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BETWEEN "VERY LARGE" AND "INFINITE": THE ASYMPTOTIC REPRESENTATION THEORY*

BY

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Dedicated to the 110th anniversary of von Neumann's birthday

Abstract. I illustrate the historical roots of the theory which I called later "Asymptotic Representation Theory" – the theory which can be considered as a part functional analysis, representation theory, and more general – probability theory, asymptotic combinatorics, the theory of random matrices, dynamics, etc. The first and very concrete example is a remarkable (and forgotten) paper by J. von Neumann, which I try here to connect with the modern theory of random matrices; the second example is a quote of an important thought of H. Weyl about the theory of symmetric groups. In the last section I give a short review of the ideas of the asymptotic representation theory, which was developed starting from the 1970s, and now became very popular. I mention several important problems, and give a list (incomplete) of references. But the reader must remember that this is just a synopsis of the "baby talk".

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1. JOHN VON NEUMANN: *Our interest... is finite but very great order of matrices...*

I start with a beautiful paper by J. von Neumann about the properties of high-order matrices, which impressed me very much many years ago.

It happened that the discovery of infinite-dimensional analysis (functional analysis) has followed very promptly the invention of multidimensional (but finite) analysis; mathematicians had no time to pay attention to a very important circumstance: the study of very high dimensions must precede that of infinite-dimensional

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objects; the understanding of "intermediate" (between finite and infinite) things is absolutely necessary for the understanding of the actual infinity. The exciting perspectives and fascinating results of the functional analysis in the first part of the 20th century eclipsed the beauty of high-dimensional analysis. My viewpoint on this was described in [16]. Many mathematicians, including the great creators of functional analysis, forewarn of that. I want to mention at least two of such warnings, which impressed me very much when I was a young mathematician and had just studied functional analysis.

I will discuss a remarkable John von Neumann's paper "Approximative properties of matrices of high finite order" [7].[1] In the Russian translation volume *Selected Papers on Functional Analysis* by von Neumann (see [8]) I gave some comments about this article and now I will use part of those remarks with some new points. This is a remarkable paper. I think it is almost forgotten and cited relatively rarely in the mathematical literature in comparison with other papers of this author, but it deserves much more attention. I remember only Glimm's reference to it in his famous paper [2] on types of factors. The main message of the paper is in the following claim:

Our interest will be concentrated in this note on the conditions in H_n and M_n [Hilbert space and unit ball in matrix norm of dimension n – comments by AV] *when n is finite but VERY GREAT. This is an approach to the study of the infinite dimensional which differs essentially from the usual one. The usual approach consists in studying an actually infinite dimensional unitary space, i.e. Hilbert space H_∞ as done. We wish to investigate instead the ASYMPTOTIC behavior of H_n and M_n for finite n, when $n \to \infty$. We think that the latter approach has been unjustifiably neglected, as compared with the former one. It is certainly not contained in it, since it permits the use of the notion of norm* [Hilbert–Schmidt norm – AV] *and normalized trace, which, owing to the factors $1/n$ possess no analoga in H_∞ and its M_n. Since Hilbert space H_∞ was conceived as a limiting case of H_n for $n \to \infty$, we feel that such a study is necessary in order to clarify to what extent H_∞ is or is not the only possible limiting case. Indeed, we think that it is not, and that investigations on operator rings by F. J. Murray and the author show that other limiting cases exist, which under many aspects are more natural ones.*

Here we see the enthusiasm of the author with his striking idea of the continuous dimension, factors of type II_1, etc., which were revolutionary at that time. But even outside of those ideas the point of view proclaimed by von Neumann is extremely important. The main result of this paper is very important even now. The author established a very interesting property of the matrices $M_n(\mathbb{C})$ of big order,

[1]One of the reasons why this article was not so widely known had been perhaps the strange place of its publication – Portugaliae Mathematica – and difficult time – time of the Second World War, 1942. Remark that von Neumann had earlier published one of his important papers in the even stranger journal – Proceedings of Tomsk University (1937). A very important paper on another classic of functional analysis, Leonid Kantorovich's "On translocation of masses", was also published in 1942, and so was not known in the West for a long time (see [17]).

which was unknown before and even now is little known. The main result illustrates how incomplete is our knowledge of the multidimensional theory of matrices; in a sense, it is much more modest than our knowledge of the infinite-dimensional case. I quote the main statement of the paper here.

THEOREM 1.1 ([7], Theorem 9.6). *For every $\delta \in \left(0, \frac{1}{2}\right)$ there exists $\epsilon = \epsilon(\delta) > 0$ such that if P_E is an orthogonal projection on a subspace E with property*

$$\delta < \frac{\dim E}{n} < 1 - \delta,$$

then for every positive integer n there exists a matrix $A \in M_n(\mathbb{C})$ with $\|A\| \leqslant 1$ such that

$$\|AP_E - P_E A\| > \epsilon.$$

This means that for all n, on the space $\mathbb{C}^n$ there exists an operator A for which there are no subspaces of intermediate dimensions which are almost invariant together with their orthogonal complements. This is a fact from the geometry and linear algebra of the *Hilbert space*. This result has no literal interpretation in the infinite-dimensional case, because the step from finite to infinite dimension changes drastically the notion of genericity. In the paper, as is usual for this author, there are many various reformulations of the main result. I suggest below some strengthened formulation and discussion using the point of view of the modern theory of random matrices.

REMARKS. 1. The method of proof was in fact based on more or less direct calculations of the Lebesgue measure of the set of matrices which do not have the property formulated in the theorem and have Hilbert–Schmidt norm less than or equal to one. It is clear from the calculations that this measure is much less than the measure of all unit balls in $M_n(\mathbb{C})$. Von Neumann called the method "volumetric method", and we can call it "the entropy method". The most difficult part of the paper is devoted to the proof of the fact that the ball in Hilbert–Schmidt norm can be a substitute for the ball in spectral norm; the direct calculation of the asymptotic of the measure unit ball in the spectral norm failed.

2. It is interesting that the problem appeared when von Neumann tried to construct an example of a new type of hyper-finite II_1 factors using the result of the paper. But soon (as mentioned in the proofs of the given paper) he proved (with F. J. Murray) that there exist non-isomorphic factors of type II_1 using another method. Moreover, one cannot obtain new examples of the factors in the way of this paper, because the hyper-finite factor of type II_1 is unique up to isomorphism; this was proved later, while the method above can give only hyper-finite ones. Nevertheless, the result of the paper has independent interest. As the author emphasized, the set of required matrices in the theorem was not constructively defined, and he could not give a constructive proof of the existence of such a matrix. I hope that after more than 70 years it is now possible to refine the problem and to give a more

constructive and simpler proof of von Neumann's result. I will mention only some interpretation of the result.

Of course, these matrices cannot be unitary or even normal. Recall that with respect to the Haar measure, almost all elements of $\mathrm{GL}(n,\mathbb{C})$ are semi-simple and have distinct eigenvalues and, as it was used in the paper, can be represented as upper-triangle (semi-diagonal as von Neumann called it) matrices with different elements on the diagonal. With respect to the (infinite) Haar measure m, we can say that almost all matrices are semi-simple and have this form. On the other hand, we can speak equivalently about the random choice of the linear independent n-vectors in n-dimensional vector spaces or points on a non-orthogonal Stiefel manifold. So we can consider the above von Neumann's result from the point of view of the theory of random matrices or random repairs. Why has the theory of random matrices appeared here? Because we have dealt with a theorem about the statistics and properties of the geometry of the configuration of the eigenvectors of an arbitrary finite-dimensional semi-simple random operator. Many papers are devoted to the study of the spectrum of random matrices in various situations, but few to the investigations of the asymptotics of interesting functionals on non-unitary and non-Hermitian matrices. This statistics of the eigenvectors for general matrices is a very intriguing subject. These questions are much deeper than the circular law for the spectra of non-self-adjoint matrices.

I want to strengthen slightly the result of the paper in the following conjecture.

CONJECTURE 1.1. *Let* $\delta > 0$, *and let* $\mathrm{Gras}_\delta = \bigcup_{k:\delta<k/n<1-\delta} G_{n,k}$, *where* $G_{n,k}$ *is the Grassmannian of all* k*-dimensional subspaces in* $\mathbb{C}^n$. *Then there exists* $\epsilon = \epsilon(\delta) > 0$ *such that for all* n

$$\int_{A\in\mathrm{GL}_1} \min_{E\in\mathrm{Gras}_\delta} \|[A,P_E]\|dm(A) = \epsilon > 0,$$

where $\mathrm{GL}_1 = \{A \in \mathrm{GL}(n,\mathbb{C}) : \|A\| \leqslant 1\}$, P_E *is the orthogonal projection to the subspace* E, m *is the normalized Haar* (*Lebesgue*) *measure on the ball* $\mathrm{GL}(n,\mathbb{C})_1$, $\|\cdot\|$ *is the normalized Hilbert–Schmidt norm in* $M_n(\mathbb{C})$, *and* $[A,P] = AP - PA$.

It is an additional interesting question, if there exists a limit distribution of the integrand as a function on $\mathrm{GL}(n,\mathbb{C})_1$; what is it and what is the infinite-dimensional interpretation of it?

An additional task is to prove this conjecture for spectral norm: perhaps it seems that the method of the paper may allow to do this. Instead of the Lebesgue measure it is natural to use the Gaussian measure on the space of matrices. Moreover, we can use only the Borel (upper diagonal) subgroup of $\mathrm{GL}(n,\mathbb{C})$ instead of the space of all matrices. Another thing is to use instead of a matrix A a configuration (repair) of n eigenvectors of the matrix A, and integrate over non-orthogonal Stiefel manifold which is the space of all nondegenerated repairs. We will not stay on this here.

The conjecture strengthens von Neumann's result in the following sense: instead of the supremum of norms we consider the integral of commutators over all matrices. In a more expressive form we can rewrite the conjecture equivalently:

$$\inf_{n\in\mathbb{N}} \int_{A\in \mathrm{GL}_1(n,\mathbb{C})} \min_{E\in \mathrm{Gras}_\delta} \{\|P_E A P_{E^\perp}\| + \|P_{E^\perp} A P_E\|\} dm(A) \equiv c = c(\delta) > 0.$$

It is interesting to find the limit distribution of the integrand (when n tends to infinity); it can happen that the limit measure is a delta measure.

There are similar problems in the spirit of the asymptotic behavior of random matrices, which can be formulated as extensions of von Neumann questions from that remarkable paper. We will return to this.[2]

2. HERMANN WEYL: *The goal of combinatorics is to find properties of typical permutations of large finite degrees*

Now I will mention H. Weyl's opinion on the role of asymptotics, concretely, about symmetric groups of high order. In a sense, this idea is also in the spirit of those by von Neumann and of asymptotic representation theory. Then I will describe the main problems that I consider as a kind of agenda for the future development of this theory. The next example is similar to the previous one, but has a different flavor. In his book *Philosophy of Mathematics and Natural Sciences*, Weyl (see [28]) wrote a special chapter about combinatorics and made a perspicacious remark about it: *Perhaps the simplest combinatorial entity is the group of permutations of n objects. This group has a different constitution for each individual number n. The question is whether there are nevertheless some asymptotic uniformities prevailing for large n or for some distinctive class of large n.* He then continued: *Mathematics has still little to tell about such a problem.*

I used this quote as an epigraph to my talk [14] at the International Congress of Mathematicians in Zürich in 1994 and, earlier, in the paper [27]. One can still consider this as a program for the future. But, nevertheless, now we can say more than in Weyl's time. In the last quarter of the 20th century, the status of combinatorics changed drastically, because its new aspects appeared under the influence of statistical physics, representation theory, geometry.

I want to illustrate this with examples from the theory of representations of symmetric groups.

One of the main objects in combinatorics are Young diagrams (or Ferrers' diagrams). The link between Young diagrams and the theory of symmetric groups is attributed to Alfred Young. In papers by F. G. Frobenius, I. Schur, and A. Young,

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[2]Our Conjecture 1.1 can be interpreted in the spirit of the theory of random matrices. It is possible to consider the Gaussian distribution on the space of all (non-Hermitian) matrices, but our functional does not concern the spectrum or singular numbers of matrices, but a more complicated characteristic of matrices. [Added in proof]

the main facts were established, including formulas for the characters, the branching rule, a link to $\mathrm{GL}(n, R)$, etc. The remarks and improvements by J. von Neumann, H. Weyl, L. Brauer created the representation theory of symmetric groups, which existed in this form for almost 100 years. But some important questions were neglected and stayed open. H. Weyl's observation shows only one of these problems.

I was never satisfied with the standard explanation why Young diagrams appear in this theory, which is that the branching rules are the same for the irreducible representations of the symmetric groups and for the Young graph. But the true explanation came from the inductive, or asymptotic, point of view.

We must pay attention to Coxeter's description of symmetric groups and to the role of the so-called Gelfand–Tsetlin commutative subalgebra of the group algebra of a symmetric group. The latter algebra (which is generated by the centers of the group algebras $\mathbb{C}[S_k]$, $k = 1, \dots, n$) appeared, in a sense, from viewing S_n as the union of the inductive family $S_1 \subset S_2 \subset \dots \subset S_n$. Its classical analogue is the Gelfand–Tsetlin basis for representations of the groups $U(n), O(n)$. But it is just the "asymptotic" point of view that I tried to use for the symmetric groups.

In a finite form, it was described in my joint paper with Okounkov [10], which contained the realization of the previous idea about new views on the representation theory of finite symmetric group [13]; see also [15], [25], and later development of those ideas which have been used in many papers as a background for the representation theory of the symmetric groups, e.g. [1]. I consider this story as a good example of how fruitful is the influence of the "almost infinite" philosophy on finite problems.

On the other hand, in the study of the representation theory of the infinite symmetric group S_∞, some aspects of the asymptotic theory of finite groups turn out to be very useful, although these subjects are completely different.

3. HOW TO CHOOSE RIGHT INFINITE OBJECTS THAT GENERALIZE A FINITE ONE

The idea about the Asymptotic Theory of Representations (it is possible to use the abbreviation ART) appeared in my mind at the beginning of the 1970s. I thought about infinite-dimensional groups and algebras in comparison with ergodic theory and combinatorics. The first contribution was the attack to the problem of limit behavior of joint distribution of the cycles of a random permutation. The results were published in the short announcement [26], communicated by Yu. Linnik (1972), who was extremely interested in the connection of this type of asymptotics and number theory, and later in two big articles [27] which were written in 1974. Then I had a more or less precise plan which included the study of the representations of symmetric groups and analysis of the paper by Thoma [11], about the characters of infinite symmetric group, which was recommended to me by I. M. Gelfand. My student (aspirant) of that time Sergey Kerov prepared his thesis (candidate dissertation) on the theme of a variant of duality theory of the algebras with involution, which I suggested in 1972. When he had finished his thesis I suggested him to work with me on what I called later ART. It was a successful choice and we started to work together on this topic in the middle of the 1970s. The main idea of my approach developed jointly with S. Kerov was to obtain the properties of the group S_∞ and its representations as asymptotic properties. One of the best examples here was the proof of Thoma's formula for characters obtaining them as the limits of sequences of irreducible characters of the finite groups S_n as $n \to \infty$. In principle, this can be done by the ergodic method (ergodic theorem), but the calculation is rather involved (see [19], [20], [4]–[6]). A more difficult question concerns the study of the corresponding factor representations, their realizations and analytical properties. There are remarkable papers by S. Kerov, G. Olshanski, A. Okounkov, and A. Borodin on the topic, and now we have much information about these problems, but some of the important questions (for instance, about representations of type II_∞) are still open (see [24]).

The central problems in the theory of locally semi-simple algebras and inductive limits of groups are the following:

1. To find the list of traces, e.g. central measures on the space of paths of the Bratteli diagram, or – invariant measures for the so-called adic transformation defined by the lexicographic ordering of paths of the Bratteli–Vershik diagram.

2. To describe the K_0-functor as a Riesz group, i.e. the ordered structure on the Grothendieck group.

3. To give various realizations of the main classes of representations of the corresponding groups and algebras, and to give asymptotic interpretation of the corresponding results of the classical theory. We have very few examples of results of that type. One of them is given in [12]. We hope that understanding of some classical procedures like the Bethe ansatz will come from asymptotic representation theory.

The first two problems were solved only for few algebras and groups. For the group S_∞, we have several proofs of Thoma's theorem on the list of characters (see Thoma [11], Vershik and Kerov [19], Okounkov [9]), but still we have no pure combinatorially-probabilistic proof which should be applied not only to Young graph, but also to a wide class of AF-algebras, including Hasse diagrams of the distributive lattices, and so on (see a recent paper [18] which contains new ideas about invariant measures). [3]

The structure of the K_0-functor for S_∞ was discovered in [21], but still has no serious applications. Both problems belong to asymptotic representation theory as well as to combinatorics and probability theory.

The asymptotic study of S_∞ and other similar groups G (like the infinite unitary group $U(\infty)$ and other inductive limits of sequences of finite or compact

[3] It has recently become clear that the standardness of the filtrations is equivalent to the property of existence of the so-called "limit shape". [Added in proof]

groups) has been concentrated in the last years near the so-called harmonic analysis, which was formulated by G. Olshanski as a bunch of problems about the decomposition of a natural representation of the double group $G \times G$ into a spectrum of irreducible representations (the so-called theory of z-measures). Studying the representations of the double (left and right) group as a method of studying the representations of the group G itself was suggested by von Neumann. A crucial fact is that the restriction of a representation of the double group to the left or right component is a representation of type II; in particular, in the case of irreducible representations of the double group, it is a factor of type II_1 or II_∞. In order to single out representations of the double group that are useful in this sense, in the case of locally finite groups G. Olshanski introduced the notion of admissible representations. This is a special class of representations of $G \times G$, and for some groups G (or locally semi-simple algebras) they are not of type I, the category of representations of $G \times G$ is of type I. For the infinite symmetric group, the description of this category was started by G. Olshanski and continued by Okounkov [9]. Nevertheless, the analysis is not yet completed; see the analysis of the types of factors in [24]. The corresponding analysis for $U(\infty)$ and other groups is only at the first stage.

The picture is different in the case of the group of infinite matrices over finite fields. It started with the paper [22] and then was continued in [23] and [3]. This theory is related to the classical results on representations of the group $\mathrm{GL}(n, F_q)$ (R. Green, D. Faddeev, A. Zelevinsky). I want to emphasize here only that the similarity between representations of this group and representations of the infinite symmetric group extends only to the so-called principal series of representations, and even in this case we have only first results.

The last remark concerns another asymptotic question in the same spirit: how to generalize correctly the classical finite concept of the Schur–Weyl duality between representations of the symmetric group S_N and the group $\mathrm{GL}(n, C)$ to the infinite case? As far as I know, in our paper [12] this question is discussed for the first time. Among many possibilities, the authors have chosen one; it will be clear later on whether this choice is justified.

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