal product* $\Box$ is a product over $\mathcal{K}$ satisfying the following three conditions.

(Q1) *Associativity*: the same as (N2).

(Q2) *Quasi-universal calculation rule for mixed moments*. There exists a family of constants

$$\{t(\pi, \lambda; \rho) |\ \rho \in \mathcal{P}(n), \rho \leqslant \pi, (\pi, \lambda) \in \mathcal{LP}(n), n \in \mathbb{N}^*\}$$

such that, for any p-tuple $(\varphi_l, \mathcal{A}_l)_{l=1}^p$ of algebraic probability spaces and $\varphi = \Box_{l=1}^p \varphi_l$, we have

$$\varphi[a_1 a_2 \ldots a_n] = \sum_{\rho \leqslant \pi} t(\pi, \lambda; \rho) \prod_{U \in \rho} \varphi_U[a_1, a_2, \ldots, a_n],$$

whenever $a_1 a_2 \ldots a_n \in \mathcal{A}_{(\pi,\lambda)}$ with $(\pi, \lambda) = \{V_1 \prec V_2 \prec \ldots \prec V_p\}$.

(Q3) *Normalization*:

$$t(1) = t(12) = t(21) = 1.$$

Here we put $t(s) := t(\pi, \lambda) := t(\pi, \lambda; \pi)$, where $s = (s_1 s_2 \ldots s_n)$ is the sequence associated with (π, λ) defined by the condition that $s_i = l$ if and only if $i \in V_l$.

THEOREM 3.2. *There exist exactly five quasi-universal products*: *tensor*, *free*, *Boolean*, *monotone*, *and anti-monotone products.*

Theorem 3.2 was proved in [3] based on the method of Speicher in [6]. The following expansion theorem (Theorem 3.3) was shown in [4] by a direct application of the theory of universal families of Ben Ghorbal and Schürmann [1].

THEOREM 3.3. *Let a natural product* $\square$ *be given. Then there exists uniquely a family of constants*

$$\{t(\pi, \lambda; \sigma) |\, \sigma \in \vec{\mathcal{P}}(n), \overline{\sigma} \leqslant \pi, (\pi, \lambda) \in \mathcal{LP}(n), n \in \mathbb{N}^*\}$$

such that, *for any* p*-tuple* $(\varphi_l, \mathcal{A}_l)_{l=1}^p$ *of algebraic probability spaces and* $\varphi = \square_{l=1}^p \varphi_l$, *we have*

$$\varphi[a_1 a_2 \ldots a_n] = \sum_{\substack{\sigma \in \vec{\mathcal{P}}(n) \\ \overline{\sigma} \leqslant \pi}} t(\pi, \lambda; \sigma) \prod_{U \in \sigma} \varphi_U[a_1, a_2, \ldots, a_n],$$

whenever $a_1 a_2 \ldots a_n \in \mathcal{A}_{(\pi,\lambda)}$.

Let $\sigma \in \vec{\mathcal{P}}(n)$ be a BGS-partition. A block $U = (i_1 i_2 \ldots i_k)$ in σ is said to be *wrong-ordered* if there exist $a, b \in \{1, 2, \ldots, k\}$ such that $a < b$ but $i_a > i_b$. A BGS-partition σ is said to be *wrong-ordered* if in σ there exists a wrong-ordered block U. Let $\{t(\pi, \lambda; \sigma)\}$ be the coefficients associated with the natural product $\square$ in Theorem 3.3. A coefficient $t(\pi, \lambda, \sigma)$ is said to be *wrong-ordered* if σ is wrong-ordered. In [4] we proved the following vanishment result. We denote it by (V).

THEOREM 3.4. *For any natural product*, *its wrong-ordered coefficients all vanish.*

Theorem 3.4 implies that any natural product is a quasi-universal product, and hence we reach for the classification Theorem 3.1 through Theorem 3.2.

However, the proof of the vanishment result (V) (Theorem 3.4) given in [4] is a cumbersome one consisting of elementary but heavy calculations, unfortunately. Therefore, for the clear understanding of the classification theorem, it is desirable to improve this heavy proof on a more light one. In Section 4 we give such a simplified proof for the vanishment result under some additional condition of positivity (P) for natural products.

Now let us define the positivity for a product $\square$. Let $\mathcal{A}$ be a *-algebra and φ be a linear functional over $\mathcal{A}$. The unitization $(\widetilde{\varphi}, \widetilde{\mathcal{A}})$ of $(\varphi, \mathcal{A})$ is the pair of a unital *-algebra $\widetilde{\mathcal{A}}$ and a unital linear functional $\widetilde{\varphi}$ over $\widetilde{\mathcal{A}}$, defind by $\widetilde{\mathcal{A}} := \mathbb{C}1_{\widetilde{\mathcal{A}}} \oplus \mathcal{A}$ with $1_{\widetilde{\mathcal{A}}}$ an artificial unit, and $\widetilde{\varphi}[1_{\widetilde{\mathcal{A}}}] = 1$, $\widetilde{\varphi}[a] = \varphi[a]$, $a \in \mathcal{A}$. If $\mathcal{A}$ is a unital *-algebra and φ is a unital linear functional over $\mathcal{A}$ ($\varphi[1_{\mathcal{A}}] = 1$), then φ is a state on $\mathcal{A}$ if and only if $\widetilde{\varphi}$ is a state on $\widetilde{\mathcal{A}}$. A *-probability space $(\mathcal{A}, \varphi)$ is a pair of a *-algebra $\mathcal{A}$ and a linear functional φ over $\mathcal{A}$ such that $\widetilde{\varphi}$ is a state on $\widetilde{\mathcal{A}}$.

A product $\Box$ over $\mathcal{K}$ is said to be *positive* if it satisfies the following condition.

(P) *Positivity.* For any *-algebras $\mathcal{A}_l$ and any functionals $\varphi_l \in \mathcal{A}_l'$ ($l = 1, 2$), $\widetilde{\varphi_1 \Box \varphi_2}$ is a state over $\widetilde{\mathcal{A}_1 \sqcup \mathcal{A}_2}$ whenever $\widetilde{\varphi}_l$ is a state over $\widetilde{\mathcal{A}}_l$ for each $l = 1, 2$.

The five products (tensor, free, Boolean, monotone, and anti-monotone) are positive.

Using the positivity (P), we can prove without heavy calculations the following Theorem 3.5 which we denote by (V^+).

THEOREM 3.5. *For any positive natural product, its wrong-ordered coefficients all vanish.*

The proof of Theorem 3.5, i.e. (V^+), will be presented in Section 4. (V^+) implies that any positive natural product is a quasi-universal product, and so we immediately reach for the following classification Theorem 3.6 through Theorem 3.2.

THEOREM 3.6. *There exist exactly five positive natural products: tensor, free, Boolean, monotone, and anti-monotone products.*

Now, by the same argument as in [2] that the four conditions (N1), (N2), (N3), and (P) imply the condition (N4), we get the next Theorem 3.7. First we need the following definition.

A *positive universal product* is a product over $\mathcal{K}$ satisfying the four conditions (N1), (N2), (N3), and (P). A product over $\mathcal{K}$ is said to be *degenerate* if $(\varphi_1 \Box \varphi_2)[a_1 a_2 \ldots a_n] = 0$ whenever $a_1 a_2 \ldots a_n \in \mathcal{A}_{(\pi,\lambda)}$ with $|\pi| \geqslant 2$.

THEOREM 3.7. *There exist exactly five non-degenerate positive universal products: tensor, free, Boolean, monotone, and anti-monotone products.*

This is the same theorem as Theorem 2.5 in [2], but this time the proof is improved so that it is dependent on Theorem 3.6, and hence on (V^+) (Theorem 3.5), and not dependent on (V) (Theorem 3.4) with heavy calculations.

4. A SIMPLE PROOF OF VANISHMENT RESULT

In this section we prove the vanishment result (V^+), i.e. Theorem 3.5, not using heavy algebraic calculations, but using the positivity.

For our purpose it is sufficient to show the following Proposition 4.1 from which we conclude that $t(\pi, \lambda; \sigma_0) = 0$ for all $\sigma_0 \in \vec{\mathcal{P}}(n) \setminus \mathcal{P}(n)$.

PROPOSITION 4.1. *Let $\Box$ be a positive natural product. Then for each $n \in \mathbb{N}^*$, each $(\pi, \lambda) \in \mathcal{LP}(n)$, and each $\sigma_0 \in \vec{\mathcal{P}}(n) \setminus \mathcal{P}(n)$ with $\overline{\sigma_0} \leqslant \pi$ there exist *-probability spaces $(\mathcal{A}_l, \varphi_l)_{l=1}^{|\pi|}$ and a sequence of elements $a_1, a_2, \ldots, a_n$ with*

$a_1a_2\dots a_n \in \mathcal{A}_{(\pi,\lambda)}$ *such that for* $\varphi = \square_{l=1}^{|\pi|}\varphi_l$ *we have*

(1) $\prod_{W\in\sigma}\varphi_{l(W)}[a_W] = \delta_{\sigma,\sigma_0}$ *for all* $\sigma \in \vec{\mathcal{P}}(n)$ *with* $\overline{\sigma} \leqslant \pi$;

(2) $\varphi\big[a_1a_2\dots a_n\big] = 0$.

At first let us give a construction of $(\mathcal{A}_l,\varphi_l)_{l=1}^{|\pi|}$ and $a_1,a_2,\dots,a_n$. Let (π,λ) and σ_0 be fixed. Suppose that $(\pi,\lambda) = \{V_1 \prec V_2 \prec \dots \prec V_{|\pi|}\}$ and $\sigma_0 = \{U_1,U_2,\dots,U_{|\sigma_0|}\}$. For each block $V \in \pi$, we put $\sigma_0(V) := \{U \in \sigma_0 | \overline{U} \subset V\}$. Then since $\overline{\sigma_0} \leqslant \pi$, we have $V = \bigcup_{U\in\sigma_0(V)}\overline{U}$.

For each $V \in \pi$, let us construct a *-probability space $(\mathcal{A}_V,\varphi_V)$ by

$$\mathcal{A}_V = \bigoplus_{U\in\sigma_0(V)}\mathcal{B}_U, \qquad \varphi_V = \frac{1}{\#\big(\sigma_0(V)\big)}\Big(\bigoplus_{U\in\sigma_0(V)}\psi_U\Big),$$

where we put, for each $U \in \sigma_0(V)$, $\mathcal{B}_U = M_{d(U)}(\mathbb{C})$ the matirix algebra, $\psi_U(\cdot) = \langle e_1^{(U)} | \cdot e_1^{(U)}\rangle$ the state over $\mathcal{B}_U$, $(e_i^{(U)})_{i=1}^{d(U)}$ the natural orthonormal basis of $\mathcal{H}_U := \mathbb{C}^{d(U)}$, and $d(U) := lg(U)$ the length of U.

On these *-probability spaces $(\mathcal{A}_l,\varphi_l) := (\mathcal{A}_{V_l},\varphi_{V_l})$, $l = 1,2,\dots,|\pi|$, we construct the operators $a_1,a_2,\dots,a_n$ ($a_1 \in \mathcal{A}_{l_1}, a_2 \in \mathcal{A}_{l_2},\dots,a_n \in \mathcal{A}_{l_n}$) as follows. For each block $U = (i_1i_2\dots i_k) \in \sigma_0$ with $\overline{U} \subset V$ and $k \geqslant 2$, we define the operators $a_{i_1},a_{i_2},\dots,a_{i_k}$ in $\mathcal{A}_V$ as the natural extensions

$$a_{i_q} := \widetilde{b_{i_q}} := b_{i_q} \oplus \Big(\bigoplus_{\substack{U' \in \sigma_0(V) \\ U' \neq U}} 0\Big)$$

of the operators $b_{i_1},b_{i_2},\dots,b_{i_k}$ in $\mathcal{B}_U$ given by

$$(b_{i_1},b_{i_2},b_{i_3},\dots,b_{i_{k-1}},b_{i_k}) \;=\; (E_{1,k},E_{k,k-1},E_{k-1,k-2},\dots,E_{3,2},E_{2,1}),$$

where $E_{i,j}$ are the matrix units in $\mathcal{B}_U$, i.e. $\langle e_k^{(U)}|E_{i,j}e_l^{(U)}\rangle = \delta_{ik}\delta_{jl}$. When $U \in \sigma_0(V)$ is a singleton block $U = (i)$, we put $a_i := \widetilde{b_i}$ with $b_i = I_U$ the identity matrix of $\mathcal{B}_U$. These operators $a_{i_1},a_{i_2},\dots,a_{i_k}$ over all $U = (i_1i_2\dots i_k) \in \sigma_0$ well-define the operators $a_1,a_2,\dots,a_n$ since $\{1,2,\dots,n\}$ is the disjoint union of all $\overline{U}$ ($U \in \sigma_0$). Finally, we normalize the operators $a_1,a_2,\dots,a_n$. Put $K := \prod_{W\in\sigma_0}\varphi_{l(W)}[a_W]$. Since $K \neq 0$, we can put $a_1' := K^{-1}a_1$ and $a_i' := a_i$ ($i \geqslant 2$). Then it can be proved that the operators $a_1',a_2',\dots,a_n'$ satisfy the desired properties (1) and (2) as follows.

P r o o f o f P r o p o s i t i o n 4.1. Let us prove the properties (1) and (2) in Proposition 4.1 separately. We remark here that the positivity is used only in the final step of the proof of property (2) in the form of the Cauchy–Schwarz inequality.

P r o o f o f p r o p e r t y (1). We examine, for general $\sigma \in \vec{\mathcal{P}}(n)$ with $\overline{\sigma} \leqslant \pi$, the value of $\prod_{W\in\sigma}\varphi_{l(W)}[a_W]$, where we put $l(W) = q$ if $\overline{W} \subset V_q \in \pi$. Concerning a BGS-partition σ with $\overline{\sigma} \leqslant \pi$, we consider the following three cases a, b and c.

Case a. There exist $W \in \sigma$, and $U, U' \in \sigma_0$ such that $\overline{W} \cap \overline{U} \neq \emptyset$, $\overline{W} \cap \overline{U'} \neq \emptyset$, $U \neq U'$. In this case there exists a common V ($\in \pi$) such that $\overline{W} \cup \overline{U} \cup \overline{U'} \subset V$. The blocks W, U, U' can be expressed as

$$W = (\iota_1\iota_2 \dots \iota_s), \quad U = (i_1 i_2 \dots i_t), \quad U' = (j_1 j_2 \dots j_u),$$

respectively. From the assumption there exist some $s_1, s_2 \in \{1, 2, \dots, s\}$, $t_0 \in \{1, 2, \dots, t\}$ and $u_0 \in \{1, 2, \dots, u\}$ such that

$$\begin{aligned} a_{\iota_{s_1}} &= a_{i_{t_0}} = \widetilde{b_{i_{t_0}}} \quad (b_{i_{t_0}} \in \mathcal{B}_U), \\ a_{\iota_{s_2}} &= a_{j_{u_0}} = \widetilde{b_{j_{u_0}}} \quad (b_{j_{u_0}} \in \mathcal{B}_{U'}). \end{aligned}$$

Since $U \cap U' = \emptyset$, we have $Aa_{\iota_{s_1}} Ba_{\iota_{s_2}} C = 0$ and $Aa_{\iota_{s_2}} Ba_{\iota_{s_1}} C = 0$ for all $A, B, C \in \mathcal{A}_V$, and hence $a_{\iota_1} a_{\iota_2} \dots a_{\iota_s} = 0$. So we get $\prod_{W' \in \sigma} \varphi_{l(W')}[a_{W'}] = 0$.

Case b. $\overline{\sigma} \leqslant \overline{\sigma_0}$ and $\sigma \neq \sigma_0$. In this case there exist $W \in \sigma$ and $U \in \sigma_0$ such that $\overline{W} \subset \overline{U}$ and $W \neq U$. By the way, W and U can be expressed as $W = (j_1 j_2 \dots j_s)$ and $U = (i_1 i_2 \dots i_t)$. Note that $\{j_1, j_2, \dots, j_s\} \subset \{i_1, i_2, \dots, i_t\}$ and $t \geqslant 2$.

Let us examine the value of $b_{j_1} b_{j_2} \dots b_{j_s} \xi$, where $\xi := e_1^{(U)}$. For the vector $b_{j_s}\xi$ to be non-zero it is necessary that $j_s = i_t$, because b_{i_t} is the only element in $\{b_{i_1}, b_{i_2}, \dots, b_{i_t}\}$ that corresponds to $E_{2,1}$ ($\in \mathcal{B}_U$). Next (when $t \geqslant 3$), for the vector $b_{j_{s-1}} b_{j_s} \xi$ to be non-zero it is necessary that $j_s = i_t$ and $j_{s-1} = i_{t-1}$, because $b_{i_{t-1}}$ is the only element in $\{b_{i_1}, b_{i_2}, \dots, b_{i_t}\}$ that corresponds to $E_{3,2}$. Repeating this argument we see that for the vector $b_{j_1} b_{j_2} \dots b_{j_s} \xi$ to be non-zero it is necessary that $(j_1 j_2 \dots j_{s-1} j_s) = (i_{t-s+1} i_{t-s+2} \dots i_{t-1} i_t)$.

Furthermore, for the expectation

$$\psi_U[b_{j_1} b_{j_2} \dots b_{j_s}] = \langle \xi | b_{j_1} b_{j_2} \dots b_{j_s} \xi \rangle$$

to be non-zero it is necessary that $(j_1 j_2 \dots j_s) = (i_1 i_2 \dots i_t)$, because b_{i_1} is the only element in $\{b_{i_1}, b_{i_2}, \dots, b_{i_t}\}$ that corresponds to $E_{1,t}$. But by assumption we have $W = (j_1 \dots j_s) \neq (i_1 \dots i_t) = U$, and so $\psi_U[b_W] = \psi_U[b_{j_1} \dots b_{j_s}] = 0$. Since

$$\varphi_{l(W)}[a_W] = \varphi_V[a_W] = \frac{1}{\#(\sigma_0(V))} \psi_U[b_W] = 0,$$

we have $\prod_{W' \in \sigma} \varphi_{l(W')}[a_{W'}] = 0$.

Case c. $\sigma = \sigma_0$. In this case it is clear that $\prod_{W \in \sigma} \varphi_{l(W)}[a_W] = K$ ($\neq 0$).

From the cases a, b, and c, we can infer that the property (1) holds for the operators $a'_1, a'_2, \dots, a'_n$.

P r o o f o f p r o p e r t y (2). Let v be the largest 'wrong' element in the set $\{1, 2, \ldots, n\}$ in the sense that

$$v := \max\bigl\{m \in \{1, 2, \ldots, n\} |\ \exists U = (i_1 i_2 \ldots i_{lg(U)}) \in \sigma_0,\\ \exists a, b \in \{1, 2, \ldots, lg(U)\} \text{ such that } a < b \text{ and } i_b < i_a = m\bigr\}.$$

Since σ_0 is wrong-ordered, the above set is non-empty, and hence the required v exists. Let U_0 be the block in σ_0 that contains v, and V_0 be the block in π that contains v. Then obviously $v \in \overline{U_0} \subset V_0$ and $lg(U_0) \geqslant 2$, and the block U_0 must be of the form

$$U_0 = (i_1 \ldots i_a \ldots i_b \ldots i_k) = (i_1 \ldots v \ldots u \ldots i_k),$$

where $a < b$ and $i_b = u < v = i_a$ for some $a, b \in \{1, 2, \ldots, k\}$ and for some $u \in \overline{U_0}$.

Now, estimate the norm of the vector (i.e. an equivalence class) $[a_v a_{v+1} \ldots a_n]$ in the GNS-representation space associated with $\varphi = \Box_{l=1}^{|\pi|} \varphi_l$. First we have from Theorem 3.3

$$\begin{aligned} \|[a_v a_{v+1} \ldots a_n]\|^2 &= \varphi[(a_v a_{v+1} \ldots a_n)^* (a_v a_{v+1} \ldots a_n)] \\ &= \varphi[a_n^* \ldots a_{v+1}^* a_v^* a_v a_{v+1} \ldots a_n] \\ &= \varphi[a_n^* \ldots a_{v+1}^* (a_v^* a_v) a_{v+1} \ldots a_n] \\ &= \sum_{\substack{\tau \in \vec{\mathcal{P}}(S) \\ \overline{\tau} \leqslant \rho}} t(\rho, \mu; \tau) \prod_{T \in \tau} \varphi_{l(T)}[c_T]. \end{aligned}$$

Here S is the linearly ordered finite set given by

$$S = \{-n, -(n-1), \ldots, -(v+1), v, v+1, \ldots, n-1, n\},$$

(ρ, μ) is the linearly ordered partion of S associated with the sequence

$$(l_n, l_{n-1}, \ldots, l_{v+1}, l_v, l_{v+1}, \ldots, l_{n-1}, l_n),$$

and c's are the operators defined by

$$c_{-n} = a_n^*,\ c_{-(n-1)} = a_{n-1}^*, \ldots, c_{-(v+1)} = a_{v+1}^*,\\ c_v = a_v^* a_v,\ c_{v+1} = a_{v+1}, \ldots, c_{n-1} = a_{n-1},\ c_n = a_n.$$

Also here we put $l(T) = q$ if it follows that $\overline{T} \cap V_q \neq \emptyset$ or $-\overline{T} \cap V_q \neq \emptyset$ with $-\overline{T} := \{-m | m \in \overline{T}\}$.

For each partition $\tau \in \vec{\mathcal{P}}(S)$ with $\overline{\tau} \leqslant \rho$, there exists a unique block $T_0 \in \tau$ such that $\overline{T_0} \ni v$. Also let R_0 be the unique block in ρ such that $R_0 \ni v$; then we have $\overline{T_0} \subset R_0$. Since $u \notin \{v, v+1, \ldots, n\}$, we have $u \notin \overline{T_0}$, and hence $\{a_u^*, a_u\} \cap \{c_m | m \in \overline{T_0}\} = \emptyset$. By the definition of the operators $a_{i_1}, \ldots, a_{i_a}, \ldots, a_{i_b}, \ldots, a_{i_k}$ $(\in \mathcal{A}_{V_0})$ based on the block $U_0 = (i_1 \ldots i_a \ldots i_b \ldots i_k)$ with $i_a = v$, $i_b = u$, we see that

$$a_u = \widetilde{E^{(U_0)}_{N+1,N}} \quad \text{with } N = (k-b)+1,$$
$$a_v = \widetilde{E^{(U_0)}_{M+1,M}} \quad \text{with } M = (k-a)+1 \text{ for } a \geqslant 2,$$
$$a_v = \widetilde{E^{(U_0)}_{1,k}} \quad \text{for } a = 1.$$

So we have $c_v = a_v^* a_v = \widetilde{E^{(U_0)}_{M,M}}$ for $a \geqslant 2$, and $c_v = a_v^* a_v = \widetilde{E^{(U_0)}_{k,k}}$ for $a = 1$. Put $c_{\overline{T_0}} := \{c_m | m \in \overline{T_0}\}$, $a_{\overline{U_0}} := \{a_m | m \in \overline{U_0}\}$, and $a^*_{\overline{U_0}} := \{a^*_m | m \in \overline{U_0}\}$. Then we have

$$\{c_v\} \subset c_{\overline{T_0}} \subset \big((a_{\overline{U_0}} \cup a^*_{\overline{U_0}}) \setminus \{a_u, a_u^*, a_v, a_v^*\}\big) \cup \{c_v\}.$$

This means that $\{\widetilde{E^{(U_0)}_{M,M}}\} \subset c_{\overline{T_0}}$ and

$$c_{\overline{T_0}} \subset \big((a_{\overline{U_0}} \cup a^*_{\overline{U_0}}) \setminus \{\widetilde{E^{(U_0)}_{N+1,N}}, \widetilde{E^{(U_0)}_{N,N+1}}, \widetilde{E^{(U_0)}_{M+1,M}}, \widetilde{E^{(U_0)}_{M,M+1}}\}\big) \cup \{\widetilde{E^{(U_0)}_{M,M}}\}$$

for $a \geqslant 2$, and that $\{\widetilde{E^{(U_0)}_{k,k}}\} \subset c_{\overline{T_0}}$ and

$$c_{\overline{T_0}} \subset \big((a_{\overline{U_0}} \cup a^*_{\overline{U_0}}) \setminus \{\widetilde{E^{(U_0)}_{N+1,N}}, \widetilde{E^{(U_0)}_{N,N+1}}, \widetilde{E^{(U_0)}_{1,k}}, \widetilde{E^{(U_0)}_{k,1}}\}\big) \cup \{\widetilde{E^{(U_0)}_{k,k}}\}$$

for $a = 1$.

Put $T_* := \big(\overline{T_0} \cup (-\overline{T_0})\big) \cap \{v, v+1, \ldots, n\}$; then $\overline{T_0} \subset T_* \cup (-T_*)$. For simplicity we denote $E^{(U_0)}_{i,j}$ by $E_{i,j}$. Concerning T_*, we consider the following three cases a, b, and c.

Case a. There exists $m \in T_*$ such that $m \notin \overline{U_0}$. In this case there exist $m' \in S$ and $U' \in \sigma_0$ such that $c_{m'} = a_m = \widetilde{b_m}$ or $c_{m'} = a_m^* = \widetilde{b_m^*}$ with $b_m \in U'$ and $U' \neq U_0$. So we have two operators b_m (or b_m^*) $\in \mathcal{B}_{U'}$ and $b_v \in \mathcal{B}_{U_0}$ with $U' \cap U_0 = \emptyset$ so that $\{\widetilde{b_v}, \widetilde{b_m}\} \subset c_{\overline{T_0}}$ or $\{\widetilde{b_v}, \widetilde{b_m^*}\} \subset c_{\overline{T_0}}$. This implies $\varphi_{l(T_0)}[c_{T_0}] = 0$.

Case b. $T_* \subset \overline{U_0}$ and $i_a = v$ $(a \geqslant 2)$. In this case let d_m $(m \in \overline{T_0})$ be the operators in $\mathcal{B}_{U_0}$ such that $c_m = \widetilde{d_m}$. Then we have in the algebra $\mathcal{B}_{U_0}$

$$\begin{aligned}\{E_{M,M}\} \subset d_{\overline{T_0}} \subset \{&E_{1,k}, E_{k,k-1}, \ldots, \widehat{E_{M+1,M}}, \ldots, \widehat{E_{N+1,N}}, \ldots, E_{3,2}, E_{2,1}\} \\ \cup \{&E_{1,2}, E_{2,3}, \ldots, \widehat{E_{N,N+1}}, \ldots, \widehat{E_{M,M+1}}, \ldots, E_{k-1,k}, E_{k,1}\} \\ \cup \{&E_{M,M}\}\end{aligned}$$

with $1 \leqslant N < M \leqslant k-1$. Here the symbol $\widehat{\ }$ denotes the omission. Since there are two gaps $\{N, N+1\}$ and $\{M, M+1\}$ in the circle graph (minus two edges)

$$\big\{\{1,2\},\{2,3\},\{3,4\},\ldots,\{k-1,k\},\{k,1\}\big\} \setminus \big\{\{N,N+1\},\{M,M+1\}\big\}$$

with $N < M$, the vertex M cannot be connected to the vertex 1. So there is no path allowing for starting from the vertex 1, passing through the vertex M, and finally arriving at the vertex 1 again. This implies that

$$\varphi_{l(T_0)}[c_{T_0}] = \frac{1}{\#\sigma_0(V_0)} \langle \xi | d_{\iota_1} \ldots d_{\iota_t} \xi \rangle = 0$$

with $T_0 = (\iota_1 \ldots \iota_t)$ and $\xi = e_1^{(U_0)}$.

Case c. $T_* \subset \overline{U_0}$ and $i_a = v$ $(a = 1)$. In this case let d_m $(m \in \overline{T_0})$ be the same as above. Then we have in $\mathcal{B}_{U_0}$

$$\begin{aligned}\{E_{k,k}\} \subset d_{\overline{T_0}} \subset \{&E_{k,k-1}, \ldots, \widehat{E_{N+1,N}}, \ldots, E_{3,2}, E_{2,1}\} \\ &\cup \{E_{1,2}, E_{2,3}, \ldots, \widehat{E_{N,N+1}}, \ldots, E_{k-1,k}\} \cup \{E_{k,k}\}.\end{aligned}$$

Since there is one gap $\{N, N+1\}$ in the linear graph (minus one edge)

$$\big\{\{1,2\},\{2,3\},\{3,4\},\ldots,\{k-1,k\}\big\} \setminus \big\{\{N,N+1\}\big\}$$

with $N \leqslant k-1$, the vertex k cannot be connected to the vertex 1. So there is no path allowing for starting from the vertex 1, passing through the vertex k, and finally arriving at the vertex 1 again. This implies that

$$\varphi_{l(T_0)}[c_{T_0}] = \frac{1}{\#\sigma_0(V_0)} \langle \xi | d_{\iota_1} \ldots d_{\iota_t} \xi \rangle = 0.$$

For each of the cases a, b, and c, we have

$$\|[a_v a_{v+1} \ldots a_n]\|^2 \ = \sum_{\substack{\tau \,\in\, \vec{\mathcal{P}}(S) \\ \overline{\tau} \,\leqslant\, \rho}} t(\rho, \mu; \tau) \prod_{T \in \tau} \varphi_{l(T)}[c_T] \ = \ 0.$$

Hence we get, by the Cauchy–Schwarz inequality,

$$|\varphi[a_1 \ldots a_{v-1} a_v \ldots a_n]| \leqslant \|[a_1 \ldots a_{v-1}]\| \, \|[a_v \ldots a_n]\| = 0,$$

from which we conclude that $\varphi[a'_1 a'_2 \ldots a'_n] = 0$. ∎

Now we have completed the simplified proof for (V^+), i.e. Theorem 3.5.

REMARK 4.1. In Proposition 4.1, the role of the associativity (N2) is not essential. Indeed, instead of iterating an associative binary product $\square : \mathcal{K}^2 \to \mathcal{K}$, we can start from an L-ary product $\square^{(L)} : \mathcal{K}^L \to \mathcal{K}$, where L is a non-empty set. Then, for any L-ary product $\square^{(L)}$ satisfying universality, the expansion theorem (Theorem 3.3) still holds in the form that the coefficients $t(\pi, \lambda; \sigma)$ in the expression are replaced with $t(l_1 l_2 \dots l_n; \sigma)$. Here $(l_1 l_2 \dots l_n)$ is the sequence of indices in L associated with $a_1, a_2, \dots, a_n$. Besides, the proof of Proposition 4.1 still works for any product $\square^{(L)}$ satisfying universality and positivity. So the vanishment result (Theorem 3.5) still holds for this "non-associative product" $\square^{(L)}$.

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