

ON FREE INFINITE DIVISIBILITY
FOR CLASSICAL MEIXNER DISTRIBUTIONS

BY

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Abstract. We prove that symmetric Meixner distributions, whose probability densities are proportional to $|\Gamma(t + ix)|^2$, are freely infinitely divisible for $0 < t \leq \frac{1}{2}$. The case $t = \frac{1}{2}$ corresponds to the law of Lévy's stochastic area whose probability density is $1/\cosh(\pi x)$. A logistic distribution, whose probability density is proportional to $1/\cosh^2(\pi x)$, is also freely infinitely divisible.

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1. INTRODUCTION

The free convolution $\mu \boxplus \nu$ of probability measures μ and ν on \mathbb{R} is the distribution of $X + Y$, where X and Y are free self-adjoint random variables following the distributions μ and ν , respectively. A probability measure ν on \mathbb{R} is said to be *freely infinitely divisible* if, for any $n \in \{1, 2, 3, \dots\}$, there exists ν_n such that

$$\nu = \underbrace{\nu_n \boxplus \dots \boxplus \nu_n}_{n \text{ times}}.$$

This concept was introduced in [17] and its basic characterization was established in [8]. The most important freely infinitely divisible distributions are Wigner's semicircle law and the free Poisson law.

Recent work has increased examples of probability measures which are infinitely divisible in both senses, classical and free: Gaussian distribution [7], chi-square distribution $(1/\sqrt{\pi x})e^{-x}1_{[0,\infty)}(x) dx$ [6], positive Boolean stable law with stability index $\alpha \in (0, \frac{1}{2}]$ [5], and Student distribution $\frac{1}{B(1/2, n-1/2)}(1+x^2)^{-n}1_{\mathbb{R}}(x) dx$ for $n = 1, 2, 3, \dots$ [10]. It is not yet clear whether a general theory of the intersection of free and classical infinite divisibility exists. We will add two more exam-

ples, Meixner distributions and the logistic distribution, which may contribute to a solution.

We will prove that symmetric *Meixner distributions*

$$\rho_t(dx) := \frac{4^t}{2\pi\Gamma(2t)} |\Gamma(t+ix)|^2 dx, \quad x \in \mathbb{R},$$

are freely infinitely divisible for $0 < t \leq \frac{1}{2}$, where $\Gamma(z)$ is the gamma function defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad z > 0.$$

The gamma function satisfies the functional relation $\Gamma(z+1) = z\Gamma(z)$, which extends Γ to a meromorphic function in \mathbb{C} with poles at $z = 0, -1, -2, -3, \dots$ (see [1], Chapter 6). The measures ρ_t are probability distributions of a Lévy process, called a *Meixner process* [16], since the characteristic function of ρ_t is given by

$$(1.1) \quad \hat{\rho}_t(z) = \left(\frac{1}{\cosh(z/2)} \right)^{2t}.$$

Hence ρ_t is classically infinitely divisible for any $t > 0$. The measure ρ_t orthogonalizes Meixner–Pollaczek polynomials $\{P_n^{(t)}(x)\}_{n=0}^\infty$ which satisfy the recurrence relation (see [11])

$$xP_n^{(t)}(x) = P_{n+1}^{(t)}(x) + \frac{n(n+2t-1)}{4} P_{n-1}^{(t)}(x), \quad n \geq 1,$$

with initial conditions $P_0^{(t)}(x) = 1$, $P_1^{(t)}(x) = x$.

If $t = \frac{1}{2}$, the measure $\rho_{1/2}$ coincides with

$$\mu_1(dx) = \frac{1}{\cosh(\pi x)} dx, \quad x \in \mathbb{R},$$

which is the law of *Lévy's stochastic area*¹

$$\frac{1}{2} \int_0^1 (B_t^1 dB_t^2 - B_t^2 dB_t^1),$$

where (B_t^1, B_t^2) is a standard two-dimensional Brownian motion [13]. The moments m_n of the rescaled measure

$$\frac{1}{2 \cosh(\pi x/2)} dx$$

¹This measure is also called the *hyperbolic secant distribution*.

are *Euler numbers* (with positive signs):

$$(m_0, m_2, m_4, m_6, m_8, \dots) = (1, 1, 5, 61, 1385, 50521, \dots), \quad m_{2n+1} = 0, \quad n \geq 0.$$

See [1], Chapter 23, for Euler numbers.

The *logistic distribution*

$$\mu_2(dx) = \frac{\pi}{2 \cosh^2(\pi x)} dx, \quad x \in \mathbb{R},$$

is known to be classically infinitely divisible [9], and we are going to prove that it is also freely infinitely divisible. This measure orthogonalizes *continuous Hahn polynomials* $\{P_n(x)\}_{n=0}^\infty$ which satisfy the recurrence relation (see [11])

$$xP_n(x) = P_{n+1}(x) + \frac{n^4}{4(4n^2 - 1)}P_{n-1}(x), \quad n \geq 1,$$

with initial conditions $P_0(x) = 1$, $P_1(x) = x$.

The moments m'_n of the rescaled measure

$$\frac{\pi}{4 \cosh^2(\pi x/2)} dx$$

are

$$(m'_0, m'_2, m'_4, m'_6, m'_8, \dots) = \left(1, \frac{1}{3}, \frac{7}{15}, \frac{31}{21}, \frac{127}{15}, \dots\right), \quad m'_{2n+1} = 0, \quad n \geq 0,$$

which can be written as $m'_n = |(2 - 2^n)B_n|$ in terms of *Bernoulli numbers* B_n (see [1]).

2. PRELIMINARIES

Let \mathbb{C}^+ and \mathbb{C}^- be the upper half-plane and the lower half-plane, respectively. Basic tools for proving free infinite divisibility of a probability measure μ are the Cauchy transform

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z - x} \mu(dx), \quad z \in \mathbb{C}^+,$$

and its reciprocal $F_\mu(z) := 1/G_\mu(z)$. Let $\Gamma_{\alpha, M}$ be a truncated cone

$$\Gamma_{\alpha, M} := \{z \in \mathbb{C}^+ : \operatorname{Im} z > M, |\operatorname{Re} z| < \alpha \operatorname{Im} z\}, \quad \alpha, M > 0.$$

The reciprocal Cauchy transform maps \mathbb{C}^+ to \mathbb{C}^+ analytically, and satisfies

$\operatorname{Im} F_\mu(z) \geq \operatorname{Im} z$ for $z \in \mathbb{C}^+$. For any $0 < \varepsilon < \alpha$ and μ , there exist $M > 0$ and a unique univalent inverse map F_μ^{-1} from $\Gamma_{\alpha-\varepsilon, (1+\varepsilon)M}$ into \mathbb{C}^+ such that $F_\mu(\Gamma_{\alpha, M}) \supset \Gamma_{\alpha-\varepsilon, (1+\varepsilon)M}$ and $F_\mu \circ F_\mu^{-1} = \operatorname{Id}$ in $\Gamma_{\alpha-\varepsilon, (1+\varepsilon)M}$ (see [8]).

Free convolution and free infinite divisibility can be characterized by the *Voiculescu transform* of μ defined by

$$(2.1) \quad \phi_\mu(z) := F_\mu^{-1}(z) - z$$

in a domain of the form $\Gamma_{\beta, L}$.

THEOREM 2.1 (Bercovici and Voiculescu [8]). (1) *The free convolution $\mu \boxplus \nu$ is a unique probability measure such that*

$$\phi_{\mu \boxplus \nu}(z) = \phi_\mu(z) + \phi_\nu(z)$$

in a common domain of the form $\Gamma_{\beta, L}$.

(2) *A probability measure μ on \mathbb{R} is freely infinitely divisible if and only if $-\phi_\mu$ analytically extends to a Pick function, i.e. an analytic function which maps \mathbb{C}^+ into $\mathbb{C}^+ \cup \mathbb{R}$.*

In terms of analytic properties of F_μ^{-1} , a useful subclass of freely infinitely divisible distributions is introduced.

DEFINITION 2.1. A probability measure μ is said to be in the class \mathcal{UI} if F_μ^{-1} defined in a domain of the form $\Gamma_{\beta, L}$ analytically extends to a univalent map in \mathbb{C}^+ . Equivalently, $\mu \in \mathcal{UI}$ if and only if there exists a simply connected open set $\mathbb{C}^+ \subset \Omega \subset \mathbb{C}$ such that

- (i) F_μ analytically extends to a univalent map in Ω ,
- (ii) $F_\mu(\Omega) \supset \mathbb{C}^+$.

This equivalence is proved just by applying the Riemann mapping theorem.

REMARK 2.1. In [4] we required F_μ to be univalent in \mathbb{C}^+ in the definition of $\mu \in \mathcal{UI}$, but this automatically follows. If F_μ^{-1} is analytic in \mathbb{C}^+ , then we have $F_\mu^{-1} \circ F_\mu(z) = z$ for $z \in \mathbb{C}^+$ by the Identity Theorem, so that F_μ is univalent in \mathbb{C}^+ .

LEMMA 2.1 (Arizmendi and Hasebe [4]). (1) *If $\mu \in \mathcal{UI}$, then μ is freely infinitely divisible.*

(2) *The class \mathcal{UI} is closed with respect to the weak convergence.*

(3) *The class \mathcal{UI} is not closed under free convolution, i.e. $\mu, \nu \in \mathcal{UI}$ does not imply $\mu \boxplus \nu \in \mathcal{UI}$.*

This class was essentially introduced in [7] to show that the normal law is freely infinitely divisible, and this class has been successfully applied to several probability measures (see [2]–[5] and [10]). Examples are presented below, mostly taken from the aforementioned references.

EXAMPLE 2.1. The following probability measures belong to \mathcal{UI} .

(1) Wigner's semicircle law

$$\mathbf{w}(dx) = \frac{1}{2\pi} \sqrt{4 - x^2} 1_{[-2,2]}(x) dx, \quad F_{\mathbf{w}}^{-1}(z) = z + \frac{1}{z}.$$

(2) The free Poisson law (or Marchenko–Pastur law)

$$\mathbf{m}(dx) = \frac{1}{2\pi} \sqrt{\frac{4-x}{x}} 1_{(0,4]}(x) dx, \quad F_{\mathbf{m}}^{-1}(z) = z + \frac{z}{z-1}.$$

(3) The Cauchy distribution

$$\mathbf{c}(dx) = \frac{1}{\pi(1+x^2)} 1_{\mathbb{R}}(x) dx, \quad F_{\mathbf{c}}^{-1}(z) = z - i.$$

(4) The beta distribution (Arizmendi and Hasebe [4])

$$\beta_a(dx) = \frac{\sin(\pi a)}{\pi a} \left(\frac{1-x}{x}\right)^a 1_{(0,1)}(x) dx, \quad F_{\beta_a}^{-1}(z) = \frac{1}{1 - (1 - a/z)^{1/a}}$$

for $\frac{1}{2} \leq |a| < 1$. $\beta_{1/2}$ is equal to \mathbf{m} up to scaling.

(5) The Gaussian distribution (Belinschi et al. [7])

$$\mathbf{g}(dx) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} 1_{\mathbb{R}}(x) dx.$$

(6) The q -Gaussian distribution (Anshelevich et al. [2])

$$\mathbf{g}_q(dx) = \frac{\sqrt{1-q}}{\pi} \sin \theta(x) \prod_{n=1}^{\infty} (1 - q^n) |1 - q^n e^{2i\theta(x)}|^2 1_{[-2/\sqrt{1-q}, 2/\sqrt{1-q}]}(x) dx$$

for $q \in [0, 1)$, where $\theta(x)$ is the solution of

$$x = \frac{2}{\sqrt{1-q}} \cos \theta, \quad \theta \in [0, \pi].$$

When $q \rightarrow 1$, \mathbf{g}_q converges weakly to \mathbf{g} , and \mathbf{g}_0 coincides with \mathbf{w} . For $q \in (0, 1)$, the density function of \mathbf{g}_q can be written as (see [12])

$$\frac{1}{2\pi} q^{-1/8} (1-q)^{1/2} \Theta_1 \left(\frac{\theta(x)}{\pi}, \frac{1}{2\pi i} \log q \right),$$

where $\Theta_1(z, \tau) := 2 \sum_{n=0}^{\infty} (-1)^n (e^{i\pi\tau})^{(n+1/2)^2} \sin(2n+1)\pi z$ is a Jacobi theta function.

(7) The ultraspherical distribution (Arizmendi and Belinschi [3])

$$\mathbf{u}_n(dx) = \frac{1}{16^n B(n + \frac{1}{2}, n + \frac{1}{2})} (4 - x^2)^{n-1/2} 1_{[-2,2]}(x) dx, \quad n = 1, 2, 3, 4, \dots,$$

where $B(p, q)$ is the beta function. The semicircle law \mathbf{w} appears in the case $n = 1$ and the normal law \mathbf{g} in the limit $n \rightarrow \infty$ if \mathbf{u}_n are suitably scaled.

(8) The Student distribution (Hasebe [10])

$$\mathbf{t}_n(dx) = \frac{1}{B(\frac{1}{2}, n - \frac{1}{2})} \frac{1}{(1 + x^2)^n} 1_{\mathbb{R}}(x) dx, \quad n = 1, 2, 3, \dots$$

\mathbf{t}_1 coincides with \mathbf{c} , and if suitably scaled, \mathbf{t}_n weakly converge to \mathbf{g} as $n \rightarrow \infty$.

(9) The Boolean stable law (Arizmendi and Hasebe [5])

$$\frac{d\mathbf{b}_\alpha^\rho}{dx} = \begin{cases} \frac{\sin(\pi\rho\alpha)}{\pi} \frac{x^{\alpha-1}}{x^{2\alpha} + 2x^\alpha \cos(\pi\rho\alpha) + 1}, & x > 0, \\ \frac{\sin(\pi(1-\rho)\alpha)}{\pi} \frac{|x|^{\alpha-1}}{|x|^{2\alpha} + 2|x|^\alpha \cos(\pi(1-\rho)\alpha) + 1}, & x < 0, \end{cases}$$

for $0 < \alpha \leq \frac{1}{2}$, $\rho \in [0, 1]$.

If $\frac{1}{2} \leq \alpha \leq \frac{2}{3}$ and $2 - \frac{1}{\alpha} \leq \rho \leq \frac{1}{\alpha} - 1$, the Boolean stable law \mathbf{b}_α^ρ (defined as above too) is still freely infinitely divisible, but not in the class \mathcal{UI} (see [5]). However, most of the known freely infinitely divisible distributions belong to \mathcal{UI} as presented above.

In order to prove $\mu \in \mathcal{UI}$, the following sufficient condition is useful.

PROPOSITION 2.1. *A probability measure μ on \mathbb{R} is in \mathcal{UI} if there exists a simple, continuous curve $\gamma = (\gamma(t))_{t \in \mathbb{R}} \subset \overline{\mathbb{C}^-}$ with the following properties:*

(A) $\lim_{t \rightarrow \infty} |\gamma(t)| = \lim_{t \rightarrow -\infty} |\gamma(t)| = \infty$;

(B) $F_\mu(\gamma) \subset \overline{\mathbb{C}^-}$;

(C) F_μ extends to an analytic function in $D(\gamma)$ and to a continuous function on $\overline{D(\gamma)}$, where $D(\gamma)$ denotes the simply connected open set containing \mathbb{C}^+ with boundary γ ;

(D) $F_\mu(z) = z + o(z)$ uniformly as $z \rightarrow \infty$, $z \in D(\gamma)$.

Proof. For $R > |\gamma(0)|$, let $t_1 := \sup\{t < 0 : |\gamma(t)| \geq R\} \in (-\infty, 0)$ and $t_2 := \inf\{t > 0 : |\gamma(t)| \geq R\} \in (0, \infty)$. The circle $\{z \in \mathbb{C} : |z| = R\}$ is divided into two arcs by $\gamma(t_1), \gamma(t_2)$ (see Figure 1), and let A be the arc which contains $\{z \in \mathbb{C}^+ : |z| = R\}$. Consider a simple closed curve γ_R consisting of the arcs $(\gamma(t))_{t \in [t_1, t_2]}$ and A . By (D), we can take $R > 0$ large enough so that $|F_\mu(z) - z| \leq \frac{1}{2}|z|$ for $z \in D(\gamma)$, $|z| > R$. By the assumption (B), F_μ maps the simple closed

curve γ_R to a curve surrounding each point of $\{z \in \mathbb{C}^+ : |z| < \frac{1}{2}R\}$ exactly once, and so the univalent map F_μ^{-1} can be defined in $\{z \in \mathbb{C}^+ : |z| < \frac{1}{2}R\}$ as the left inverse map of $F_\mu|_{D(\gamma_R)}$ which maps numbers with large positive imaginary parts to numbers with large positive imaginary parts. Here $D(\gamma_R)$ is the bounded Jordan domain surrounded by γ_R . Letting $R \rightarrow \infty$, we conclude by analytic continuation that F_μ^{-1} exists in \mathbb{C}^+ as a univalent map. ■

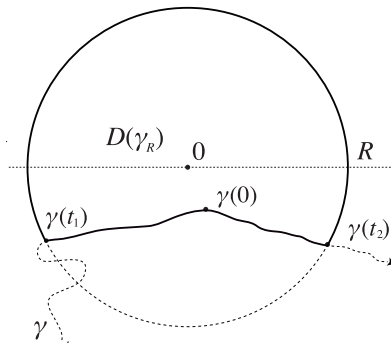


FIGURE 1

REMARK 2.2. Note that the map $F_\mu|_{D(\gamma_R)}$ may not be univalent in the whole of $D(\gamma_R)$. The fact that each point of $\{z \in \mathbb{C}^+ : |z| < \frac{1}{2}R\}$ has rotation number 1 implies that there exists a subset S_R (which is in fact open and simply connected) of $D(\gamma_R)$ such that F_μ is univalent in S_R and that $F_\mu(S_R) = \{z \in \mathbb{C}^+ : |z| < \frac{1}{2}R\}$.

3. PROOF FOR MEIXNER DISTRIBUTIONS

We present some properties of Meixner distributions.

(1) ρ_t is a probability measure for $t > 0$ because

$$\begin{aligned} \int_{\mathbb{R}} |\Gamma(t + ix)|^2 dx &= \int_{\mathbb{R}} \left| \int_0^\infty s^{t+ix-1} e^{-s} ds \right|^2 dx = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{tu-e^u} e^{ixu} du \right|^2 dx \\ &= 2\pi \int_{\mathbb{R}} e^{2tu-2e^u} du = 2\pi \int_0^\infty \left(\frac{s}{2}\right)^{2t} e^{-s} \frac{ds}{s} = \frac{2\pi\Gamma(2t)}{4^t}, \end{aligned}$$

where Plancherel’s theorem was used in the third equality.

(2) $\rho_{1/2}$ coincides with μ_1 thanks to the formula $\Gamma(1 - z)\Gamma(z) = \pi/\sin(\pi z)$.

(3) By the residue theorem, $G_t := G_{\rho_t}$ has the series expansion

$$G_t(z) = \frac{4^t}{\Gamma(2t)} \sum_{n=0}^\infty \frac{(-1)^n \Gamma(n + 2t)}{n!} \cdot \frac{1}{z + i(t + n)},$$

which is convergent for $0 < t \leq 1/2$.

(4) For any compact set $I \subset \mathbb{R}$, there is $M > 0$ such that

$$|\Gamma(t + zi)\Gamma(t - zi)| \leq M e^{-\pi|x|} |x|^{2t-1}, \quad z = x + yi, \quad |x| \geq 1, \quad t, y \in I.$$

This estimate follows from Stirling's formula.

(5) The density function of ρ_t is symmetric, and moreover strictly decreasing on $[0, \infty)$ as the following calculation shows. We have

$$\frac{d}{dx} |\Gamma(t + xi)|^2 = -2 |\Gamma(t + xi)|^2 \operatorname{Im} \psi(t + xi)$$

by using the digamma function $\psi(z) = (d/dz) \log \Gamma(z)$. It is known that $\psi(z) = -\gamma - \sum_{n=0}^{\infty} (1/(z+n) - 1/(n+1))$, where γ is Euler's constant, and so

$$\operatorname{Im} \psi(t + xi) = \sum_{n=0}^{\infty} \frac{x}{(t+n)^2 + x^2} > 0 \quad \text{for } x > 0.$$

We do not use the series expansion of $G_t(z)$; instead the following recursive relation is useful.

PROPOSITION 3.1. *It follows that*

$$(3.1) \quad G_t(z - ti) = \frac{1}{z} + \frac{it}{z} G_{t+1/2} \left(z + \left(\frac{1}{2} - t \right) i \right), \quad \operatorname{Im} z > t, \quad t > 0.$$

The iterative use of this relation extends G_t to a meromorphic function in \mathbb{C} with poles at $-(t+n)i$, $n = 0, 1, 2, \dots$

PROOF. Assume $t > \frac{1}{2}$. Because $\Gamma(t + iz)\Gamma(t - iz)$ does not have a pole in $\{z \in \mathbb{C} : -\frac{1}{2} \leq \operatorname{Im} z \leq 0\}$ and vanishes rapidly as $\operatorname{Re} z \rightarrow \infty$ (see the above property (4)), we get

$$\begin{aligned} G_t \left(z - \frac{i}{2} \right) &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{1}{z - (x + \frac{i}{2})} \Gamma(t + ix)\Gamma(t - ix) dx \\ &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{1}{z - x} \Gamma \left(t + \frac{1}{2} + ix \right) \Gamma \left(t - \frac{1}{2} - ix \right) dx, \quad \operatorname{Im} z > \frac{1}{2}. \end{aligned}$$

Using the basic relation $z\Gamma(z) = \Gamma(z+1)$, we obtain

$$\begin{aligned} G_t \left(z - \frac{i}{2} \right) &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{\Gamma \left(t + \frac{1}{2} + ix \right) \Gamma \left(t + \frac{1}{2} - ix \right)}{(z-x)(t - \frac{1}{2} - ix)} dx \\ &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{1}{z + (t - \frac{1}{2})i} \left(\frac{1}{t - \frac{1}{2} - ix} - \frac{1}{iz - ix} \right) \left| \Gamma \left(t + \frac{1}{2} + ix \right) \right|^2 dx \\ &= \frac{ti}{z + (t - \frac{1}{2})i} \cdot \frac{4^{t+1/2}}{2\pi\Gamma(2t+1)} \int_{\mathbb{R}} \frac{1}{z-x} \left| \Gamma \left(t + \frac{1}{2} + ix \right) \right|^2 dx \\ &\quad + \frac{1}{z + (t - \frac{1}{2})i} \cdot \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{|\Gamma(t + \frac{1}{2} + ix)|^2}{t - \frac{1}{2} - ix} dx. \end{aligned}$$

In the last integral, we can again apply the formula $z\Gamma(z) = \Gamma(z+1)$, and moreover we deform the contour \mathbb{R} to $\mathbb{R} + \frac{i}{2}$:

$$\begin{aligned} \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \frac{|\Gamma(t + \frac{1}{2} + ix)|^2}{t - \frac{1}{2} - ix} dx \\ &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \Gamma\left(t + \frac{1}{2} + ix\right) \Gamma\left(t - \frac{1}{2} - ix\right) dx \\ &= \frac{4^t}{2\pi\Gamma(2t)} \int_{\mathbb{R}} \Gamma(t + ix) \Gamma(t - ix) dx = 1. \end{aligned}$$

The above calculations amount to

$$G_t\left(z - \frac{i}{2}\right) = \frac{1}{z + (t - \frac{1}{2})i} + \frac{it}{z + (t - \frac{1}{2})i} G_{t+1/2}(z),$$

which holds for any $t > 0$ since $G_t(z)$ depends on $t > 0$ real analytically. The desired relation (3.1) follows from the simple replacement of z by $z + (\frac{1}{2} - t)i$. The right-hand side of (3.1) is meromorphic in $\{z \in \mathbb{C} : \text{Im } z > t - \frac{1}{2}\}$ with pole at zero, so that G_t extends to a meromorphic function in $\{z \in \mathbb{C} : \text{Im } z > t - \frac{1}{2}\}$. Next we can write $G_{t+1/2}$ in terms of G_{t+1} , and so iteratively G_t can be written in terms of $G_{t+n/2}$ for any $n \in \mathbb{N}$. This procedure extends G_t to a meromorphic function in \mathbb{C} with poles at $-(t+n)i$, $n = 0, 1, 2, \dots$ ■

LEMMA 3.1. *If a probability measure μ has a density $p(x)$ such that $p(x) = p(-x)$, $p'(x) \leq 0$ for a.e. $x > 0$ and $\lim_{x \rightarrow \infty} p(x) \log x = 0$, then it follows that $\text{Re } G_\mu(x + yi) > 0$ for $x, y > 0$.*

P r o o f. The claim follows from the computation

$$\begin{aligned} \text{Re } G_\mu(x + yi) &= \int_{\mathbb{R}} \frac{x - u}{(x - u)^2 + y^2} p(u) du \\ &= -\frac{1}{2} \int_{\mathbb{R}} \left(\frac{\partial}{\partial u} \log((x - u)^2 + y^2) \right) p(u) du \\ &= \frac{1}{2} \int_{\mathbb{R}} \log((x - u)^2 + y^2) p'(u) du \\ &= \frac{1}{2} \int_0^\infty \log\left(\frac{(x + u)^2 + y^2}{(x - u)^2 + y^2}\right) (-p'(u)) du > 0, \quad x, y > 0. \end{aligned}$$

The property $p'(-u) = -p'(u)$ was used at the final equality. ■

THEOREM 3.1. *The Meixner distribution ρ_t is in \mathcal{UI} for $0 < t \leq \frac{1}{2}$.*

Proof. We may assume that $0 < t < \frac{1}{2}$ since the set \mathcal{UI} is closed with respect to the weak convergence. We will check conditions (A)–(D) of Proposition 2.1 for $F_t(z) := 1/G_t(z)$ and $\gamma_t := \{x - ti : x \in \mathbb{R}\}$. Condition (A) is trivial. To prove (B), we use Proposition 3.1:

$$\operatorname{Im} G_t(x - ti) = \frac{t}{x} \operatorname{Re} G_{t+1/2} \left(x + \left(\frac{1}{2} - t \right) i \right).$$

Since $(d/dx)|\Gamma(t + \frac{1}{2} + xi)|^2 < 0$ for $x > 0$, we can apply Lemma 3.1 to the measure $\rho_{t+1/2}$, to assert that $\operatorname{Re} G_{t+1/2}(x + (\frac{1}{2} - t)i) > 0$ for $x > 0$. Hence $\operatorname{Im} G_t(x - ti) > 0$ for $x > 0$ and also for $x < 0$ by symmetry. Hence condition (B) holds since $-ti$ is a pole of G_t .

By Proposition 3.1, G_t is a meromorphic function, and so is F_t . If G_t had a zero in $\overline{D(\gamma_t)}$, there would be a point $z_0 \in \mathbb{C}^+ \cup \mathbb{R} \setminus \{0\}$ such that $G_t(z_0 - ti) = 0$. This implies that $1 + tiG_{t+1/2}(z_0 + (\frac{1}{2} - t)i) = 0$, and so $G_{t+1/2}(z_0 + (\frac{1}{2} - t)i) = -\frac{i}{t} \in \mathbb{C}^+$. This is a contradiction because $G_{t+1/2}$ maps \mathbb{C}^+ into \mathbb{C}^- . Thus condition (C) is proved.

Condition (D) can be checked as follows. Let $p_t(x)$ be the density function of ρ_t . In the integral $\int_{\mathbb{R}} (z - x)^{-1} \rho_t(dx)$, one is allowed to replace the contour \mathbb{R} by $C_t := \{x - \frac{3t}{2}i : -\infty < x < -\frac{3t}{2}\} \cup \{-\frac{3t}{2}i + \frac{3t}{2}e^{i\theta} : 0 \leq \theta \leq \pi\} \cup \{x - \frac{3t}{2}i : \frac{3t}{2} < x < \infty\}$. Thus we obtain

$$\int_{\mathbb{R}} \frac{1}{z - x} \rho_t(dx) = \int_{C_t} \frac{1}{z - w} p_t(w) dw.$$

Clearly, $1 = \int_{\mathbb{R}} p_t(x) dx = \int_{C_t} p_t(w) dw$, so we have

$$1 - zG_t(z) = \int_{C_t} \frac{1}{w - z} wp_t(w) dw.$$

If z tends to infinity satisfying $z \in D(\gamma_t)$, then $1 - zG_t(z)$ tends to zero by the Lebesgue convergence theorem. This implies

$$\left| \frac{F_t(z) - z}{z} \right| \rightarrow 0,$$

the conclusion. ■

REMARK 3.1. *The proof uses the inequality $\operatorname{Re} G_{t+1/2}(x + yi) > 0$ for $x, y > 0$. If this inequality holds even for negative y , then we can prove the free infinite divisibility of ρ_t for $t > \frac{1}{2}$ too.*

REMARK 3.2. The free cumulant sequence $(r_n(\mu))_{n=1}^{\infty}$ of a probability measure μ with finite moments of all orders can be defined as the coefficients of series expansion of $F_{\mu}^{-1}(z) - z$:

$$F_{\mu}^{-1}(z) - z = \sum_{n=1}^{\infty} \frac{r_n(\mu)}{z^{n-1}}$$

(see [14], Remark 16.18). The free infinite divisibility of ρ_t ($0 < t \leq \frac{1}{2}$) implies that the corresponding free cumulant sequence is conditionally nonnegative definite, i.e. the $N \times N$ matrix $(r_{m+n}(\rho_t))_{m,n=1}^N$ is nonnegative definite for any $N \geq 1$ (see Theorem 13.16 of [14]).² If $t = \frac{1}{2}$, the free cumulants up to the 10th order are given by

$$(r_2(\mu_2), r_4(\mu_1), r_6(\mu_1), \dots) = (1, 3, 38, 947, 37394, \dots), \quad r_{2n+1}(\mu_1) = 0, \quad n \geq 0.$$

This sequence can be found in [15].

4. PROOF FOR THE LOGISTIC DISTRIBUTION

The free infinite divisibility of the logistic distribution μ_2 is proved with direct computation of the Cauchy transform. From the residue theorem it turns out that

$$\begin{aligned} (4.1) \quad G_{\mu_2}(z) &= \sum_{n=1}^{\infty} \frac{i}{(z + (n - \frac{1}{2})i)^2} \\ &= \sum_{n=1}^{\infty} \frac{2x(y + n - \frac{1}{2})}{[x^2 + (y + n - \frac{1}{2})^2]^2} + i \sum_{n=1}^{\infty} \frac{x^2 - (y + n - \frac{1}{2})^2}{[x^2 + (y + n - \frac{1}{2})^2]^2}, \\ &\quad z = x + yi \in \mathbb{C}^+. \end{aligned}$$

Now we take $\gamma_{1/2} := \{x - i/2 : x \in \mathbb{R}\}$. The imaginary part of G_{μ_2} on $\gamma_{1/2}$ can be written as

$$g(x) := \operatorname{Im} G_{\mu_2}\left(x - \frac{i}{2}\right) = \sum_{n=0}^{\infty} \frac{x^2 - n^2}{(x^2 + n^2)^2}.$$

Fortunately, g can be written by elementary functions.

LEMMA 4.1. The function g is given by

$$g(x) = \frac{1}{2} \left(\frac{1}{x^2} + \left(\frac{\pi}{\sinh(\pi x)} \right)^2 \right).$$

²If a measure μ has a compact support, the free infinite divisibility is equivalent to the conditional nonnegative definiteness of free cumulants. This equivalence can be extended to a measure with finite moments of all orders when the moment problem is determinate.

Proof. It is known that

$$\frac{1}{\sinh(\pi x)} = \frac{1}{\pi x} - \frac{\pi}{6}x + O(x^3) \quad \text{as } x \rightarrow 0,$$

and so

$$\left(\frac{\pi}{\sinh(\pi x)}\right)^2 = \frac{1}{x^2} + O(1) \quad \text{as } x \rightarrow 0.$$

The poles of $(\pi/\sinh(\pi x))^2$ are at $x = ni$ ($n \in \mathbb{Z}$), and the function

$$\left(\frac{\pi}{\sinh(\pi x)}\right)^2 - \sum_{n=-\infty}^{\infty} \frac{1}{(x - ni)^2}$$

does not have a singular point. This function is bounded by a constant on \mathbb{C} , and so equal to a constant, which is actually zero as is known from the limit $x \rightarrow \infty$. Hence

$$\begin{aligned} \left(\frac{\pi}{\sinh(\pi x)}\right)^2 &= \sum_{n=-\infty}^{\infty} \frac{1}{(x - ni)^2} = \frac{1}{x^2} + \sum_{n=1}^{\infty} \left(\frac{1}{(x - ni)^2} + \frac{1}{(x + ni)^2}\right) \\ &= \frac{1}{x^2} + 2 \sum_{n=1}^{\infty} \frac{x^2 - n^2}{(x^2 + n^2)^2}, \end{aligned}$$

leading to the conclusion. ■

We easily find that $g(x) > 0$ for $x \neq 0$ thanks to Lemma 4.1, and the function F_{μ_2} vanishes at $-i/2$ since it is a pole of G_{μ_2} . Hence condition (B) is satisfied.

The following properties can be proved from (4.1):

- (i) $\operatorname{Re} G_{\mu_2}(x + yi) > 0$ for $x > 0$ and $y \geq -\frac{1}{2}$;
- (ii) $\operatorname{Im} G_{\mu_2}(yi) < 0$ for $y > -\frac{1}{2}$.

Thus G_{μ_2} does not have a zero in $\overline{D(\gamma_{1/2})}$, and so F_{μ_2} is analytic in $D(\gamma_{1/2})$, continuous on $\overline{D(\gamma_{1/2})}$. Consequently, $\gamma_{1/2} = \{x - i/2 : x \in \mathbb{R}\}$ satisfies condition (C).

Condition (D) is proved similarly to the case of ρ_t .

OPEN PROBLEMS. The authors have not been able to solve the following questions.

(a) Free infinite divisibility for Meixner distributions ρ_t in the case $t > \frac{1}{2}$ and for non-symmetric Meixner distributions.

(b) Free infinite divisibility for the measure with density

$$\frac{2\pi}{2^r B\left(\frac{r}{2}, \frac{r}{2}\right)} \left(\frac{1}{\cosh(\pi x)}\right)^r \quad \text{for } r > 0, r \neq 1, 2.$$

(c) Characterization of the class \mathcal{UI} in terms of free Lévy measures.

(d) Combinatorial meaning of the free cumulant sequence of ρ_t , in particular of $\rho_{1/2}$.

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