PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 33, Fasc. 2 (2013), pp. 201–212

CONVERGENCE OF THE FOURTH MOMENT AND INFINITE DIVISIBILITY

BY

OCTAVIO ARIZMENDI (SAARBRÜCKEN)

Abstract. In this note we prove that, for infinitely divisible laws, convergence of the fourth moment to 3 is sufficient to ensure convergence in law to the Gaussian distribution. Our results include infinitely divisible measures with respect to classical, free, Boolean and monotone convolution. A similar criterion is proved for compound Poissons with jump distribution supported on a finite number of atoms. In particular, this generalizes recent results of Nourdin and Poly (2012).

2000 AMS Mathematics Subject Classification: Primary: 46L54; Secondary: 60E07.

Key words and phrases: Bercovici–Pata bijections, Boolean convolution, free convolution, monotone convolution, infinite divisibility.

1. INTRODUCTION AND STATEMENT OF RESULTS

In a seminal paper, Nualart and Peccati [20] proved a convergence criterion for multiple integrals in a fixed chaos with respect to the classical Brownian motion to the standard normal distribution $\mathcal{N}(0,1)$, which gives a drastic simplification for the so-called method of moments. More precisely, let $(W_t)_{t\geqslant 0}$ be a standard Brownian motion. For every square-integrable function f on $\mathbb{R}_+^m$ we denote by $I_m^W(f)$ the m-th multiple Wiener–Itô stochastic integral of f with respect to W.

THEOREM 1.1 (Nualart and Peccati [20]). *Let $\{X_n = I_m^W(f_n)\}_{n>0}$ be a sequence of multiple Wiener–Itô integrals in a fixed m-chaos with $E[X_n^2] \to 1$ and denote by μ_{X_n} the distribution of X_n. Then the following statements are equivalent*:

1. $E[X_n^4] \to 3$;
2. $\mu_{X_n} \to \mathcal{N}(0,1)$.

In free probability, the standard semicircle distribution $\mathcal{S}(0,1)$ plays the role of the Gaussian distribution. Recently, it was proved by Kemp et al. [13] that the Nualart–Peccati criterion also holds for the free Brownian motion $(S_t)_{t\geqslant 0}$ and its multiple Wigner integrals $I_m^S(f)$.

THEOREM 1.2 (Kemp et al. [13]). *Let* $\{X_n := I^S_m(f_n)\}_{n>0}$ *be a sequence of multiple Wigner integrals in a fixed* m*-chaos with* $E[X_n^2] \to 1$ *and denote by* μ_{X_n} *the distribution of* X_n*. Then the following statements are equivalent*:

1. $E[X_n^4] \to 2$;
2. $\mu_{X_n} \to \mathcal{S}(0,1)$.

In this paper we prove analogous results to Theorems 1.1 and 1.2 in the setting of infinitely divisible laws. Let $ID(*)$ and $ID(\boxplus)$ denote the classes of probability measures which are infinitely divisible with respect to classical convolution $*$ and free convolution $\boxplus$, respectively.

THEOREM 1.3. *Let* $\{\mu_n = \mu_{X_n}\}_{n>0}$ *be a sequence of probability measures with variance one and mean zero such that* $\mu_n \in ID(*)$*. If* $E[X_n^4] \to 3$*, then* $\mu_{X_n} \to \mathcal{N}(0,1)$.

THEOREM 1.4. *Let* $\{\mu_n = \mu_{X_n}\}_{n>0}$ *be a sequence of probability measures with variance one and mean zero such that* $\mu_n \in ID(\boxplus)$*. If* $E[X_n^4] \to 2$*, then* $\mu_{X_n} \to \mathcal{S}(0,1)$.

To complete the picture we show that the monotone probability of Muraki also fits in our framework, namely, the analogue of Theorems 1.3 and 1.4 is also true in the monotone case.

THEOREM 1.5. *Let* $\{\mu_n = \mu_{X_n}\}_{n>0}$ *be a sequence of* $\rhd$*-infinitely divisible probability measures with common variance one and mean zero and denote by* $\mathcal{A}(0,1)$ *the arcsine distribution with mean zero and variance one. If* $E[X_n^4] \to 1.5$*, then* $\mu_{X_n} \to \mathcal{A}(0,1)$.

Moreover, we can extend our results to compound Poisson distributions whose Lévy measure has finite support. We only state the free version for the sake of clarity.

THEOREM 1.6. *Let* $\mu_{X_n} \in ID(\boxplus)$ *be random variables and let us denote by* $\pi_{\boxplus}(\lambda,\nu)$ *the free compound Poisson measure with rate* λ *and jump distribution* $\nu := \sum_{i=1}^k a_i\delta_{b_i}$*. If* $E[X_n^k] \to m_k\big(\pi(\lambda)\big)$ *for* $i = 1,\dots,2k+2$*, then* $\mu_{X_n} \to \pi_{\boxplus}(\lambda,\nu)$.

We want to emphasize that our approach relies on a third notion of non-commutative independence (Boolean independence) and the so-called Bercovici–Pata bijections Λ, $\Lambda^{\rhd}$, and $\mathbb{B}$ (see Section 2). This gives another example on how this third notion of independence regarded sometimes as uninteresting because of its simplicity can provide a better understanding in other notions of independence.

Some natural questions arise from the theorems above: What is the relation between Theorems 1.1 and 1.2, and Theorems 1.3 and 1.4? Multiple integrals are in general not infinitely divisible.[1] However, this is true in the first or second chaos.

[1] A counterexample for the free case is given by the third Chebyshev polynomial of a semicircle, i.e., $x = s^3 - 2s$.

In particular, from Theorem 1.6 we may recover Theorem 4.3 in Nourdin and Poly [19]. Another interesting question coming from Theorem 1.5 is if Theorem 1.1 is also valid for multiple integrals with respect to monotone Brownian motion; to the knowledge of the author this is still an open question.

2. PRELIMINARIES

2.1. The Cauchy transfom. We denote by $\mathcal{M}$ the set of Borel probability measures on $\mathbb{R}$. The upper half-plane and the lower half-plane are denoted by $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively.

Let $G_\mu(z) = \int_{\mathbb{R}} \big(1/(z-x)\big)\mu(dx)$ $(z \in \mathbb{C}^+)$ be the Cauchy transform of $\mu \in \mathcal{M}$.

The relation between weak convergence and the Cauchy transform is the following (see, e.g., [2]).

PROPOSITION 2.1. *Let μ_1 and μ_2 be two probability measures on $\mathbb{R}$ and*

$$d(\mu_1, \mu_2) = \sup\{|G_{\mu_1}(z) - G_{\mu_2}(z)|\,; \Im(z) \geqslant 1\}.$$

Then d is a distance which defines a metric for the weak topology of probability measures. In particular, $G_\mu(z)$ is bounded in $\{z : \Im(z) \geqslant 1\}$.

In other words, a sequence of probability measures $\{\mu_n\}_{n\geqslant 1}$ on $\mathbb{R}$ converges weakly to a probability measure μ on $\mathbb{R}$ if and only if for all z with $\Im(z) \geqslant 1$ we have

$$\lim_{n\to\infty} G_{\mu_n}(z) = G_\mu(z).$$

2.2. The Jacobi parameters. Let μ be a probability measure with all the moments. The Jacobi parameters $\gamma_m = \gamma_m(\mu) \geqslant 0, \beta_m = \beta_m(\mu) \in \mathbb{R}$ are defined by the recursion

$$xP_m(x) = P_{m+1}(x) + \beta_m P_m(x) + \gamma_{m-1}P_{m-1}(x),$$

where the polynomials $P_{-1}(x) = 0$, $P_0(x) = 1$, and $(P_m)_{m\geqslant 0}$ is a sequence of orthogonal monic polynomials with respect to μ, that is,

$$\int_{\mathbb{R}} P_m(x)P_n(x)\mu(dx) = 0 \quad \text{if } m \neq n.$$

A measure μ is supported on m points if and only if $\gamma_{m-1} = 0$ and $\gamma_n > 0$ for $n = 0, 1, \ldots, m-2$.

The Cauchy transform may be expressed as a continued fraction in terms of the Jacobi parameters as follows:

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z-t}\mu(dt) = \cfrac{1}{z - \beta_0 - \cfrac{\gamma_0}{z - \beta_1 - \cfrac{\gamma_1}{z - \beta_2 - \ldots}}}.$$

In the case when μ has $2n+2$ moments we can still make an orthogonalization procedure until the level n. In this case the Cauchy transform has the form

$$G_\mu(z) = \cfrac{1}{z-\beta_0-\cfrac{\gamma_0}{z-\beta_1-\cfrac{\gamma_1}{\ddots \over z-\beta_n-\gamma_n G_\nu(z)}}}, \tag{2.1}$$

where ν is a probability measure.

2.3. Different notions of convolution. In non-commutative probability, there exist various notions of independence. In this paper we will focus on the notions of independence coming from universal products as classified by Muraki [16]: tensor (classical), free, Boolean, and monotone independence.

2.3.1. *Classical convolution.* Recall that the *classical convolution* of two probability measures μ_1, μ_2 on $\mathbb{R}$ is defined as the probability measure $\mu_1 * \mu_2$ on $\mathbb{R}$ such that $C_{\mu_1 * \mu_2}(t) = C_{\mu_1}(t) + C_{\mu_2}(t), t \in \mathbb{R}$, where $C_\mu(t) = \log \hat{\mu}(t)$ with $\hat{\mu}(t)$ being the characteristic function of μ. Classical convolution corresponds to the sum of tensor independent random variables: $\mu_a * \mu_b = \mu_{a+b}$ for independent random variables a and b. The (classical) cumulants are the coefficients $c_n = c_n(\mu)$ in the series expansion

$$C_\mu(t) = \sum_{n=1}^{\infty} = \frac{c_n}{n!} t^n.$$

Similar convolutions and related transforms exist for the free, Boolean, and monotone theories.

2.3.2. *Free convolution.* Free convolution was defined in [23] for compactly supported probability measures and later extended in [14] for the case of finite variance, and in [6] for the general unbounded case. Let $G_\mu(z)$ be the Cauchy transform of $\mu \in \mathcal{M}$ and let $F_\mu(z)$ be its reciprocal $1/G_\mu(z)$. It was proved in Bercovici and Voiculescu [6] that there are positive numbers η and M such that F_μ has a right inverse F_μ^{-1} defined on the region $\Gamma_{\eta,M} := \{z \in \mathbb{C}^+; |\mathrm{Re}(z)| < \eta \mathrm{Im}(z), |z| > M\}$.

The Voiculescu transform of μ is defined by $\phi_\mu(z) = F_\mu^{-1}(z) - z$ on any region of the form $\Gamma_{\eta,M}$ where F_μ^{-1} is defined.

The *free additive convolution* of two probability measures μ_1, μ_2 on $\mathbb{R}$ is the probability measure $\mu_1 \boxplus \mu_2$ on $\mathbb{R}$ such that

$$\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z) \quad \text{for } z \in \Gamma_{\eta_1,M_1} \cap \Gamma_{\eta_2,M_2}.$$

Free additive convolution corresponds to the sum of two free random variables: $\mu_a \boxplus \mu_b = \mu_{a+b}$ for free random variables a and b. The free cumulants [21] are the coefficients $\kappa_n = \kappa_n(\mu)$ in the series expansion

$$\phi_\mu(z) = \sum_{n=1}^{\infty} \kappa_n z^{1-n}. \tag{2.2}$$

2.3.3. *Boolean convolution.* The *Boolean convolution* [22] of two probability measures μ_1, μ_2 on $\mathbb{R}$ is defined as the probability measure $\mu_1 \uplus \mu_2$ on $\mathbb{R}$ such that the transform $K_\mu(z) = z - F_\mu(z)$ (usually called *self-energy*) satisfies

$$K_{\mu_1 \uplus \mu_2}(z) = K_{\mu_1}(z) + K_{\mu_2}(z), \quad z \in \mathbb{C}^+.$$

Boolean convolution corresponds to the sum of Boolean-independent random variables. Boolean cumulants are defined as the coefficients $r_n = r_n(\mu)$ in the series

$$K_\mu(z) = \sum_{n=1}^{\infty} r_n z^{1-n}. \tag{2.3}$$

2.3.4. *Monotone convolution.* The monotone convolution was defined in [15] and extended to unbounded measures in [10]. The *monotone convolution* of two probability measures μ_1, μ_2 on $\mathbb{R}$ is defined as the probability measure $\mu_1 \rhd \mu_2$ on $\mathbb{R}$ such that

$$F_{\mu_1 \rhd \mu_2}(z) = F_{\mu_1}\big(F_{\mu_2}(z)\big), \quad z \in \mathbb{C}^+.$$

Monotone convolution corresponds to the sum of monotone independent random variables. Recently, Hasebe and Saigo [12] have defined a notion of monotone cumulants $(h_n)_{n\geqslant 1}$ which satisfy the relation $h_n(\mu^{\rhd k}) = k h_n(\mu)$.

2.3.5. *Moment-cumulant formulas.* For a measure μ its classical, free, Boolean, and monotone cumulants, i.e., $(c_n)_{n\geqslant 1}$, $(\kappa_n)_{n\geqslant 1}$, $(r_n)_{n\geqslant 1}$, $(h_n)_{n\geqslant 1}$, satisfy the moment-cumulant formulas

$$m_n = \sum_{\pi \in \mathcal{P}(n)} c_\pi^a = \sum_{\pi \in NC(n)} \kappa_\pi^a = \sum_{\pi \in \mathcal{I}(n)} r_\pi^a = \sum_{(\pi,\lambda) \in \mathcal{M}(n)} \frac{h_\pi^a}{|\pi|!}, \tag{2.4}$$

where, for a sequence of complex numbers $(f_n)_{n\geqslant 1}$ and a partition $\pi = \{V_1, \ldots, V_i\}$, we define $f_\pi := f_{|V_1|} \ldots f_{|V_i|}$ and $|\pi|$ is the number of blocks of the partition π and where $\mathcal{P}(n), NC(n), \mathcal{I}(n), \mathcal{M}(n)$ denote the set of all, non-crossing, interval, and monotone partitions (see [12], [21], [22]), respectively. We note here that convergence of the first n moments is equivalent to the convergence of the first n cumulants.

2.4. Infinite divisibility

DEFINITION 2.1. Let us assume that $\circledast$ is one of the above convolutions, namely, $\circledast \in \{*, \uplus, \boxplus, \rhd\}$. A probability measure μ is said to be *$\circledast$-infinitely divisible* if for each $n \in \mathbb{N}$ there exists $\mu_n \in \mathcal{M}$ such that $\mu = \mu_n^{\circledast n}$. We will denote by $ID(\circledast)$ the set of $\circledast$-infinitely divisible measures.

Recall that a probability measure μ is infinitely divisible in the classical sense if and only if its classical cumulant transform $\log\hat{\mu}$ has the Lévy–Khintchine representation

$$\log\hat{\mu}(u) = i\gamma u - \int_{\mathbb{R}} \left(e^{iut} - 1 - \frac{iut}{1+t^2}\right)\frac{1+t^2}{t^2}\sigma(dt), \quad u \in \mathbb{R}, \tag{2.5}$$

where $\gamma \in \mathbb{R}$ and σ is a finite measure on $\mathbb{R}$. If this representation exists, the pair (γ, σ) is determined in a unique way and is called the (*classical*) *generating pair* of μ. In this case we denote μ by $\rho_{*}^{\gamma,\sigma}$.

From the Voiculescu transform one has a representation analogous to Lévy–Khintchine's. Bercovici and Voiculescu [6] proved that a probability measure μ is $\boxplus$-infinitely divisible if and only if there exist a finite measure σ on $\mathbb{R}$ and a real constant γ such that

$$\phi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t}\sigma(dt), \qquad z \in \mathbb{C}^+. \tag{2.6}$$

The pair (γ, σ) is called the *free* generating pair of μ and we denote μ by $\rho_{\boxplus}^{\gamma,\sigma}$.

For the Boolean case, things are easier. As shown by Speicher and Woroudi [22], any probability measure is infinitely divisible with respect to the Boolean convolution and there is also a Boolean Lévy–Khintchine representation. Indeed, it follows then by the general Nevanlinna–Pick theory that for any probability measure μ there exist a real constant γ and a finite measure σ on $\mathbb{R}$ such that

$$K_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t}\,\sigma(dt), \qquad z \in \mathbb{C}^+. \tag{2.7}$$

The pair (γ, σ) is called the *Boolean* generating pair of μ and we denote μ by $\rho_{\uplus}^{\gamma,\sigma}$.

A characterization of $\triangleright$-infinitely divisible measures was done by Muraki [17] and Belinschi [4]. A probability measure μ belongs to $ID(\triangleright)$ if and only if there exists a composition semigroup of reciprocal Cauchy transforms $F_{s+t} = F_s \circ F_t = F_t \circ F_s$ and $F_1 = F_\mu$. In this case the map $t \mapsto F_t(z)$ is differentiable for each fixed z in $\mathbb{R}$ and we define the mapping A_μ on $\mathbb{C}^+$ by

$$A_\mu(z) = \frac{d}{dt}\Big|_{t=0} F_t(z), \qquad z \in \mathbb{C}^+.$$

For mappings of this form there exist $\gamma \in \mathbb{R}$ and a finite measure σ such that

$$A_\mu(z) = -\gamma - \int_{\mathbb{R}} \frac{1+tz}{z-t}\,\sigma(dt). \tag{2.8}$$

This is the Lévy–Khintchine formula for monotone convolution and in this case we denote μ by $\rho_{\triangleright}^{\gamma,\sigma}$. The monotone cumulants h_n are the coefficients in the series

$$-A_\mu(z) = \sum_{n=1}^{\infty} h_n z^{1-n}. \tag{2.9}$$

An important class of infinitely divisible measures is the class of compound Poisson distributions since any infinitely divisible measure can be approximated by compound Poissons.

DEFINITION 2.2. Let $\circledast \in \{*, \uplus, \boxplus, \triangleright\}$. Denote by $k_n^{\circledast}$ the cumulants with respect to the convolution $\circledast$. If $k_n^{\circledast}(\mu) = \lambda m_n(\nu)$ for some $\lambda > 0$ and some distribution ν we say that μ is a $\circledast$-*compound Poisson distribution* with rate λ and jump distribution ν and denote μ by $\pi_{\circledast}(\lambda, \nu)$. If $\lambda = 1$ and $\nu = \delta_1$, then $k_n^{\circledast}(\mu) = 1$ for all $n \in \mathbb{N}$, and we call μ simply a $\circledast$-*Poisson*.

2.5. Bercovici–Pata bijections. From the various Lévy–Khintchine representations it is readily seen that there is a bijective correspondence between the infinite divisible measures with respect to the different notions of independence. These bijections are called the *Bercovici–Pata bijections*, since they were studied by Bercovici and Pata [5] in relation to limit theorems and domain of attractions.

DEFINITION 2.3. 1. The (*classical-to-free*) *Bercovici–Pata bijection* $\Lambda : ID(*) \to ID(\boxplus)$ is defined by the application $\rho_*^{\gamma,\sigma} \mapsto \rho_{\boxplus}^{\gamma,\sigma}$.

2. The (*Boolean-to-free*) *Bercovici–Pata bijection* $\mathbb{B} : \mathcal{M} \to ID(\boxplus)$ is defined by the application $\rho_{\uplus}^{\gamma,\sigma} \mapsto \rho_{\boxplus}^{\gamma,\sigma}$.

3. The (*classical-to-monotone*) *Bercovici–Pata bijection* $\Lambda^{\triangleright} : ID(*) \to ID(\triangleright)$ is defined by the application $\rho_*^{\gamma,\sigma} \mapsto \rho_{\triangleright}^{\gamma,\sigma}$.

The weak continuity of Λ and Λ^{-1} was proved in [3]. On the other hand, the weak continuity of $\mathbb{B}$ and $\mathbb{B}^{-1}$ follows from the continuity of the free and Boolean convolution powers since $\mathbb{B}(\mu) = (\mu^{\boxplus 2})^{\uplus 1/2}$. Finally, the weak continuity of $\Lambda^{\triangleright}$ was proved in Hasebe [11]. In summary, the arrows of the following commutative diagram are weakly continuous:

$$\begin{array}{ccccc} & & ID(\uplus) = \mathcal{M} & & \\ & \swarrow^{\mathbb{B}} & \downarrow & \searrow & \\ ID(\boxplus) & \xrightarrow{\Lambda^{-1}} & ID(*) & \xrightarrow{\Lambda^{\triangleright}} & ID(\triangleright) \end{array}$$

REMARK 2.1. *It follows from* (2.2) *and* (2.3) *that the Boolean cumulants of μ are free cumulants of its image under the Boolean Bercovici–Pata bijection $\mathbb{B}$, namely, $r_n(\mu) = \kappa_n\big(\mathbb{B}(\mu)\big)$. Similarly, $c_n(\mu) = \kappa_n\big(\Lambda(\mu)\big)$ and $c_n(\mu) = h_n\big(\Lambda^{\triangleright}(\mu)\big)$.*

3. CONVERGENCE TO THE GAUSSIAN DISTRIBUTION

For a random variable X with all moments, mean zero and variance one, let us denote by $S_n^*(X) = (X_1 + X_2 + \ldots + X_n)/\sqrt{n}$ the normalized sum of n independent copies of X. The so-called central limit theorem states that $S_n^*(X)$ converges, as $n \to \infty$, to the standard normal distribution $\mathcal{N}(0,1)$.

On the other hand, the free central limit theorem (see [8], [23]) states that the normalized sum of free copies of X converges weakly to the standard semicircle distribution $\mathcal{S}(0,1)$ with density

$$\frac{1}{2\pi}\sqrt{4-x^2}, \quad x \in [-2,2].$$

Similarly, the Boolean central limit theorem (see [22]) states that the normalized sum of *Boolean*-independent copies of X converges weakly to the Bernoulli distribution, $\mathbf{b} := 1/2\delta_{-1} + 1/2\delta_1$.

For monotone independence, the limiting distribution for the central limit theorem (see [15]) is the arcsine distribution with density

$$\frac{1}{\pi\sqrt{2-x^2}}, \quad x \in [-\sqrt{2}, \sqrt{2}].$$

The standard proof for convergence to any of these "Gaussian" distributions consists in showing the convergence of *all* the moments. In this section we will prove that, when staying among infinitely divisible laws, convergence of the fourth moment is enough, namely we will prove Theorems 1.3, 1.4, and 1.5.

The main observation is that the following simple lemma together with the continuity properties of the Bercovici–Pata bijections gives the desired results.

LEMMA 3.1. *Let X_n be random variables with variance one and mean zero. If $E(X_n^4) \to 1$, then $\mu_{X_n} \to b$, a symmetric Bernoulli* $\mathbf{b}$.

Proof. Let $Y_n = X_n^2$. Then $E[Y_n] \to 1$ and $E[Y_n^2] \to 1$. This means that $\mathrm{Var}(Y_n) \to 0$, and thus $Y_n \to 1$ in L^2. By the condition $E[X_n] = 0$ we see that $\mu_{X_n} \to \mathbf{b}$. ∎

Now, we can prove Theorems 1.3, 1.4, and 1.5 which we state again for the convenience of the reader.

THEOREM 3.1. *Let $\{\mu_{X_n}\}_{n>0}$ be a sequence of probability measures with variance one and mean zero.*

(1) *If $\mu_{X_n} \in ID(*)$ and $E[X_n^4] \to 3$, then $\mu_{X_n} \to \mathcal{N}(0,1)$.*
(2) *If $\mu_{X_n} \in ID(\boxplus)$ and $E[X_n^4] \to 2$, then $\mu_{X_n} \to \mathcal{S}(0,1)$.*
(3) *If $\mu_{X_n} \in ID(\rhd)$ and $E[X_n^4] \to 1.5$, then $\mu_{X_n} \to \mathcal{A}(0,1)$.*

Proof. We first prove (2). Recall from Definition 2.3 that $\mathbb{B}$ stands for the Boolean-to-free Bercovici–Pata bijection. Assume $\mu_n = \mathbb{B}(\nu_n)$ for some ν_n. Then, by Remark 2.1, ν_n has variance one and mean zero and $m_4(\nu_n) \to 1$. The previous lemma applies and yields that $\nu_n \to \mathbf{b}$. By the continuity of $\mathbb{B}$, we deduce that $\mathbb{B}\big((\nu_n)\big) \to \mathbb{B}(\mathbf{b}) = \mathcal{S}(0,1)$. Parts (1) and (3) follow the same lines by changing $\mathbb{B}$ by $\mathbb{B} \circ \Lambda^{-1}$ and $\mathbb{B} \circ \Lambda^{-1} \circ \Lambda^{\rhd}$, respectively. ∎

EXAMPLE 3.1 (*Boolean central limit theorem*). Let X_i be Boolean independent identically distributed random variables with $E(X_i) = 0$ and $E(X_i^2) = 1$. Then, the random variable $Y_n = (X_1 + X_2 + \ldots + X_n)/\sqrt{n}$ is infinitely divisible. Moreover, $E(Y_n) = 0$, $E(Y_n^2) = 1$, and $E(Y_n^4) = r_2 + r_4/n \to r_2 = 1$. Thus, by Lemma 3.1, we see that $Y_n \to \mathbf{b}$.

EXAMPLE 3.2 (*Convergence of the Poisson distribution to the normal distribution*). Let X_n be a random variable with distribution $\pi(n, \delta_1)$. Then the random variable $Y_n = (X_n - n)/\sqrt{n}$ converges weakly to $\mathcal{N}(0,1)$. Indeed, $Y_n = (X_n - n)/\sqrt{n}$ is infinitely divisible. Moreover, $E(Y_n)=0$, $E(Y_n^2)=1$, and $E(Y_n^4) = c_4/n + 3c_2^2 = 1/n + 3 \to 3$. Hence, by Theorem 3.1, we see that $Y_n \to \mathcal{N}(0,1)$.

A similar argument proves the following criteria for approximation to the Poisson distributions. This shall be compared to the results in [18].

PROPOSITION 3.1. *Let us fix* $\circledast \in \{*, \uplus, \boxplus, \triangleright\}$, *let* $\mu_n \in ID(\circledast)$ *be a sequence of probability measures such that* $\mu_n \in ID(\circledast)$ *for all* n *and denote by* $\pi_{\circledast}$ *the* $\circledast$*-Poisson measure. If* $\kappa_i(X_n) \to 1$ *for* $i = 1, 2, 3, 4$, *then* $\mu_{X_n} \to \pi_{\circledast}$.

Proof. The Boolean Poisson $\pi_{\uplus}$ has distribution $1/2\delta_0 + 1/2\delta_1$. Thus one needs to show that convergence of the first fourth moments is enough to ensure weak convergence, but this is just a simple modification of Lemma 3.1. From this point the proof of Theorem 3.1 is also valid here by just replacing $\mathbf{b}$ by $1/2\delta_0 + 1/2\delta_1$. ∎

Since the proof given above seems to be produced *ad hoc*, one may ask if a more general phenomenon is hidden or this case is very particular. This is the content of the next section.

4. CONVERGENCE TO COMPOUND POISSON DISTRIBUTIONS

In this section we consider convergence to compound Poisson and, more generally, convergence to freely infinitely divisible measure with Lévy measure whose support is finite.

We first prove the following basic lemma which replaces Lemma 3.1.

LEMMA 4.1. *Assume that* μ *is a probability measure with Jacobi parameters* $\{\gamma_i, \beta_i\}_{i=1}^k$ *with* $\gamma_k = 0$. *If* $\{\mu_n\}_{n\in\mathbb{N}}$ *is a sequence of probability measures such that* $\gamma_i(\mu_n) \to \gamma_i(\mu)$ *and* $\beta_i(\mu_n) \to \beta_i(\mu)$ *for all* $i = 0, 1, \ldots, k$, *then* μ_n *converges weakly to* μ.

Proof. Let G_μ be the Cauchy transform of μ_n. By equation (2.1) we can expand G_μ as a continued fraction as follows:

$$G_{\mu_n}(z) = \cfrac{1}{z - \beta_0 - \cfrac{\gamma_0}{\ddots \over z - \beta_n - \gamma_n G_{\nu\mu}(z)}},$$

where ν is some probability measure. Now, recall that $G_{\nu\mu}(z)$ is bounded in the set $\{z|\Im(z) \geqslant 1\}$, and thus, since $\gamma_n \to 0$, we see that $\gamma_n G_{\nu\mu}(z) \to 0$. This implies the pointwise convergence

$$G_{\mu_n}(z) \to \cfrac{1}{z - \beta_0 - \cfrac{\gamma_0}{\cfrac{\ddots}{z - \beta_n}}}$$

in the set $\{z|\Im(z) \geqslant 1\}$, which then implies the weakly convergence $\mu_n \to \mu$. ∎

Now we are in a position of proving the main result of the section which contains Theorem 1.5 as a special case.

THEOREM 4.1. *Let* $\circledast \in \{*, \uplus, \boxplus, \rhd\}$ *and* $\rho_{\circledast}^{\gamma,\sigma} \in ID(\circledast)$ *with* $\circledast$*-Lévy pair* (γ, σ). *Furthermore, assume that* $\sigma = \sum_{i=1}^{k} a_i \delta_{b_i}$. *If* $\{\mu_n\}_{n\in\mathbb{N}}$ *is a sequence of* $\circledast$*-infinitely probability measures such that* $m_n(\mu) \to m_n(\rho_{\circledast}^{\gamma,\sigma})$ *for* $i = 1, \ldots, 2k+2$, *then* μ_n *converges weakly to* $\rho_{\circledast}^{\gamma,\sigma}$.

Proof. The proof is a mere modification of the proof of Theorem 1.1. The only further observations to be made in this case are the following. First, if $\rho_{\circledast}^{\gamma,\sigma}$ is a $\uplus$-infinitely probability measure with $\uplus$-Lévy pair (γ, σ) and $\sigma = \sum_{i=1}^{k} a_i \delta_{b_i}$, then $\rho_{\circledast}^{\gamma,\sigma}$ is of the form $\sum_{i=1}^{k+1} a_i \delta_{b_i}$. This easily follows from the Boolean Lévy–Khintchine representation. Second, the convergence of the first $2k+2$ moments is equivalent to the convergence of the first $2k+2$ Jacobi parameters. Thus Lemma 4.1 settles the Boolean case. Finally, we may use again the continuity of the Bercovici–Pata bijection to get the result for the other cases. ∎

As an example we get the analogue of Theorem 1.1 in Deya and Nourdin [9] for the so-called tetilla law.

EXAMPLE 4.1. Consider the distribution $\mathcal{T}$ with density

$$f(t) = \frac{\sqrt{3}}{2\pi|t|}\left(\frac{3t^2+1}{9h(t)} - h(t)\right), \qquad |t| \leqslant \sqrt{(11+5\sqrt{5})/2},$$

where

$$h(t) = \sqrt[3]{\frac{18t^2+1}{27} + \sqrt{\frac{t^2(1+11t^2-t^4)}{27}}}.$$

This is the distribution of free commutator $s_1 s_2 + s_2 s_1$, where s_1 and s_2 are free semicircle variables. If $X_n \in ID(\boxplus)$ is a sequence of random variables such that $E[X_n^i] \to m_i(\mu)$ for $i = 1, 2, \ldots, 6$, then X_n converges in distribution to $\mathcal{T}$. Indeed, $\mathcal{T}$ is a free compound Poisson $\pi_{\boxplus}(\lambda, \nu)$ with $\lambda = 2$ and $\nu = \mathbf{b}$; thus by Theorem 1.5 we get the desired result.

Finally, as a direct consequence of Theorem 4.1 we recover a theorem of Nourdin and Poly [19].

COROLLARY 4.1 ([19], Theorem 4.3). *Let $f \in L^2_s(\mathbb{R}^2_+)$ with $0 \leqslant \operatorname{rank}(f) < \infty$, let $\mu_0 \in \mathbb{R}$, and let $A \sim \mathcal{S}(0, \mu_0^2)$ be independent of the underlying free Brownian motion S. Assume that $|\mu_0| + \|f\|_{L^2(\mathbb{R}^2_+)} > 0$ and set*

$$Q(x) = x^{2(1+\mathbf{1}_{\{\mu_0 \neq 0\}})} \prod_{i=1}^{a(f)} \big(x - \lambda_i(f)\big)^2.$$

Let $\{F_n\}_{n \geqslant 1}$ be a sequence of double Wigner integrals. Then, as $n \to \infty$, we have

(i) $F_n \to A + I_2^S(f)$ *in law*

if and only if the following are satisfied:

(ii-a) $\kappa_2(F_n) \to \kappa_2\big(A + I_2^S(f)\big)$;

(ii-b) $\sum_{r=3}^{\deg Q} \frac{Q^{(r)}(0)}{r!} \kappa_r(F_n) \to \sum_{r=3}^{\deg Q} \frac{Q^{(r)}(0)}{r!} \kappa_r\big(I_2^S(f)\big)$;

(ii-c) $\kappa_r(F_n) \to \kappa_r\big(I_2^S(f)\big)$ *for $a(f)$ consecutive values of r such that $r \geqslant 2(1 + \mathbf{1}_{\{\mu_0 \neq 0\}})$.*

P r o o f. The fact the (i) implies (ii) follows from the boundedness of the sequence. To prove that (ii) implies (i) we note that since F_n is in the first and second chaos, F_n is freely infinitely divisible. Moreover, F has a Lévy pair (γ, σ) with σ with finite support. We note here that (ii-a) corresponds to the Gaussian part. ∎

REFERENCES

[1] O. Arizmendi, T. Hasebe, and N. Sakuma, *On the law of free subordinators*, ALEA Lat. Am. J. Probab. Math. Stat. 10 (1) (2013), pp. 271–291.

[2] Z. D. Bai and J. W. Silverstein, *Spectral Analysis of Large Dimensional Random Matrices*, second edition, Springer, New York 2009.

[3] O. E. Barndorff-Nielsen and S. Thorbjørnsen, *Classical and free infinite divisibility and Lévy processes*, in: *Quantum Independent Increment Processes II. Quantum Lévy Processes, Classical Probability and Applications to Physics*, U. Franz and M. Schürmann (Eds.), Springer, 2006, pp. 33–160.

[4] S. T. Belinschi, *Complex analysis methods in noncommutative probability*, available in arXiv:math/0602343v1.

[5] H. Bercovici and V. Pata, *Stable laws and domains of attraction in free probability theory* (With an appendix by Philippe Biane), Ann. of Math. (2) 149 (3) (1999), pp. 1023–1060.

[6] H. Bercovici and D. Voiculescu, *Lévy–Hinčin type theorems for multiplicative and additive free convolution*, Pacific J. Math. 153 (2) (1992), pp. 217–248.

[7] H. Bercovici and D. Voiculescu, *Free convolution of measures with unbounded support*, Indiana Univ. Math. J. 42 (3) (1993), pp. 733–773.

[8] M. Bożejko, *On $\Lambda(p)$ sets with minimal constant in discrete noncommutative groups*, Proc. Amer. Math. Soc. 51 (1975), pp. 407–412.

[9] A. Deya and I. Nourdin, *Convergence of Wigner integrals to the tetilla law*, ALEA 9 (2012), pp. 101–127.

[10] U. Franz and N. Muraki, *Markov property of monotone Lévy processes*, in: *Infinite Dimensional Harmonic Analysis III: Proceedings of the Third German-Japanese Symposium. University of Tübingen, Germany, 15–20 September 2003*, H. Heyer, T. Hirai, T. Kawazoe, and K. Saitô (Eds.), World Scientific, 2005, pp. 37–57. math.PR/0401390.
[11] T. Hasebe, *Monotone convolution semigroups*, Studia Math. 200 (2010), pp. 175–199.
[12] T. Hasebe and H. Saigo, *The monotone cumulants*, Ann. Inst. H. Poincaré Probab. Statist. 47 (4) (2011), pp. 1160–1170.
[13] T. Kemp, I. Nourdin, G. Peccati, and R. Speicher, *Wigner chaos and the fourth moment*, Ann. Probab. 40 (4) (2012), pp. 1577–1635.
[14] H. Maassen, *Addition of freely independent random variables*, J. Funct. Anal. 106 (1992), pp. 409–438.
[15] N. Muraki, *Monotonic independence, monotonic central limit theorem and monotonic law of small numbers*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 4 (2001), pp. 39–58.
[16] N. Muraki, *The five independences as natural products*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 6 (2003), pp. 337–371.
[17] N. Muraki, *Monotonic convolution and monotonic Lévy–Hincin formula*, preprint, 2000.
[18] I. Nourdin and G. Peccati, *Poisson approximations in the free Wigner chaos*, Ann. Probab. 41 (4) (2013), pp. 2709–2723.
[19] I. Nourdin and G. Poly, *Convergence in law in the second Wiener/Wigner chaos*, Electron. Comm. Probab. 17 (36) (2012).
[20] D. Nualart and G. Peccati, *Central limit theorems for sequences of multiple stochastic integrals*, Ann. Probab. 33 (1) (2005), pp. 177–193.
[21] R. Speicher, *Multiplicative functions on the lattice of non-crossing partitions and free convolution*, Math. Ann. 298 (1994), pp. 611–628.
[22] R. Speicher and R. Woroudi, *Boolean convolution*, in: *Free Probability Theory*, D. Voiculescu (Ed.), Fields Inst. Commun. 12, Amer. Math. Soc., 1997, pp. 267–280.
[23] D. Voiculescu, *Symmetries of some reduced free product C*-algebras*, in: *Operator Algebras and Their Connections with Topology and Ergodic Theory*, Lecture Notes in Math., Vol. 1132, Springer, Berlin–New York 1985, pp. 556–588.
[24] D. Voiculescu, *Addition of certain non-commutating random variables*, J. Funct. Anal. 66 (1986), pp. 323–346.

Department of Mathematics
Saarland University
Saarbrücken, Germany
E-mail: octarech@gmail.com

Received on 10.3.2013;
revised version on 13.4.2013