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AN INEQUALITY FOR NORMS OF POISSON WICK PRODUCTS

BY

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Abstract. An inequality for the norms of Poisson Wick products, involving second quantization operators, will be presented.

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1. INTRODUCTION

In this paper we present an inequality for the norms of Poisson Wick products. The inequality involves the second quantization operator of a constant times the identity operator.

The paper is structured as follows. In Section 2 we review a minimal background about the Charlier polynomials, exponential functions, second quantization operator, and Poisson Wick product. In Section 3 we prove an inequality for the norms of Poisson Wick products.

2. BACKGROUND

Let $\mathbb{N} \cup \{0\}$ be the set of nonnegative integers, and let a be a fixed positive number. The Poisson probability measure with mean a is the discrete probability measure defined as

$$P_a(B) = \sum_{n \in B} \frac{a^n}{n!} e^{-a} \tag{2.1}$$

for any subset B of $\mathbb{N} \cup \{0\}$. The Poisson probability measure P_a has finite moments of all orders. Thus for each $n \geqslant 0$, the random variable $g_n : \mathbb{N} \cup \{0\} \to \mathbb{R}$ defined by

$$g_n(x) = x^n \tag{2.2}$$

belongs to $L^2(\mathbb{N} \cup \{0\}, P_a)$. Applying the Gram–Schmidt orthogonalization procedure to the sequence $\{g_0, g_1, g_2, \dots\} \subset L^2(\mathbb{N} \cup \{0\}, P_a)$, we obtain a sequence of orthogonal polynomial random variables $\{C_{0,a}, C_{1,a}, C_{2,a}, \dots\}$ called the *Charlier polynomials*. These polynomials are monic, i.e. normalized in such a way that their leading coefficient is equal to one. For each $n \geqslant 0$, the formula for the n-th Charlier polynomial is (see [6])

$$C_{n,a}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a^{n-k} (x)_k \tag{2.3}$$

for all $x \in \mathbb{N} \cup \{0\}$, where

$$(x)_k = \begin{cases} 0 & \text{if } x < k, \\ x!/(x-k)! & \text{if } x \geqslant k. \end{cases} \tag{2.4}$$

For every $n \geqslant 0$, the square of the L^2-norm of n-th Charlier polynomial is

$$E\left[C_{n,a}^2\right] = n! a^n,$$

where E denotes the expectation. Since the Charlier polynomials form an orthogonal basis of $L^2(\mathbb{N} \cup \{0\}, P_a)$, we conclude that for every random variable $f \in L^2(\mathbb{N} \cup \{0\}, P_a)$ there is a unique sequence $\{b_n\}_{n \geqslant 0}$ of complex numbers such that

$$f = \sum_{n=0}^{\infty} b_n C_{n,a}, \tag{2.5}$$

where this equality is understood in the L^2 sense. Moreover, the square of the L^2-norm of f is

$$\|f\|_2^2 = \sum_{n=0}^{\infty} n! a^n |b_n|^2 < \infty.$$

For every complex number t, we define the exponential function:

$$\mathcal{E}_t(x) := \sum_{n=0}^{\infty} \frac{t^n}{n! a^n} C_{n,a}(x). \tag{2.6}$$

The exponential functions have been defined so far in terms of the Charlier polynomials, but we can easily compute their pointwise formula in the following way.

For all complex t and nonnegative integer x, we have

$$
\begin{aligned}
\mathcal{E}_t(x) &= \sum_{n=0}^{\infty} \frac{t^n}{n!a^n} C_{n,a}(x) \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!a^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a^{n-k} (x)_k \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!a^k} (x)_k \sum_{n=k}^{\infty} (-1)^{n-k} \frac{t^{n-k}}{(n-k)!} \\
&= \sum_{k=0}^{x} \frac{t^k}{k!a^k} \cdot \frac{x!}{(x-k)!} \cdot e^{-t} \\
&= \sum_{k=0}^{x} \binom{x}{k} \frac{t^k}{a^k} e^{-t} \\
&= \left(1 + \frac{t}{a}\right)^x e^{-t}.
\end{aligned} \tag{2.7}
$$

It is not hard to see that, for all $t \in \mathbb{C}$ and $1 \leqslant p < \infty$, the exponential function $\mathcal{E}_t$ belongs to $L^p(\mathbb{N} \cup \{0\}, P_a)$. Moreover, the vector space spanned by $\{\mathcal{E}_t\}_{t \in \mathbb{C}}$ is dense in $L^p(\mathbb{N} \cup \{0\}, P_a)$ for all $1 \leqslant p < \infty$.

We define the *Wick product* of two Charlier polynomials as

$$C_{m,a} \diamond C_{n,a} := C_{m+n,a} \tag{2.8}$$

for all nonnegative integers m and n.

We can extend the definition of the Wick product in a bilinear way, defining for every polynomial function $f = \sum_{m \leqslant M} a_m C_{m,a}$ and $g = \sum_{n \leqslant N} b_n C_{n,a}$, where $\{a_m\}_{m \leqslant M}$ and $\{b_n\}_{n \leqslant N}$ are finite sequences of complex numbers,

$$f \diamond g := \sum_{k=0}^{M+N} \big(\sum_{m+n=k} a_m b_n \big) C_{k,a}. \tag{2.9}$$

It is not hard to see that if we consider the subspace of $L^2(\mathbb{N} \cup \{0\}, P_a)$ determined as

$$V := \big\{ f = \sum_{m=0}^{\infty} a_m C_{m,a} \mid \forall m \geqslant 0, a_m \in \mathbb{C}, \sum_{m=0}^{\infty} m! 2^m a^m |a_m|^2 < \infty \big\},$$

then for all $f = \sum_{m=0}^{\infty} a_m C_{m,a} \in V$ and $g = \sum_{m=0}^{\infty} b_m C_{m,a} \in V$, where a_m and b_m are complex numbers for all $m \geqslant 0$, we can define the Wick product of f and g as follows:

$$f \diamond g := \sum_{k=0}^{\infty} \big(\sum_{m+n=k} a_m b_n \big) C_{k,a}. \tag{2.10}$$

The series on the right-hand side of (2.10) is convergent in $L^2(\mathbb{N} \cup \{0\}, P_a)$ for f and g in V, and thus $f \diamond g \in L^2(\mathbb{N} \cup \{0\}, P_a)$. In fact, we consider the formula (2.10) to be the definition of the Wick product not only for two functions from the space V, but also for any two functions f and g from $L^2(\mathbb{N} \cup \{0\}, P_a)$ for which the series on the right-hand side of (2.10) converges in the L^2 sense.

It is easy to see that every exponential function belongs to the space V, and for all complex numbers s and t, we have the following formula (see [3]):

$$\mathcal{E}_s \diamond \mathcal{E}_t = \mathcal{E}_{s+t}. \tag{2.11}$$

The idea behind the formula (2.11) is the same as the idea behind the formula, from calculus, $e^{\lambda x} \cdot e^{\lambda y} = e^{\lambda(x+y)}$ for any x, y, and λ complex numbers. In calculus, we can prove that $e^{\lambda x} \cdot e^{\lambda y} = e^{\lambda(x+y)}$ using the Taylor series formula for the exponential function, computing the product of two series, and using the Newton binomial formula. To understand this, one must compare the following two similar formulas:

$$\mathcal{E}_t = \sum_{n=0}^{\infty} \frac{(t/a)^n}{n!} C_{1,a}^{\diamond n}$$

and

$$e^{\lambda x} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} x^n.$$

For every complex number c, we define the *second quantization operator* $\Gamma(cI)$, of c times the identity operator I, first for each Charlier polynomial $C_{n,a}$ with $n \geqslant 0$, as follows:

$$\Gamma(cI) C_{n,a} := c^n C_{n,a}. \tag{2.12}$$

As before, we extend this definition in a linear way, defining formally for every $f = \sum_{n \geqslant 0} b_n C_{n,a}$ in $L^2(\mathbb{N} \cup \{0\}, P_a)$, where $\{b_n\}_{n \geqslant 0}$ is a sequence of complex numbers, $\Gamma(cI) f$ as

$$\Gamma(cI) f := \sum_{n=0}^{\infty} b_n c^n C_{n,a}. \tag{2.13}$$

Again, in order for $\Gamma(cI) f$ to be in $L^2(\mathbb{N} \cup \{0\}, P_a)$, the last series must be convergent in the L^2 sense. This is true for every exponential function, and we have

$$\Gamma(cI) \mathcal{E}_t = \mathcal{E}_{ct} \tag{2.14}$$

for every c and t in $\mathbb{C}$.

3. A POINTWISE FORMULA AND INEQUALITY FOR THE NORMS OF POISSON WICK PRODUCTS

We have seen in the previous section that the extension of the definition of the Wick product, from two Charlier polynomials to two L^2-functions, raises some questions about the convergence of the series used in the definition. In this section we present an inequality that ensures that the Wick product of two L^p random variables belongs to the space $L^p(\mathbb{N} \cup \{0\}, P_a)$ for $1 \leqslant p \leqslant \infty$.

THEOREM 3.1. *Let P_a be the Poisson probability measure with mean a. Let α and β be positive numbers such that*

$$\alpha + \beta = 1.$$

Let f and g be functions in $L^1(\mathbb{N} \cup \{0\}, P_a)$. Then, for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\{[\Gamma(\alpha I)f] \diamond [\Gamma(\beta I)g]\}(n) = \sum_{k=0}^{\infty} \frac{(a\beta)^k}{k!} e^{-a\beta} \sum_{l=0}^{\infty} \frac{(a\alpha)^l}{l!} e^{-a\alpha} \sum_{p+q=n} \binom{n}{p} \alpha^p \beta^q f(p+k) g(q+l). \tag{3.1}$$

This implies that, if $f \geqslant 0$ and $g \geqslant 0$, then

$$[\Gamma(\alpha I)f] \diamond [\Gamma(\beta I)g] \geqslant 0. \tag{3.2}$$

Moreover, if f and g are in $L^r(\mathbb{N} \cup \{0\}, P_a)$ for some $1 \leqslant r \leqslant \infty$, then

$$\|\Gamma(\alpha I)f \diamond \Gamma(\beta I)g\|_r \leqslant \|f\|_r \cdot \|g\|_r. \tag{3.3}$$

P r o o f. It is enough to check (3.1) for $f = \mathcal{E}_s$ and $g = \mathcal{E}_t$, where s and t are arbitrary complex numbers. Indeed, for a fixed $n \geqslant 0$, we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(a\beta)^k}{k!} e^{-a\beta} \sum_{l=0}^{\infty} \frac{(a\alpha)^l}{l!} e^{-a\alpha} \sum_{p+q=n} \binom{n}{p} \alpha^p \beta^q \mathcal{E}_s(p+k) \mathcal{E}_t(q+l) \\
&\quad = \sum_{k=0}^{\infty} \frac{(a\beta)^k}{k!} e^{-a\beta} \sum_{l=0}^{\infty} \frac{(a\alpha)^l}{l!} e^{-a\alpha} \\
&\qquad \times \sum_{p+q=n} \binom{n}{p} \alpha^p \beta^q \left(1 + \frac{s}{a}\right)^{p+k} \left(1 + \frac{t}{a}\right)^{q+l} e^{-s-t} \\
&\quad = e^{-s-t} \sum_{k=0}^{\infty} \frac{(a\beta)^k}{k!} \left(1 + \frac{s}{a}\right)^k e^{-a\beta} \sum_{l=0}^{\infty} \frac{(a\alpha)^l}{l!} \left(1 + \frac{t}{a}\right)^l e^{-a\alpha} \\
&\qquad \times \sum_{p+q=n} \binom{n}{p} \left[\alpha\left(1 + \frac{s}{a}\right)\right]^p \left[\beta\left(1 + \frac{t}{a}\right)\right]^q \\
&\quad = e^{-s-t} e^{-a(\alpha+\beta)} \sum_{k=0}^{\infty} \frac{[a\beta(1+s/a)]^k}{k!} \sum_{l=0}^{\infty} \frac{[a\alpha(1+t/a)]^l}{l!} \\
&\qquad \times \left[\alpha\left(1 + \frac{s}{a}\right) + \beta\left(1 + \frac{t}{a}\right)\right]^n.
\end{aligned}$$

We use now the fact that $\alpha + \beta = 1$, and obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(a\beta)^k}{k!} e^{-a\beta} \sum_{l=0}^{\infty} \frac{(a\alpha)^l}{l!} e^{-a\alpha} \sum_{p+q=n} \binom{n}{p} \alpha^p \beta^q \mathcal{E}_s(p+k)\mathcal{E}_t(q+l) \\
&\qquad = e^{-s-t} e^{-a} e^{a\beta(1+s/a)} e^{a\alpha(1+t/a)} \left[(\alpha+\beta) + \frac{\alpha s + \beta t}{a}\right]^n \\
&\qquad = \left(1 + \frac{\alpha s + \beta t}{a}\right)^n e^{-s(1-\beta)} e^{-t(1-\alpha)} \\
&\qquad = \left(1 + \frac{\alpha s + \beta t}{a}\right)^n e^{-s\alpha} e^{-t\beta} \\
&\qquad = \left(1 + \frac{\alpha s + \beta t}{a}\right)^n e^{-(\alpha s + \beta t)} \\
&\qquad = \mathcal{E}_{\alpha s + \beta t}(n) \\
&\qquad = \left[\mathcal{E}_{\alpha s} \diamond \mathcal{E}_{\beta t}\right](n) \\
&\qquad = \left\{[\Gamma(\alpha I)\mathcal{E}_s] \diamond [\Gamma(\beta I)\mathcal{E}_t]\right\}(n).
\end{aligned}$$

Since

$$\sum_{k=0}^{\infty} \frac{(a\beta)^k}{k!} e^{-a\beta} = 1,$$
$$\sum_{l=0}^{\infty} \frac{(a\alpha)^l}{l!} e^{-a\alpha} = 1,$$

and

$$\sum_{p+q=n} \binom{n}{p} \alpha^p \beta^q = 1$$

for any $n \geqslant 0$, by using three times Jensen's inequality for the convex function $x \mapsto |x|^r$, we have, for all $r \geqslant 1$,

$$\begin{aligned}
|\{\Gamma(\alpha I) f \diamond \Gamma(\beta I) g\}(n)|^r \leqslant{}& \sum_{k=0}^{\infty} \frac{(a\beta)^k}{k!} e^{-a\beta} \sum_{l=0}^{\infty} \frac{(a\alpha)^l}{l!} e^{-a\alpha} \\
&\times \sum_{p+q=n} \binom{n}{p} \alpha^p \beta^q |f(p+k)|^r |g(q+l)|^r.
\end{aligned}$$

Multiplying both sides of this inequality by the positive weight $(a^n/n!)e^{-a}$ and summing up from $n = 0$ to ∞, we obtain

$$\begin{aligned}
\|\Gamma(\alpha I) f \diamond \Gamma(\beta I) g\|_r^r \leqslant{}& \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a\beta)^k}{k!} e^{-a\beta} \sum_{l=0}^{\infty} \frac{(a\alpha)^l}{l!} e^{-a\alpha} \\
&\times \sum_{p+q=n} \binom{n}{p} \alpha^p \beta^q |f(p+k)|^r |g(q+l)|^r \frac{a^n}{n!} e^{-a}.
\end{aligned}$$

Observe first that the $n!$ from $\binom{n}{p}$ cancels with the $n!$ from $a^n/n!$. If we make the changes of variables $u := p + k$ and $v := q + l$, and rearrange the order of summation (Tonelli's theorem), we obtain

$$\begin{aligned}\|\Gamma(\alpha I)f \diamond \Gamma(\beta I)g\|_r^r &\leqslant \sum_{u=0}^{\infty} \frac{a^u}{u!} e^{-a} |f(u)|^r \sum_{v=0}^{\infty} \frac{a^v}{v!} e^{-a} |g(v)|^r \\ &\quad \times \sum_{k=0}^{u} \frac{u!}{k!(u-k)!} \beta^k \alpha^{u-k} \sum_{l=0}^{v} \frac{v!}{l!(v-l)!} \alpha^l \beta^{v-l} \\ &= \sum_{u=0}^{\infty} \frac{a^u}{u!} e^{-a} |f(u)|^r \sum_{v=0}^{\infty} \frac{a^v}{v!} e^{-a} |g(v)|^r \\ &\quad \times (\beta + \alpha)^u (\alpha + \beta)^v \\ &= \sum_{u=0}^{\infty} \frac{a^u}{u!} e^{-a} |f(u)|^r \sum_{v=0}^{\infty} \frac{a^v}{v!} e^{-a} |g(v)|^r \cdot 1^u \cdot 1^v \\ &= \|f\|_r^r \cdot \|g\|_r^r. \quad \blacksquare\end{aligned}$$

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