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CONVOLUTIONS OF GENERALIZED WHITE NOISE FUNCTIONALS

BY

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Abstract. We study a general definition of convolution products of test white noise functionals, of which the consistency property is examined. As an application of the consistency property of the convolution product we study an extension of the convolution to generalized white noise functionals. We also study relations between the convolution and generalized Fourier–Gauss and generalized Fourier–Mehler transforms.

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1. INTRODUCTION

In the study of (analytic) Wiener integral, the notion of convolution has been introduced by Yeh in [21], and then, by a similar method, Huffman et al. in [7] studied a new type convolution in terms of analytic Wiener integral. Since then the convolution products have been studied by many authors.

On the other hand, Hida [6] introduced the white noise theory to give rigorous meaning of white noise as the time derivative of the Brownian motion. Then the white noise theory has been extensively developed with wide applications to stochastic calculus, mathematical finance and mathematical physics, etc. In the white noise theory, Kuo [13] (see also [14]) introduced the convolution product and studied relation between the convolution product and Fourier transform. Recently, Obata and Ouerdiane [18] examined a different type convolution in the white noise theory.

Recently, the authors in [8], motivated by [21] and [7], studied a new type of convolution of functions on abstract Wiener space, of which the convolution

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products included the convolution products investigated in [21] and [7]. In [8], the authors focused on the study of relation between the convolution and generalized Fourier–Gauss transform with operator parameters.

In this paper, we examine the convolution products studied in [8] in the white noise setting. We focus on the study of possibility of extension of the convolution product to generalized white noise functionals. For our purpose, we first study the consistency property (see Theorem 4.2) of the convolutions, and then analyze the extension of the convolution to generalized white noise functionals. Also, we study relations between the convolution and generalized Fourier–Gauss and generalized Fourier–Mehler transforms.

This paper is organized as follows. In Section 2, we recall basic notions of white noise functionals. In Section 3, we recall well-known results in white noise operator theory, which are necessary for our study, see [17] and [14]. In Section 4, we introduce a general definition of convolution products of test white noise functionals. Then we study a consistency property of the convolution product and also investigate a relation between the convolution and generalized Fourier–Gauss transform. In Section 5, we consider basic properties of the convolution product of generalized white noise functionals and prove a relation between the convolution and generalized Fourier–Mehler transform. Finally, to unify our convolution and Kuo's convolution, we suggest a new type of convolution of generalized white noise functionals, of which the study is in progress.

2. WHITE NOISE FUNCTIONALS

Let $H_{\mathbb{R}}$ be a real separable Hilbert space with inner product $\langle\cdot,\,\cdot\rangle$ and let A be a positive self-adjoint operator on $H_{\mathbb{R}}$ such that there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ and an increasing sequence $\{l_n\}_{n=1}^{\infty}$ with $l_1 > 1$ and $\sum_{n=1}^{\infty} l_n^{-2} < \infty$ satisfying

$$Ae_n = l_n e_n, \qquad n = 1, 2, \ldots$$

We note that $\rho := \|A^{-1}\|_{\mathrm{OP}} = l_1^{-1} < 1$ and $\|A^{-1}\|_{\mathrm{HS}}^2 < \infty$. Then, by the standard construction from $H_{\mathbb{R}}$ and A (see [12], [14], [17]), we have a Gelfand triple

$$E_{\mathbb{R}} \subset H_{\mathbb{R}} \subset E_{\mathbb{R}}^*, \tag{2.1}$$

where $E_{\mathbb{R}}^*$ is the strong dual space of $E_{\mathbb{R}}$. In fact, the topology of $E_{\mathbb{R}}$ is defined by the Hilbertian norms $\{|\cdot|_p \equiv |A^p\cdot|\}_{p\geqslant 0}$, where $|\cdot|$ is the norm generated by $\langle\cdot,\,\cdot\rangle$, and then $E_{\mathbb{R}}$ becomes a countable Hilbert nuclear space. By taking the complexification of (2.1) we have the complex Gelfand triple

$$E \subset H \subset E^*,$$

i.e., $E = E_{\mathbb{R}} + iE_{\mathbb{R}}$ and $H = H_{\mathbb{R}} + iH_{\mathbb{R}}$. The canonical bilinear form on $E^* \times E$ is denoted by $\langle\cdot,\,\cdot\rangle$ again.

The (boson) Fock space $\Gamma(H)$ over H is defined by

$$\Gamma(H) = \left\{\phi = (f_n)_{n=0}^{\infty} : f_n \in H^{\widehat{\otimes} n},\ \|\phi\|^2 = \sum_{n=0}^{\infty} n!|f_n|_0^2 < \infty\right\},$$

where $H^{\widehat{\otimes} n}$ is the n-fold symmetric tensor product of H and $H^{\widehat{\otimes} 0} = \mathbb{C}$. Let $\Gamma(A)$ be the second quantization of the operator A defined by

$$\Gamma(A)\phi = (A^{\otimes n} f_n)_{n=0}^{\infty}, \quad \phi = (f_n)_{n=0}^{\infty} \in \Gamma(H),$$

and then $\Gamma(A)$ is a positive self-adjoint operator in $\Gamma(H)$ with $\|\Gamma(A)^{-1}\|_{\mathrm{OP}} < 1$ and $\|\Gamma(A)^{-1}\|_{\mathrm{HS}}^2 < \infty$. By the standard construction from $\Gamma(H)$ and $\Gamma(A)$, we have a Gelfand triple

$$(E) \subset \Gamma(H) \subset (E)^*.$$

In fact, the (projective) topology of (E) is determined by the family $\{\|\cdot\|_p\}_{p\geqslant 0}$ of norms defined by

$$\|\phi\|_p^2 = \sum_{n=0}^{\infty} n!|f_n|_p^2, \quad \phi = (f_n)_{n=0}^{\infty} \in (E).$$

It is known that for each $\Phi \in (E)^*$ there exists a unique sequence $(F_n)_{n=0}^{\infty}$ with $F_n \in (E^{\otimes n})_{\mathrm{sym}}^*$ such that

$$\langle\langle \Phi,\, \phi\rangle\rangle = \sum_{n=0}^{\infty} n!\, \langle F_n,\, f_n\rangle\,, \quad \phi = (f_n)_{n=0}^{\infty} \in (E);$$

in this case, Φ means $(F_n)_{n=0}^{\infty}$.

It follows from the Bochner–Minlos theorem that there exists a unique probability measure μ on $E_{\mathbb{R}}^*$ such that its characteristic function is given by

$$\exp\left(-\frac{1}{2}|\xi|_0^2\right) = \int_{E_{\mathbb{R}}^*} e^{i\langle x,\, \xi\rangle} \mu(dx), \quad \xi \in E_{\mathbb{R}}.$$

The above probability measure μ is called the *standard Gaussian measure* on $E_{\mathbb{R}}^*$ and the probability space $(E_{\mathbb{R}}^*, \mu)$ is referred to as the (standard) *Gaussian space*. The unitary isomorphism between $L^2(E_{\mathbb{R}}^*, \mu; \mathbb{C})$ and $\Gamma(H)$, called the *Wiener–Itô–Segal isomorphism*, is uniquely determined by the correspondence

$$\Gamma(H) \ni \phi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots\right)$$
$$\leftrightarrow \phi_\xi(x) = \exp\left(\langle x,\, \xi\rangle - \tfrac{1}{2}\langle \xi,\, \xi\rangle\right) \in L^2(E_{\mathbb{R}}^*, \mu; \mathbb{C}),$$

where ϕ_ξ is called an *exponential vector* (or *coherent state*) and $\phi_\xi(x)$ is said to be the *Gaussianization* of ϕ_ξ. In general, the Gaussianization of $\phi \in (E)$ is given by

$$\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:,\, f_n\right\rangle, \quad x \in E_{\mathbb{R}}^*, \tag{2.2}$$

where $:x^{\otimes n}:$ is the n-fold *Wick tensor product* of x (see [14] and [17]).

3. WHITE NOISE OPERATORS

Let $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ be the space of all continuous linear operators from a locally convex space $\mathfrak{X}$ into another locally convex space $\mathfrak{Y}$. An element of $\mathcal{L}\big((E), (E)^*\big)$ is called a *white noise operator* or *generalized operator*.

Since $\{\phi_{\xi_1} \otimes \ldots \otimes \phi_{\xi_m} : \xi_i \in E,\ i = 1, 2, \ldots, m\}$ spans a dense subspace of $(E)^{\otimes m}$, every $\Xi \in \mathcal{L}\big((E)^{\otimes m}, \big((E)^{\otimes n}\big)^*\big)$ is uniquely determined by the function $G : E^{m+n} \to \mathbb{C}$ defined by

$$G(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n) = \langle\!\langle \Xi(\phi_{\xi_1} \otimes \ldots \otimes \phi_{\xi_m}),\ \phi_{\eta_1} \otimes \ldots \otimes \phi_{\eta_n} \rangle\!\rangle \tag{3.1}$$

for $\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n \in E$. In particular, for $\Xi \in \mathcal{L}\big((E), (E)^*\big)$, the form given as in (3.1) is denoted by $\widehat{\Xi}$ and called the *symbol of* Ξ. Also, for $\Phi \in \mathcal{L}\big(\mathbb{C}, (E)^*\big) \cong \mathcal{L}\big((E), \mathbb{C}\big) \cong (E)^*$, the form given as in (3.1) is denoted by $S(\Phi)$ and called the S*-transform of* Φ.

THEOREM 3.1 (Ji and Obata [10]). *A Gâteaux-entire function* $G : E^{\otimes m+n} \to \mathbb{C}$ *is expressed in the form* (3.1) *with* $\Xi \in \mathcal{L}\big((E)^{\otimes m}, \big((E)^{\otimes n}\big)^*\big)$ *if and only if there exist constant numbers* $C \geqslant 0$, $K \geqslant 0$, *and* $p \geqslant 0$ *such that*

$$|G(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n)|^2 \leqslant C \exp\big\{K\big(\sum_{j=1}^{m} |\xi_j|_p^2 + \sum_{k=1}^{n} |\eta_k|_p^2\big)\big\}$$

for any $\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n \in E$. *Moreover,* $\Xi \in \mathcal{L}\big((E)^{\otimes m}, (E)^{\otimes n}\big)$ *if and only if for any* $\epsilon > 0$ *and* $p \geqslant 0$ *there exist constant numbers* $C \geqslant 0$ *and* $q \geqslant 0$ *such that*

$$|G(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n)|^2 \leqslant C \exp\big\{\epsilon\big(\sum_{j=1}^{m} |\xi_j|_{p+q}^2 + \sum_{k=1}^{n} |\eta_k|_{-p}^2\big)\big\} \tag{3.2}$$

for any $\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n \in E$.

For more study of analytic characterization theorems in white noise theory, we refer to [19], [15], [16], and [2].

For each $\kappa_{l,m} \in (E^{\otimes l+m})^*$, applying Theorem 3.1 we can see that there exists an operator $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}\big((E), (E)^*\big)$, called an *integral kernel operator*, such that

$$\widehat{\Xi_{l,m}(\kappa_{l,m})}(\xi, \eta) = \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E.$$

Note that $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}\big((E), (E)\big)$ if and only if $\kappa_{l,m} \in E^{\otimes l} \otimes (E^{\otimes m})^*$. For $f \in E^*$, we write

$$\Xi_{0,1}(f) = a(f), \quad \Xi_{1,0}(f) = a^*(f).$$

The operators $a(f)$ and $a^*(f)$ are called the *annihilation* and *creation operator*, respectively, where U^* is the adjoint operator of the given linear operator U with respect to the canonical bilinear form.

Let $\tau(K)$ be the corresponding distribution to $K \in \mathcal{L}(E, E^*)$ under the canonical isomorphism $\mathcal{L}(E, E^*) \cong (E \otimes E)^*$, i.e.,

$$\langle \tau(K), \eta \otimes \xi \rangle = \langle K\xi, \eta \rangle, \quad \xi, \eta \in E.$$

For each $K \in \mathcal{L}(E, E^*)$, the *generalized Gross Laplacian* $\Delta_{\mathrm{G}}(K) \in \mathcal{L}\big((E), (E)\big)$ is defined by

$$\Delta_{\mathrm{G}}(K) = \Xi_{0,2}\big(\tau(K)\big),$$

and then

$$\widehat{\Delta_G(K)}(\xi, \eta) = \langle K\xi, \xi \rangle \, e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E,$$

see [3]. In particular, $\Delta_{\mathrm{G}} \equiv \Delta_{\mathrm{G}}(I)$ is called the *Gross Laplacian* [5].

For given $\Phi, \Psi \in (E)^*$, applying Theorem 3.1, we can easily see that there exists a unique $\Phi \diamond \Psi \in (E)^*$ such that

$$S(\Phi \diamond \Psi)(\xi) = S(\Phi)(\xi) S(\Psi)(\xi), \quad \xi \in E.$$

In this case, the vector $\Phi \diamond \Psi$ is called the *Wick product* of $\Phi, \Psi \in (E)^*$. For each $\Phi \in (E)^*$, we associate the *Wick multiplication operator* $M_\Phi^\diamond$ by

$$M_\Phi^\diamond \Psi = \Phi \diamond \Psi, \quad \Psi \in (E)^*.$$

The operator symbol of the Wick multiplication operator is

$$\widehat{M_\Phi^\diamond}(\xi, \eta) = \langle \Phi, \phi_\eta \rangle \, e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E;$$

for more detail, see [4], [14], and [18].

For each $U \in \mathcal{L}(E, E^*)$ and $V \in \mathcal{L}(E, E)$, by applying Theorem 3.1 we can easily see that there exists a unique operator

$$\mathcal{G}_{U,V} \in \mathcal{L}\big((E), (E)\big)$$

such that

$$\widehat{\mathcal{G}_{U,V}}(\xi, \eta) = \exp\big(\tfrac{1}{2}\langle U\xi, \xi \rangle + \langle V\xi, \eta \rangle\big), \quad \xi, \eta \in E.$$

The operator $\mathcal{G}_{U,V}$ is called the *generalized Fourier–Gauss transform* [3] and its adjoint operator $\mathcal{F}_{U,V} = \mathcal{G}_{U,V}^* \in \mathcal{L}\big((E)^*, (E)^*\big)$ is called the *generalized Fourier–Mehler transform*. Then we have

$$\begin{aligned} \mathcal{G}_{U,V} &= \Gamma(V) \exp\big(\tfrac{1}{2}\Delta_{\mathrm{G}}(U)\big), \\ \mathcal{F}_{U,V} &= \exp\big(\tfrac{1}{2}\Delta_{\mathrm{G}}^*(U)\big)\Gamma(V^*). \end{aligned} \tag{3.3}$$

4. CONVOLUTIONS OF TEST WHITE NOISE FUNCTIONALS

In this section, we will define convolution operators of test white noise functionals.

4.1. Translation and dilation operators. For each $y \in E^*_{\mathbb{R}}$, the *translation operator* $\mathcal{T}_y$ on (E) is defined by

$$\mathcal{T}_y\phi(x) = \phi(x+y), \qquad x \in E^*_{\mathbb{R}}.$$

In fact, for $\xi \in E$ we have

$$\mathcal{T}_y\phi_\xi = e^{\langle y,\, \xi\rangle}\phi_\xi = e^{a(y)}\phi_\xi, \qquad \widehat{\mathcal{T}_y}(\xi,\eta) = e^{\langle y,\, \xi\rangle + \langle \xi,\, \eta\rangle}.$$

Therefore, by applying Theorem 3.1, we can see that $\mathcal{T}_y \in \mathcal{L}\big((E),(E)\big)$. If $y \in E_{\mathbb{R}}$ and $\phi \in (E)$ is given as in (2.2), then by direct calculation by using (2.18) and (2.19) in [17], we get

$$\mathcal{T}_y\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, \sum_{k=0}^{\infty} \frac{(n+k)!}{n!k!} y^{\otimes k} \widehat{\otimes}^k f_{n+k} \right\rangle, \tag{4.1}$$

where $\widehat{\otimes}^k$ is the left contraction (see [17]), and for any $p, q \in \mathbb{R}$ with $p \geqslant q$ and $r \geqslant 0$, applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\|\mathcal{T}_y\phi\|^2_{-q} &\leqslant \sum_{n=0}^{\infty} n! \left(\sum_{k=0}^{\infty} \frac{(n+k)!}{n!k!} \rho^{(p-q)k} |y|_p^k |f_{n+k}|_{-q} \right)^2 \\
&\leqslant \sum_{n=0}^{\infty} n! \left(\sum_{k=0}^{\infty} \frac{(n+k)!}{n!k!} \rho^{(p-q)k + r(n+k)} |y|_p^k |f_{n+k}|_{-(q-r)} \right)^2 \\
&\leqslant \|\phi\|^2_{-q+r} \left(\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{2^{n+k}}{k!} \rho^{2(p-q+r)k + 2rn} |y|_p^{2k} \right) \\
&= \big(\sum_{n=0}^{\infty} (2\rho^{2r})^n \big) \exp(2\rho^{2(p-q+r)} |y|_p^2) \|\phi\|^2_{-q+r}.
\end{aligned}$$

Therefore, for any $y \in E^*$, $\mathcal{T}_y$ is well defined as in (4.1), and if $y \in E$, then $\mathcal{T}_y$ can be extended as a continuous linear operator from $(E)^*$ onto itself.

For each $T \in \mathcal{L}(E^*, E^*)$, applying Theorem 3.1 we can easily see that there exists an operator $\mathcal{D}_T \in \mathcal{L}\big((E),(E)\big)$ such that

$$\widehat{\mathcal{D}_T}(\xi,\eta) = \exp\big(\langle T^*\xi,\, \eta\rangle + \tfrac{1}{2}\langle (TT^* - I)\xi,\, \xi\rangle\big), \qquad \xi,\eta \in E,$$

which implies

$$\mathcal{D}_T\phi_\xi(x) = \exp\big(\tfrac{1}{2}\langle (TT^*-I)\xi,\, \xi\rangle\big)\phi_{T^*\xi} = \phi_\xi(Tx), \qquad x \in E^*_{\mathbb{R}},\ \xi \in E.$$

Therefore, $\mathcal{D}_T$ is called the *dilation*. Moreover, we have

$$\mathcal{D}_T = \Gamma(T^*)\exp\big(\tfrac{1}{2}\Delta_{\mathrm{G}}(TT^*-I)\big) = \mathcal{G}_{TT^*-I,T^*}.$$

For each $T \in \mathcal{L}(E^*, E^*)$ and $y \in E^*$, put

$$\mathcal{R}_{T,y} = \mathcal{D}_T \mathcal{T}_y. \tag{4.2}$$

Then, for any $\xi \in E$, we have

$$\mathcal{R}_{T,y}\phi_\xi(x) = \phi_\xi(Tx + y) = \phi_{T^*\xi}(x)\phi_\xi(y) \exp\big(\tfrac{1}{2}\langle TT^*\xi,\, \xi\rangle\big), \tag{4.3}$$

which implies

$$\widehat{\mathcal{R}_{T,y}}(\xi, \eta) = \exp\big(\langle T^*\xi,\, \eta\rangle + \tfrac{1}{2}\langle (TT^* - I)\xi,\, \xi\rangle + \langle y,\, \xi\rangle\big), \quad \xi, \eta \in E.$$

Thus we have the following expression:

$$\mathcal{R}_{T,y} = \Gamma(T^*) \exp\big(\tfrac{1}{2}\Delta_{\mathrm{G}}(TT^* - I)\big) e^{a(y)} \in \mathcal{L}\big((E), (E)\big). \tag{4.4}$$

THEOREM 4.1. *For each $y \in E$, the translation $\mathcal{R}_{I,y}$ can be extended to $(E)^*$ as the operator in $\mathcal{L}\big((E)^*, (E)^*\big)$.*

Proof. For each $y \in E$, the translation operator $\mathcal{T}_y$ can be extended to $(E)^*$ as the operator in $\mathcal{L}\big((E)^*, (E)^*\big)$, and $\mathcal{R}_{I,y} = T_y$ from (4.2). Therefore, the proof is immediate. ∎

4.2. Convolutions of test white noise functionals. We start with the following lemma for the existence of the operator $C_{A,B,C,D}$ for $A, B, C, D \in \mathcal{L}(E^*, E^*)$.

LEMMA 4.1. *Let $A, B, C, D \in \mathcal{L}(E^*, E^*)$. Then there exists a unique operator $C_{A,B,C,D} \in \mathcal{L}\big((E) \otimes (E), (E)\big)$ such that, for any $\xi_1, \xi_2, \eta_1 \in E$,*

$$\begin{aligned} &\langle\!\langle C_{A,B,C,D}(\phi_{\xi_1} \otimes \phi_{\xi_2}),\, \phi_{\eta_1}\rangle\!\rangle \\ &\quad = \exp\big(\langle B^*\xi_1 + D^*\xi_2,\, \eta_1\rangle + \langle (CA^* + DB^*)\xi_1,\, \xi_2\rangle\big) \\ &\qquad \times \exp\big(\tfrac{1}{2}\langle (AA^* + BB^* - I)\xi_1,\, \xi_1\rangle + \tfrac{1}{2}\langle (CC^* + DD^* - I)\xi_2,\, \xi_2\rangle\big). \end{aligned} \tag{4.5}$$

Proof. Since $A, B, C, D \in \mathcal{L}(E^*, E^*)$, we get $A^*, B^*, C^*, D^* \in \mathcal{L}(E, E)$ and

$$AA^* + BB^* - I, CC^* + DD^* - I \in \mathcal{L}(E, E^*).$$

Therefore, for any $p, q, r \geqslant 0$, we have

$$\begin{aligned} \left|\frac{1}{2}\left\langle (AA^* + BB^* - I)\,\xi,\, \xi\right\rangle\right| &\leqslant \left|\frac{1}{2}\left(AA^* + BB^* - I\right)\xi\right|_{-p} |\xi|_p \\ &\leqslant D_p|\xi|_p^2 \leqslant D_p\rho^q|\xi|_{p+q}^2, \end{aligned} \tag{4.6}$$

$$\begin{aligned} |\langle B^*\xi,\, \eta\rangle| &\leqslant |B^*\xi|_p|\eta|_{-p} \leqslant K_{p,q}|\xi|_{p+q}|\eta|_{-p} \\ &\leqslant \frac{1}{4\epsilon}K_{p,q}^2\rho^{2r}|\xi|_{p+q+r}^2 + \epsilon|\eta|_{-p}^2 \end{aligned} \tag{4.7}$$

for any $\epsilon > 0$ and some nonnegative constants $D_p \geqslant 0$ and $K_{p,q} \geqslant 0$. Indeed, for any $\epsilon > 0$ there exists $q, r \geqslant 0$ such that $D_p \rho^q \leqslant \epsilon$ and $K_{p,q}^2 \rho^{2r} \leqslant 4\epsilon^2$.

Let us denote the right-hand side of (4.5) by $G(\xi_1, \xi_2, \eta_1)$. Then, applying inequalities (4.6) and (4.7), we can see that $G(\xi_1, \xi_2, \eta_1)$ satisfies (3.2) with $m = 2$ and $n = 1$. Therefore, by Theorem 3.1, there exists a unique operator $C_{A,B,C,D} \in \mathcal{L}\big((E) \otimes (E), (E)\big)$ such that

$$\langle\langle C_{A,B,C,D}(\phi_{\xi_1} \otimes \phi_{\xi_2}),\, \phi_{\eta_1} \rangle\rangle = G(\xi_1, \xi_2, \eta_1), \qquad \xi_1, \xi_2, \eta_1 \in E,$$

which completes the proof. ■

LEMMA 4.2. *Let $A, B, C, D \in \mathcal{L}(E^*, E^*)$. Then we have*

$$C_{A,B,C,D}(\phi_{\xi_1} \otimes \phi_{\xi_2})(y) = \langle\langle \mathcal{R}_{A,By}\phi_{\xi_1},\, \mathcal{R}_{C,Dy}\phi_{\xi_2} \rangle\rangle, \qquad y \in E_{\mathbb{R}}^*,$$

for $\xi_1, \xi_2 \in E$.

P r o o f. For any $\xi_1, \xi_2 \in E$, by applying (4.3), we obtain

$$\begin{aligned} &\langle\langle \mathcal{R}_{A,By}\phi_{\xi_1},\, \mathcal{R}_{C,Dy}\phi_{\xi_2} \rangle\rangle \\ &\quad = \exp\big[\tfrac{1}{2}\big(\langle (AA^* - I)\xi_1,\, \xi_1\rangle + \langle (CC^* - I)\xi_2,\, \xi_2\rangle\big) + \langle CA^*\xi_1,\, \xi_2\rangle\big] \\ &\qquad \times \exp\big(\tfrac{1}{2}\,\langle B^*\xi_1 + D^*\xi_2,\, B^*\xi_1 + D^*\xi_2\rangle\big)\phi_{B^*\xi_1 + D^*\xi_2}(y) \\ &\quad = \exp\big(\tfrac{1}{2}\langle (AA^* + BB^* - I)\,\xi_1,\, \xi_1\rangle + \tfrac{1}{2}\langle (CC^* + DD^* - I)\,\xi_2,\, \xi_2\rangle\big) \\ &\qquad \times \exp\big(\langle (CA^* + DB^*)\xi_1,\, \xi_2\rangle\big)\phi_{B^*\xi_1 + D^*\xi_2}(y), \end{aligned} \tag{4.8}$$

which completes the proof. ■

By Lemma 4.2, for any $\phi, \psi \in (E)$, we can write

$$\phi *_{A,B,C,D} \psi = C_{A,B,C,D}(\phi \otimes \psi),$$

which is called the *convolution* of ϕ and ψ.

THEOREM 4.2. *Let $A, A', B, B', C, C', D, D' \in \mathcal{L}(E^*, E^*)$. Then we have*

$$\langle\langle \phi *_{A,B,C,D} \psi, \varphi \rangle\rangle = \langle\langle \phi, \psi *_{A',B',C',D'} \varphi \rangle\rangle, \qquad \phi, \psi, \varphi \in (E), \tag{4.9}$$

if and only if

(C1) $B^* = D'$,
(C2) $B' = CA^* + DB^*$,
(C3) $D^* = C'A'^* + D'B'^*$,
(C4) $A'A'^* + B'B'^* = CC^* + DD^*$,
(C5) $AA^* + BB^* = C'C'^* + D'D'^* = I$.

P r o o f. By (4.5), for any $\xi, \eta, \zeta \in E$, we have

$$\begin{aligned}(4.10)\quad &\langle\langle C_{A,B,C,D}(\phi_\xi \otimes \phi_\eta), \phi_\zeta\rangle\rangle \\ &= \exp\big(\langle B^*\xi + D^*\eta,\, \zeta\rangle + \langle (CA^* + DB^*)\xi,\, \eta\rangle\big) \\ &\quad\times \exp\big(\tfrac{1}{2}\langle (AA^* + BB^* - I)\xi,\, \xi\rangle + \tfrac{1}{2}\langle (CC^* + DD^* - I)\eta,\, \eta\rangle\big),\end{aligned}$$

$$\begin{aligned}(4.11)\quad &\langle\langle \phi_\xi, C_{A',B',C',D'}(\phi_\eta \otimes \phi_\zeta)\rangle\rangle \\ &= \exp\big(\langle B'^*\eta + D'^*\zeta,\, \xi\rangle + \langle (C'A'^* + D'B'^*)\eta,\, \zeta\rangle\big) \\ &\quad\times \exp\big(\tfrac{1}{2}\langle (A'A'^* + B'B'^* - I)\eta,\, \eta\rangle + \tfrac{1}{2}\langle (C'C'^* + D'D'^* - I)\zeta,\, \zeta\rangle\big).\end{aligned}$$

Therefore, by the continuity of the convolution operator, (4.9) holds if and only if, for any $\xi, \eta, \zeta \in E$,

$$\langle\langle C_{A,B,C,D}(\phi_\xi \otimes \phi_\eta),\, \phi_\zeta\rangle\rangle = \langle\langle \phi_\xi, C_{A',B',C',D'}(\phi_\eta \otimes \phi_\zeta)\rangle\rangle$$

if and only if the conditions (C1)–(C5) hold. ∎

COROLLARY 4.1. *Let* $A, B, C, D \in \mathcal{L}(E^*, E^*)$ *satisfy the following*:
(D1) $AA^* + BB^* = CC^* + DD^* = I$,
(D2) $CA^* + DB^* = 0$,
(D3) $D^*D + B^*B = I$.
Then we have

$$\langle\langle \phi *_{A,B,C,D} \psi,\, \varphi\rangle\rangle = \langle\langle \phi,\, \psi *_{I,0,D^*,B^*} \varphi\rangle\rangle, \qquad \phi, \psi, \varphi \in (E).$$

P r o o f. The proof is immediate from Theorem 4.2. ∎

For each of the given $A, B, C, D \in \mathcal{L}(E^*, E^*)$ satisfying the conditions (D1), (D2), and (D3), the convolutions $*_{A,B,C,D}$ and $*_{I,0,D^*,B^*}$ will be denoted by $*^l_{B,D}$ and $*^r_{B,D}$, respectively. Then, by Corollary 4.1, we have

$$(4.12)\qquad \langle\langle \phi *^l_{B,D} \psi, \varphi\rangle\rangle = \langle\langle \phi, \psi *^r_{B,D} \varphi\rangle\rangle, \qquad \phi, \psi, \varphi \in (E).$$

EXAMPLE 4.1. (1) Let $P, Q \in \mathcal{L}(E, E)$ be such that $P^* = P$, $Q^* = Q$, $PQ = QP$, and $P^2 + Q^2 = I$. Then $A = P$, $B = C = Q$, and $D = -P$ satisfy the conditions (D1), (D2), and (D3), and then, by (4.12), we have

$$\langle\langle \phi *^l_{Q,-P} \psi, \varphi\rangle\rangle = \langle\langle \phi, \psi *^r_{Q,-P} \varphi\rangle\rangle, \qquad \phi, \psi, \varphi \in (E).$$

(2) For the case of $A = B = C = 1/\sqrt{2}$ and $D = -1/\sqrt{2}$, the convolution $*^l_{1/\sqrt{2},-1/\sqrt{2}}$ coincides with the convolution studied in [21] which is called the *Yeh convolution* and denoted by $*^l_{\mathrm{Y}}$. We write $C^l_{\mathrm{Y}} = C_{1/\sqrt{2},1/\sqrt{2},1/\sqrt{2},-1/\sqrt{2}}$.

(3) If we take $A = I$ and $B = 0$, then we have $A' = D, B' = C, C' = I, D' = 0$ by (C1)–(C5) with $C, D \in \mathcal{L}(E^*, E^*)$. In particular, the convolution $*_{I,0,I,I}$ coincides with the convolution studied by Obata and Ouerdiane in [18].

THEOREM 4.3. *Let $B, D \in \mathcal{L}(E, E)$. There exists a unique operator $C^r_{B,D} \in \mathcal{L}\big((E)^* \otimes (E), (E)\big)$ such that*

$$C^r_{B,D}(\psi \otimes \varphi) = \psi *^r_{B,D} \varphi$$

for $\psi, \varphi \in (E)$.

P r o o f. Consider the function $G : E \times E \times E \to E$ defined by

$$G(\xi_1, \eta_1, \eta_2) = \exp(\langle B^*\eta_1 + D^*\eta_2,\, \xi_1\rangle), \quad \xi_1, \eta_1, \eta_2 \in E.$$

Then, applying similar arguments to those used in the proof of Lemma 4.1 and Theorem 3.1, we infer that there exists a unique operator $\Xi \in \mathcal{L}\big((E), (E) \otimes (E)\big)$ such that

$$\langle\langle \Xi(\phi_{\xi_1}),\, \phi_{\eta_1} \otimes \phi_{\eta_2} \rangle\rangle = G(\xi_1, \eta_1, \eta_2), \quad \xi_1, \eta_1, \eta_2 \in E.$$

On the other hand, by the kernel theorem, we have the following topological isomorphisms:

$$\mathcal{L}\big((E)^* \otimes (E), (E)\big) \overset{\overset{J_1}{\to}}{\cong} (E) \otimes (E) \otimes (E)^* \overset{\overset{J_2}{\to}}{\cong} \mathcal{L}\big((E), (E) \otimes (E)\big).$$

Put

$$C^r_{B,D} = J_1^{-1}\left(J_2^{-1}(\Xi)\right).$$

Then, for any $\xi_1, \eta_1, \eta_2 \in E$, we obtain

$$\begin{aligned}\langle\langle C^r_{B,D}(\phi_{\eta_2} \otimes \phi_{\xi_1}), \phi_{\eta_1} \rangle\rangle &= \langle\langle J_1(C^r_{B,D}), \phi_{\eta_1} \otimes \phi_{\eta_2} \otimes \phi_{\xi_1} \rangle\rangle \\ &= \Big\langle\Big\langle J_2\big(J_1(C^r_{B,D})\big)(\phi_{\xi_1}),\, \phi_{\eta_1} \otimes \phi_{\eta_2} \Big\rangle\Big\rangle = \langle\langle \Xi(\phi_{\xi_1}),\, \phi_{\eta_1} \otimes \phi_{\eta_2} \rangle\rangle \\ &= \exp(\langle B^*\eta_1 + D^*\eta_2,\, \xi_1\rangle) = \langle\langle C_{A,B,C,D}(\phi_{\eta_1} \otimes \phi_{\eta_2}),\, \phi_{\xi_1} \rangle\rangle.\end{aligned}$$

Therefore, for any $\xi_1, \eta_1, \eta_2 \in E$, we have

$$\langle\langle C^r_{B,D}(\phi_{\eta_2} \otimes \phi_{\xi_1}), \phi_{\eta_1} \rangle\rangle = \langle\langle \phi_{\eta_1}, \phi_{\eta_2} *^r_{B,D} \phi_{\xi_1} \rangle\rangle = \langle\langle \phi_{\eta_2} *^r_{B,D} \phi_{\xi_1}, \phi_{\eta_1} \rangle\rangle,$$

which implies the assertion. ■

COROLLARY 4.2. *There exists a unique operator $C^r_{\mathrm{Y}} \in \mathcal{L}\big((E)^* \otimes (E), (E)\big)$ such that*

$$C^r_{\mathrm{Y}}(\psi \otimes \varphi) = \psi *^r_{\mathrm{Y}} \varphi$$

for $\psi, \varphi \in (E)$, where $*^r_{\mathrm{Y}} = *_{I,0,-1/\sqrt{2},1/\sqrt{2}}$.

P r o o f. The proof is immediate from Theorem 4.3. ■

THEOREM 4.4. *Let* $A, B, C, D \in \mathcal{L}(E^*, E^*)$ *satisfy the conditions* (D1), (D2), *and* (D3). *Then we have*

$$\phi *^l_{B,D} \psi = \big(\Gamma(B^*)\phi\big) \diamond \big(\Gamma(D^*)\psi\big), \quad \phi, \psi \in (E).$$

P r o o f. From (4.10) we can easily see that, for any $\xi, \eta \in E$,

$$\phi_\xi *^l_{B,D} \phi_\eta = \big(\Gamma(B^*)\phi_\xi\big) \diamond \big(\Gamma(D^*)\phi_\eta\big).$$

Therefore, the proof is immediate from the continuity of the convolution $*^l_{B,D}$ and the second quantization. ∎

COROLLARY 4.3. *We have*

$$\phi *^l_{\mathrm{Y}} \psi = \left(\Gamma\left(\frac{1}{\sqrt{2}}I\right)\phi\right) \diamond \left(\Gamma\left(-\frac{1}{\sqrt{2}}I\right)\psi\right), \quad \phi, \psi \in (E).$$

P r o o f. The proof is immediate from Theorem 4.4. ∎

THEOREM 4.5. *Let* $A, B, C, D \in \mathcal{L}(E^*, E^*)$ *and assume that* $U \in \mathcal{L}(E, E^*)$, *and* $V \in \mathcal{L}(E, E)$. *If* U *is symmetric and*

$$CA^* + D(I + U - V^*V)B^* = 0,$$

then, for any $\phi, \psi \in (E)$, *we have*

(4.13)
$$\mathcal{G}_{U,V}(\phi *_{B,D} \psi) = (\mathcal{G}_{AA^*+B(I+U)B^*-I,VB^*}\phi)(\mathcal{G}_{CC^*+D(I+U)D^*-I,VD^*}\psi),$$

where the right-hand side is the pointwise multiplication.

P r o o f. For any $\xi, \eta \in E$, using (4.8), we have

$$\begin{aligned}\phi_\xi *^l_{B,D} \phi_\eta = \exp\big(&\tfrac12\langle(AA^* + BB^* - I)\xi,\, \xi\rangle \\ &+ \tfrac12\langle(CC^* + DD^* - I)\eta,\, \eta\rangle + \langle(CA^* + DB^*)\xi,\, \eta\rangle\big)\phi_{B^*\xi+D^*\eta},\end{aligned}$$

and then we obtain

$$\begin{aligned}&\mathcal{G}_{U,V}(\phi_\xi *^l_{B,D} \phi_\eta)\\ &= \exp\big(\tfrac12\langle(AA^* + BB^* - I)\xi,\, \xi\rangle + \tfrac12\langle(CC^* + DD^* - I)\eta,\, \eta\rangle\\ &\quad + \langle(CA^* + DB^*)\xi,\, \eta\rangle\big)\exp\big(\tfrac12\langle U(B^*\xi + D^*\eta),\, B^*\xi + D^*\eta\rangle\big)\phi_{V(B^*\xi+D^*\eta)}\\ &= \exp\big[\tfrac12\big\langle\big(AA^*+B(I + U)B^* - I\big)\xi,\, \xi\big\rangle+\tfrac12\big\langle\big(CC^*+D(I + U)D^* - I\big)\eta,\, \eta\big\rangle\big]\\ &\quad \times \phi_{VB^*\xi}\phi_{VD^*\eta}\\ &= (\mathcal{G}_{AA^*+B(I+U)B^*-I,VB^*}\phi_\xi)(\mathcal{G}_{CC^*+D(I+U)D^*-I,VD^*}\phi_\eta),\end{aligned}$$

which completes the proof of (4.13). ∎

REMARK 4.1. A general study of relations between the convolution and the generalized Fourier–Gauss transform in (standard) Wiener space can be found in [8].

COROLLARY 4.4. *Let* $A, B, C, D \in \mathcal{L}(E^*, E^*)$ *satisfy the conditions* (D1), (D2), *and* (D3). *Then, for any* $V \in \mathcal{L}(E, E)$ *and* $\phi, \psi \in (E)$, *we have*

$$\mathcal{G}_{V^*V,V}(\phi *^l_{B,D} \psi) = (\mathcal{G}_{BV^*VB^*,VB^*}\phi)\,(\mathcal{G}_{DV^*VD^*,VD^*}\psi),$$

where the right-hand side is the pointwise multiplication.

P r o o f. The proof is immediate from Theorem 4.5. ■

COROLLARY 4.5. *Let* $\alpha \in \mathbb{C}$. *Then, for any* $\phi, \psi \in (E)$, *we have*

$$\mathcal{G}_{\alpha^2,\alpha}(\phi *^l_{\Upsilon} \psi) = (\mathcal{G}_{\alpha^2/2,\alpha/\sqrt{2}}\phi)(\mathcal{G}_{\alpha^2/2,-\alpha/\sqrt{2}}\psi),$$

where $\mathcal{G}_{\alpha,\beta} = \mathcal{G}_{\alpha I,\beta I}$.

P r o o f. The proof is immediate from Corollary 4.4. ■

THEOREM 4.6. *Let* $A, B, C, D \in \mathcal{L}(E^*, E^*)$ *satisfy the conditions* (D1), (D2), *and* (D3). *Then, for any* $\psi, \varphi \in (E)$, *and* $\zeta \in E$, *we have*

$$\psi *^r_{B,D} \phi_\zeta = \langle\!\langle \psi,\, \Gamma(D)\phi_\zeta \rangle\!\rangle\, \Gamma(B)\phi_\zeta, \tag{4.14}$$

$$\psi *^r_{B,D} \varphi = \big(\Gamma(B) \circ (M^\diamond_{\Gamma(D^*)\psi})^*\big)(\varphi). \tag{4.15}$$

P r o o f. For any $\eta, \zeta \in E$, by (4.11),

$$\phi_\eta *^r_{B,D} \phi_\zeta = e^{\langle \eta,\, D\zeta\rangle}\phi_{B\zeta} = \langle\!\langle \phi_\eta,\, \Gamma(D)\phi_\zeta \rangle\!\rangle\, \Gamma(B)\phi_\zeta.$$

Therefore, the proof of (4.14) is immediate by continuity. From (4.14), for any $\zeta, \xi \in E$, we obtain

$$\begin{aligned} S(\psi *^r_{B,D} \phi_\zeta)(\xi) &= \langle\!\langle \Gamma(D^*)\psi,\, \phi_\zeta \rangle\!\rangle\, \langle\!\langle \Gamma(B^*)\phi_\xi,\, \phi_\zeta \rangle\!\rangle \\ &= \langle\!\langle \Gamma(D^*)\psi \diamond \Gamma(B^*)\phi_\xi,\, \phi_\zeta \rangle\!\rangle \\ &= S\Big(\big(\Gamma(B) \circ (M^\diamond_{\Gamma(D^*)\psi})^*\big)(\phi_\zeta)\Big)(\xi). \end{aligned}$$

Therefore, (4.15) is immediate from continuity. ■

REMARK 4.2. If $B = D = I$, then from (4.15) we have

$$\psi *^r_{I,I} \varphi = (M^\diamond_\psi)^*(\varphi).$$

Therefore, the convolution $*^r_{I,I}$ coincides with the convolution studied in [18].

Let $B, D \in \mathcal{L}(E, E)$. Then, by Theorem 4.3, the convolution $*^r_{B,D}$ can be extended to $(E)^* \otimes (E)$ and we denote the extension by the same symbol. Then, for any $\Phi \in (E)^*$ and $\phi \in (E)$, we have

$$\Phi *^r_{B,D} \phi = C^r_{B,D}(\Phi \otimes \phi), \qquad \Phi *^r_{\Upsilon} \phi = C^r_{\Upsilon}(\Phi \otimes \phi).$$

5. CONVOLUTIONS OF GENERALIZED WHITE NOISE FUNCTIONALS

In this section, we study extensions of convolutions to generalized white noise functionals.

THEOREM 5.1. *Let $A, C \in \mathcal{L}(E^*, E^*)$ and $B, D \in \mathcal{L}(E, E)$ satisfy the conditions* (D1), (D2), *and* (D3). *The operator $C^l_{B,D} \equiv C_{A,B,C,D}$ can be extended to $(E)^* \otimes (E)^*$ as a continuous linear operator in $\mathcal{L}\big((E)^* \otimes (E)^*, (E)^*\big)$, of which the extension is denoted by the same symbol.*

P r o o f. By the dual property, it is enough to see that $(C^l_{B,D})^*$ belongs to $\mathcal{L}\big((E), (E) \otimes (E)\big)$. For any ξ_1, η_1, η_2, we have

$$\begin{aligned}\langle\langle (C^l_{B,D})^*(\phi_{\xi_1}), \phi_{\eta_1} \otimes \phi_{\eta_2}\rangle\rangle &= \langle\langle C^l_{B,D}(\phi_{\eta_1} \otimes \phi_{\eta_2}), \phi_{\xi_1}\rangle\rangle \\ &= \langle\langle \phi_{\eta_1} *^l_{B,D} \phi_{\eta_2}, \phi_{\xi_1}\rangle\rangle = \exp(\langle B^*\eta_1 + D^*\eta_2,\, \xi_1\rangle).\end{aligned}$$

Then, by applying similar arguments to those used in the proof of Lemma 4.1, we see that $(C^l_{B,D})^* \in \mathcal{L}\big((E), (E) \otimes (E)\big)$. Therefore, by the dual property, we obtain $C^l_{B,D} \in \mathcal{L}\big((E)^* \otimes (E)^*, (E)^*\big)$. ∎

COROLLARY 5.1. *The operator C^l_{Y} can be extended to $(E)^* \otimes (E)^*$ as a continuous linear operator in $\mathcal{L}\big((E)^* \otimes (E)^*, (E)^*\big)$, of which the extension is denoted by the same symbol.*

P r o o f. The proof is immediate from Theorem 5.1. ∎

Let $A, C \in \mathcal{L}(E^*, E^*)$ and $B, D \in \mathcal{L}(E, E)$ satisfy the conditions (D1), (D2), and (D3). For each $\Phi, \Psi \in (E)^*$, the convolution $\Phi *^l_{B,D} \Psi \in (E)^*$ is defined by

$$\Phi *^l_{B,D} \Psi = C^l_{B,D}(\Phi \otimes \Psi).$$

THEOREM 5.2. *Let $A, C \in \mathcal{L}(E^*, E^*)$ and $B, D \in \mathcal{L}(E, E)$ satisfy the conditions* (D1), (D2), *and* (D3). *Then, for any $\Phi, \Psi \in (E)^*$ and $\varphi \in (E)$, we have*

$$\langle\langle \Phi *^l_{B,D} \Psi, \varphi\rangle\rangle = \langle\langle \Phi, \Psi *^r_{B,D} \varphi\rangle\rangle.$$

P r o o f. The proof is immediate by the definitions of the convolutions $*^l_{B,D}$ and $*^r_{B,D}$. ∎

COROLLARY 5.2. *For any $\Phi, \Psi \in (E)^*$ and $\varphi \in (E)$, we have*

$$\langle\langle \Phi *^l_{\mathrm{Y}} \Psi, \varphi\rangle\rangle = \langle\langle \Phi, \Psi *^r_{\mathrm{Y}} \varphi\rangle\rangle.$$

P r o o f. The proof is immediate from Theorem 5.2. ∎

THEOREM 5.3. *Let $A, C \in \mathcal{L}(E^*, E^*)$ and $B, D \in \mathcal{L}(E, E)$ satisfy the conditions* (D1), (D2), *and* (D3). *Then, for any $\Phi, \Psi \in (E)^*$, we have*

$$\Phi *^l_{B,D} \Psi = \big(\Gamma(B^*)\Phi\big) \diamond \big(\Gamma(D^*)\Psi\big). \tag{5.1}$$

P r o o f. By (4.14), we have

$$\Psi *^{r}_{B,D} \phi_\xi = \langle\langle \Gamma(D^*)\Psi,\, \phi_\xi \rangle\rangle\, \Gamma(B)\phi_\xi, \qquad \xi \in E.$$

Therefore, we obtain

$$S(\Phi *^{l}_{B,D} \Psi)(\xi) = \langle\langle \Phi, \Psi *^{r}_{B,D} \phi_\xi \rangle\rangle = S\big(\Gamma(D^*)\Psi\big)(\xi) S\big(\Gamma(B^*)\Phi\big)(\xi),$$

which implies (5.1). ■

The following theorem gives a relation between the convolution $*^{l}_{B,D}$ and the Fourier–Mehler transform.

THEOREM 5.4. *Let* $A, C \in \mathcal{L}(E^*, E^*)$ *and* $B, D \in \mathcal{L}(E, E)$ *satisfy the conditions* (D1), (D2), *and* (D3). *Let* $U \in \mathcal{L}(E, E^*)$ *be symmetric. Then*, *for any* $\Phi, \Psi \in (E)^*$, *we have*

$$(\mathcal{F}_{U,I}\Phi) *^{l}_{B,D} (\mathcal{F}_{U,I}\Psi) = \mathcal{F}_{B^*UB+D^*UD,I}(\Phi *^{l}_{B,D} \Psi). \tag{5.2}$$

P r o o f. Note that for any symmetric $U \in \mathcal{L}(E, E^*)$ and any $V \in \mathcal{L}(E, E)$, $\xi \in E$,

$$\Gamma(V) \exp\big(\Delta_{\mathrm{G}}(V^*UV)\big)\phi_\xi = \exp(\langle V^*UV\xi,\, \xi\rangle)\phi_{V\xi} = \exp\big(\Delta_{\mathrm{G}}(U)\big)\Gamma(V)\phi_\xi,$$

which implies that

$$\Gamma(V) \exp\big(\Delta_{\mathrm{G}}(V^*UV)\big) = \exp\big(\Delta_{\mathrm{G}}(U)\big)\Gamma(V).$$

Also, we note that for any $y \in E^*$,

$$\exp\big(a(y)\big) \exp\big(\Delta_{\mathrm{G}}(U)\big) = \exp\big(\Delta_{\mathrm{G}}(U)\big) \exp\big(a(y)\big).$$

Therefore, by (4.4) and (3.3), we obtain

$$\begin{aligned}
&(\Psi *^{r}_{B,D} \mathcal{G}_{D^*UD,I}\phi)(y)\\
&= \Big\langle\Big\langle \Psi, \Gamma(D) \exp\big(\tfrac{1}{2}\Delta_{\mathrm{G}}(D^*D - I)\big) \exp\big(a(B^*y)\big) \exp\big(\tfrac{1}{2}\Delta_{\mathrm{G}}(D^*UD)\big)\phi \Big\rangle\Big\rangle\\
&= \Big\langle\Big\langle \Psi, \Gamma(D) \exp\big(\tfrac{1}{2}\Delta_{\mathrm{G}}(D^*UD)\big) \exp\big(\tfrac{1}{2}\Delta_{\mathrm{G}}(D^*D - I)\big) \exp\big(a(B^*y)\big)\phi \Big\rangle\Big\rangle\\
&= \Big\langle\Big\langle \Psi, \exp\big(\tfrac{1}{2}\Delta_{\mathrm{G}}(U)\big)\Gamma(D) \exp\big(\tfrac{1}{2}\Delta_{\mathrm{G}}(D^*D - I)\big) \exp\big(a(B^*y)\big)\phi \Big\rangle\Big\rangle\\
&= \big((\mathcal{F}_{U,I}\Psi) *^{r}_{B,D} \phi\big)(y),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}_{U,I}(\phi_\xi *^{r}_{B,D} \phi_\eta) &= \exp\big(\langle D^*\xi,\, \eta\rangle + \tfrac{1}{2}\langle B^*UB\eta,\, \eta\rangle\big)\phi_{B\eta}\\
&= \phi_\xi *^{r}_{B,D} (\mathcal{G}_{B^*UB^*,I}\phi_\eta),
\end{aligned}$$

which implies that

$$\mathcal{G}_{U,I}(\Psi *^r_{B,D} \varphi) = \Psi *^r_{B,D} (\mathcal{G}_{B^*UB,I}\varphi), \quad \Psi \in (E)^*, \ \varphi \in (E).$$

Therefore, we obtain

$$\langle\!\langle (\mathcal{F}_{U,I}\Phi) *^l_{B,D} (\mathcal{F}_{U,I}\Psi), \varphi \rangle\!\rangle = \langle\!\langle \Phi, \mathcal{G}_{U,I}(\mathcal{F}_{U,I}\Psi *^r_{B,D} \varphi) \rangle\!\rangle$$

$$= \langle\!\langle \Phi, (\Psi *^r_{B,D} \mathcal{G}_{B^*UB,I}\mathcal{G}_{D^*UD,I}\varphi) \rangle\!\rangle = \langle\!\langle \Phi, (\Psi *^r_{B,D} \mathcal{G}_{B^*UB+D^*UD,I}\varphi) \rangle\!\rangle$$

$$= \langle\!\langle \mathcal{F}_{B^*UB+D^*UD,I}(\Phi *^l_{B,D} \Psi), \varphi \rangle\!\rangle,$$

which completes the proof of (5.2). ■

COROLLARY 5.3. *Let $\alpha \in \mathbb{C}$. Then, for any $\Phi, \Psi \in (E)^*$, we have*

$$(\mathcal{F}_{\alpha,1}\Phi) *^l_{\mathrm{Y}} (\mathcal{F}_{\alpha,1}\Psi) = \mathcal{F}_{\alpha,1}(\Phi *^l_{\mathrm{Y}} \Psi).$$

Proof. The proof is immediate from Theorem 5.4. ■

REMARK 5.1. In [13] (see also [14]), Kuo introduced the convolution $\Phi * \Psi$ of generalized white noise functionals $\Phi, \Psi \in (E)$, which is defined by

$$\Phi * \Psi = \Phi \diamond \Psi \diamond g_{-2},$$

where $g_{-2} \in (E)$ is such that $S(g_2)(\xi) = e^{\langle \xi, \xi \rangle/2}$ for any $\xi \in E$. As a generalization of Kuo's convolution, the authors in [9] with the notion of convolution of white noise operators studied the convolution $\Phi *_F \Psi$ of generalized white noise functionals $\Phi, \Psi \in (E)$, which is defined by

$$\Phi *_F \Psi = \Phi \diamond \Psi \diamond F$$

for given $F \in (E)$. Our study in this paper and the studies in [13] and [9] suggest a general type of convolution of generalized white noise functionals defined by

$$\Phi *_{B,D;F} \Psi = \big(\Gamma(B)\Phi\big) \diamond \big(\Gamma(D)\Psi\big) \diamond F, \quad \Phi, \Psi \in (E)^*,$$

for given $B, D \in \mathcal{L}(E^*, E^*)$ and $F \in (E)^*$. The study of the new convolution is in progress and will appear in a separated paper.

REMARK 5.2. The time-varying convolution [11] describing the output of a linear time-varying system and the affine convolution [1] (wavelet transform, more generally coherent state transform), which is a main tool in the time frequency analysis [20] originated in signal analysis or quantum mechanics, are special cases of our convolution. Therefore, the differential equations of convolution type will be useful for the study of time-varying and time-scaling systems.

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