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HERZ–SCHUR MULTIPLIERS AND NON-UNIFORMLY BOUNDED REPRESENTATIONS OF LOCALLY COMPACT GROUPS

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Abstract. Let G be a second countable, locally compact group and let φ be a continuous Herz–Schur multiplier on G. Our main result gives the existence of a (not necessarily uniformly bounded) strongly continuous representation π of G on a Hilbert space $\mathscr{H}$, together with vectors $\xi, \eta \in \mathscr{H}$, such that $\varphi(y^{-1}x) = \langle \pi(x)\xi, \pi(y^{-1})^*\eta \rangle$ for $x, y \in G$ and $\sup_{x \in G} \|\pi(x)\xi\| \cdot \sup_{y \in G} \|\pi(y^{-1})^*\eta\| = \|\varphi\|_{M_0A(G)}$. Moreover, we obtain control over the growth of the representation in the sense that $\|\pi(g)\| \leqslant \exp\left(\frac{c}{2}\mathrm{d}(g,e)\right)$ for $g \in G$, where $e \in G$ is the identity element, c is a constant, and d is a metric on G.

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1. INTRODUCTION

Let us assume that Y is a non-empty set. A function $\psi : Y \times Y \to \mathbb{C}$ is called a *Schur multiplier* if for every operator $A = (a_{x,y})_{x,y \in Y} \in \mathbf{B}\big(\ell^2(Y)\big)$ the matrix $\big(\psi(x,y)a_{x,y}\big)_{x,y \in Y}$ represents an operator from $\mathbf{B}\big(\ell^2(Y)\big)$ (this operator is denoted by $M_\psi A$). If ψ is a Schur multiplier it follows from the closed graph theorem that $M_\psi \in \mathbf{B}\big(\mathbf{B}\big(\ell^2(Y)\big)\big)$, and one refers to $\|M_\psi\|$ as the *Schur norm* of ψ and denotes it by $\|\psi\|_S$.

Let G be a locally compact group. In [6], Herz introduced a class of functions on G, which was later called the class of *Herz–Schur multipliers* on G. By the introduction to [1], a continuous function $\varphi : G \to \mathbb{C}$ is a Herz–Schur multiplier if and only if the function

$$\hat{\varphi}(x,y) = \varphi(y^{-1}x) \quad (x, y \in G) \tag{1.1}$$

is a Schur multiplier, and the *Herz–Schur norm* of φ is given by

$$\|\varphi\|_{HS} = \|\hat{\varphi}\|_S.$$

In [3] De Cannière and Haagerup introduced the Banach algebra $MA(G)$ of *Fourier multipliers* of G, consisting of functions $\varphi : G \to \mathbb{C}$ such that

$$\varphi\psi \in A(G) \quad (\psi \in A(G)),$$

where $A(G)$ is the *Fourier algebra* of G as introduced by Eymard in [4]. The norm of φ (denoted by $\|\varphi\|_{MA(G)}$) is given by considering φ as an operator on $A(G)$. According to Proposition 1.2 in [3] a Fourier multiplier of G can also be characterized as a continuous function $\varphi : G \to \mathbb{C}$ such that

$$\lambda(g) \overset{M_\varphi}{\mapsto} \varphi(g)\lambda(g) \quad (g \in G)$$

extends to a σ-weakly continuous operator (still denoted by M_φ) on the group von Neumann algebra ($\lambda : G \to \mathbf{B}\big(L^2(G)\big)$ is the *left regular representation* and the group von Neumann algebra is the closure of the span of $\lambda(G)$ in the weak operator topology). Moreover, one has $\|\varphi\|_{MA(G)} = \|M_\varphi\|$. The Banach algebra $M_0A(G)$ of *completely bounded Fourier multipliers* of G consists of the Fourier multipliers φ of G, for which M_φ is completely bounded. Let $\|\varphi\|_{M_0A(G)} = \|M_\varphi\|_{\mathrm{cb}}$.

In [1] Bożejko and Fendler show that the completely bounded Fourier multipliers coincide isometrically with the continuous Herz–Schur multipliers. In [7] Jolissaint gives a short and self-contained proof of this result in the following form.

PROPOSITION 1.1 ([1], [7]). *Let G be a locally compact group and assume that $\varphi : G \to \mathbb{C}$ and $k \geqslant 0$ are given. Then the following are equivalent*:

(i) *φ is a completely bounded Fourier multiplier of G with $\|\varphi\|_{M_0A(G)} \leqslant k$.*

(ii) *φ is a continuous Herz–Schur multiplier on G with $\|\varphi\|_{HS} \leqslant k$.*

(iii) *There exist a Hilbert space $\mathscr{H}$ and two bounded, continuous maps $P, Q : G \to \mathscr{H}$ such that*

$$\varphi(y^{-1}x) = \langle P(x), Q(y)\rangle \quad (x, y \in G)$$

and

$$\|P\|_\infty \|Q\|_\infty \leqslant k,$$

where

$$\|P\|_\infty = \sup_{x\in G} \|P(x)\| \quad \textit{and} \quad \|Q\|_\infty = \sup_{y\in G} \|Q(y)\|.$$

By a *representation* $(\pi, \mathscr{H})$ of a locally compact group G on a Hilbert space $\mathscr{H}$ we mean a homomorphism of G into the invertible elements of $\mathbf{B}(\mathscr{H})$. A representation $(\pi, \mathscr{H})$ of G is said to be *uniformly bounded* if

$$\sup_{g\in G} \|\pi(g)\| < \infty$$

and one usually writes $\|\pi\|$ for $\sup_{g\in G}\|\pi(g)\|$. If $g\mapsto\pi(g)$ is continuous with respect to the strong operator topology on $\mathbf{B}(\mathscr{H})$ then we say that $(\pi,\mathscr{H})$ is *strongly continuous*. Let $(\pi,\mathscr{H})$ be a strongly continuous, uniformly bounded representation of G. Then, according to Theorem 2.2 of [3], any *coefficient of* $(\pi,\mathscr{H})$ is a continuous Herz–Schur multiplier, i.e.,

$$g \overset{\varphi}{\mapsto} \langle \pi(g)\xi,\eta\rangle \quad (g\in G)$$

is a continuous Herz–Schur multiplier with

$$\|\varphi\|_{M_0A(G)} \leqslant \|\pi\|^2\|\xi\|\|\eta\|$$

for any $\xi,\eta\in\mathscr{H}$ (note that this result also follows as a corollary to Proposition 1.1). U. Haagerup has shown that on the non-abelian free groups there are Herz–Schur multipliers which cannot be realized as coefficients of uniformly bounded representations. The proof by Haagerup has remained unpublished, but Pisier has later given a different proof, cf. [8]. Haagerup's proof can be modified to prove the corresponding result for the connected, real rank one, simple Lie groups with finite center; cf. [9], Theorem 3.6.

Strictly speaking, the requirement that the above representations be uniformly bounded is not fully needed in order to construct a continuous Herz–Schur multiplier. From Proposition 1.1 it follows that it is enough to require that

$$\sup_{x\in G}\|\pi(x)\xi\|<\infty \quad \text{and} \quad \sup_{y\in G}\|\pi(y^{-1})^*\eta\|<\infty. \tag{1.2}$$

In Theorem 1.1 of [2] Bożejko and Fendler show that for countable discrete groups all Herz–Schur multipliers can be realized as coefficients of representations satisfying a condition similar to (1.2). More specifically, they show that if φ is a Hermitian Herz–Schur multiplier on a countable discrete group Γ, then there exist a representation $(\pi,\mathscr{H})$ and vectors $\xi,\eta\in\mathscr{H}$ such that

$$\varphi(y^{-1}x)=\langle\pi(x)\xi,\pi(y)\eta\rangle \quad (x,y\in\Gamma)$$

with

$$\sup_{x\in\Gamma}\|\pi(x)\xi\|\leqslant\|\varphi\|^{1/2}_{M_0A(\Gamma)} \quad \text{and} \quad \sup_{y\in\Gamma}\|\pi(y)\eta\|\leqslant\|\varphi\|^{1/2}_{M_0A(\Gamma)}.$$

Furthermore, they give a quantitative bound on $\|\pi(g)\|$ for $g\in\Gamma$ and note that the same holds for non-Hermitian Herz–Schur multipliers by including a $\sqrt{2}$ factor in the bound for $\sup_{x\in\Gamma}\|\pi(x)\xi\|$ and $\sup_{y\in\Gamma}\|\pi(y)\eta\|$. In Section 2 we present a generalization of the result by Bożejko and Fendler to second countable, locally compact groups (Theorem 2.1 and Corollary 2.1) and prove our main result, which is the following

THEOREM 1.1. *Let G be a second countable, locally compact group and let* d *be a proper, left invariant metric on G which has* at most exponential growth*, i.e.,*

$$\mu\big(B_n(e)\big) \leqslant a \cdot e^{bn} \quad (n \in \mathbb{N})$$

for some constants $a, b > 0$, where μ is a left invariant Haar measure on G and $B_n(e) = \{g \in G : \mathrm{d}(g, e) < n\}$ is the open ball of radius n centered at the identity element $e \in G$. Then for any continuous Herz–Schur multiplier φ on G there exist a strongly continuous representation $(\pi, \mathscr{H})$ and vectors $\xi, \eta \in \mathscr{H}$ such that

$$\varphi(y^{-1}x) = \langle \pi(x)\xi, \pi(y^{-1})^*\eta \rangle \quad (x, y \in G)$$

with

$$\sup_{x \in G} \|\pi(x)\xi\| = \|\varphi\|_{M_0A(G)}^{1/2} \quad \textit{and} \quad \sup_{y \in G} \|\pi(y^{-1})^*\eta\| = \|\varphi\|_{M_0A(G)}^{1/2}.$$

Moreover, for every fixed $c > b$, $(\pi, \mathscr{H})$ can be chosen such that

$$\|\pi(g)\| \leqslant e^{\frac{c}{2} \cdot \mathrm{d}(g,e)} \quad (g \in G).$$

Note that the existence of a proper, left invariant metric with at most exponential growth on a second countable, locally compact group is guaranteed by [5].

2. COEFFICIENTS OF NON-UNIFORMLY BOUNDED REPRESENTATIONS

Second countability guarantees the existence of a proper, left invariant metric, cf. the Theorem in [10]. Actually, according to Haagerup and Przybyszewska [5] one can choose this metric, d, to have *at most exponential growth*, i.e.,

$$\mu\big(B_n(e)\big) \leqslant a \cdot e^{bn} \quad (n \in \mathbb{N}) \tag{2.1}$$

for some constants $a, b > 0$, where μ is a left invariant Haar measure on G.

Inspired by the proof of Theorem 1.1 in [2], we state and prove Theorem 2.1 for *Hermitian* Herz–Schur multipliers, i.e., Herz–Schur multipliers φ for which $\varphi^* = \varphi$, where

$$\varphi^*(g) = \overline{\varphi(g^{-1})} \quad (g \in G).$$

The non-Hermitian case is treated in Corollary 2.1.

THEOREM 2.1. *If φ is a continuous Hermitian Herz–Schur multiplier on a second countable, locally compact group G, and* d *is a proper, left invariant metric on G satisfying* (2.1) *for some $a, b > 0$ (which exists according to* [5]*), then there exist a strongly continuous representation $(\pi, \mathscr{H})$ and vectors $\xi, \eta \in \mathscr{H}$ such that*

$$\varphi(y^{-1}x) = \langle \pi(x)\xi, \pi(y)\eta \rangle \quad (x, y \in G)$$

with

$$\sup_{x\in G}\|\pi(x)\xi\| = \|\varphi\|^{1/2}_{M_0A(G)} \quad \textit{and} \quad \sup_{y\in G}\|\pi(y)\eta\| = \|\varphi\|^{1/2}_{M_0A(G)}.$$

Moreover, for every fixed $c > b$, $(\pi, \mathscr{H})$ *can be chosen such that*

$$\|\pi(g)\| \leqslant e^{\frac{c}{2}\cdot \mathrm{d}(g,e)} \quad (g \in G).$$

Before we proceed with the proof of Theorems 1.1 and 2.1 we need the following application of [5], which was communicated to us by Haagerup.

LEMMA 2.1. *If G is a second countable, locally compact group, then there exist a positive function $h \in L^1(G)$ with $\|h\|_1 = 1$, and a positive function c on G such that*

$$\frac{1}{c(g)}\int_G f(z)h(z)\mathrm{d}\mu(z) \leqslant \int_G f(z)h(gz)\mathrm{d}\mu(z) \leqslant c(g)\int_G f(z)h(z)\mathrm{d}\mu(z)$$

for $g \in G$ and any positive $f \in L^\infty(G)$, where μ is the Haar measure on G. Moreover, we may use

$$h(g) = \frac{e^{-c\cdot \mathrm{d}(g,e)}}{\int_G e^{-c\cdot \mathrm{d}(x,e)}\mathrm{d}\mu(x)} \quad \textit{and} \quad c(g) = e^{c\cdot \mathrm{d}(g,e)} \quad (g \in G)$$

for $c > b$, when d is a proper, left invariant metric on G satisfying (2.1).

P r o o f. Let μ be a left invariant Haar measure on G and let d be a proper, left invariant metric on G satisfying (2.1). We claim that

$$0 < \int_G e^{-c\cdot \mathrm{d}(g,e)}\mathrm{d}\mu(g) < \infty.$$

Put $E_1 = B_1(e)$ and define inductively

$$E_n = B_n(e) \setminus B_{n-1}(e) \quad (n \geqslant 2).$$

Then G is the disjoint union of E_n for $n \in \mathbb{N}$ and

$$e^{-cn} \leqslant e^{-c\cdot \mathrm{d}(g,e)} \leqslant e^{-c(n-1)} \quad (g \in E_n).$$

Hence,

$$\begin{aligned}\int_G e^{-c\cdot \mathrm{d}(g,e)}\mathrm{d}\mu(g) &= \sum_{n=1}^{\infty}\int_{E_n} e^{-c\cdot \mathrm{d}(g,e)}\mathrm{d}\mu(g) \leqslant \sum_{n=1}^{\infty} e^{-c(n-1)}\mu(E_n)\\ &\leqslant e^c\sum_{n=1}^{\infty} e^{-cn}\mu\big(B_n(e)\big) \leqslant ae^c\sum_{n=1}^{\infty} e^{(b-c)n} < \infty\end{aligned}$$

because $c > b$.

By the reverse triangle inequality we see that

$$|\mathrm{d}(z,g^{-1})-\mathrm{d}(z,e)|\leqslant \mathrm{d}(e,g^{-1})\qquad (g,z\in G).$$

Using left invariance of the metric one finds that

$$|\mathrm{d}(gz,e)-\mathrm{d}(z,e)|\leqslant \mathrm{d}(g,e)\qquad (g,z\in G).$$

This implies

$$\frac{1}{c(g)}e^{-c\cdot \mathrm{d}(z,e)}\leqslant e^{-c\cdot \mathrm{d}(gz,e)}\leqslant c(g)e^{-c\cdot \mathrm{d}(z,e)}\qquad (g,z\in G),$$

which is easily seen to complete the proof. ■

LEMMA 2.2. *Assume that G is a second countable, locally compact group, that $\mathscr{H}$ is a Hilbert space, and $R: G\to \mathscr{H}$ is bounded and continuous. Let $R': G\to L^2(G,\mathscr{H},\mu)$ be given by*

$$R'(x)(z)=\sqrt{h(z)}R(z^{-1}x)\qquad (x,z\in G),$$

where $h\in L^{-1}(G)$ is chosen as in Lemma 2.1. Then R' is bounded and continuous, with $\|R'(x)\|_2\leqslant \|R\|_\infty$ for all $x\in G$. Also, let $\mathscr{K}_R=\overline{\mathrm{span}}\{R'(x)\ :\ x\in G\}$ be a sub-Hilbert space of $L^2(G,\mathscr{H},\mu)$. Then there exists a unique representation $(\pi_R,\mathscr{K}_R)$ such that

$$\pi_R(g)R'(x)=R'(gx)\qquad (g,x\in G).$$

Moreover,

$$\|\pi_R(g)\|\leqslant e^{\frac{c}{2}\cdot \mathrm{d}(g,e)}\qquad (g\in G)$$

and the representation is strongly continuous.

Proof. From Lebesgue's dominated convergence theorem it follows easily that R' is continuous. To see that R' is bounded, note that

$$\|R'(x)\|_2^2=\int_G h(z)\|R(z^{-1}x)\|^2\mathrm{d}\mu(z)\leqslant \|R\|_\infty^2\qquad (x\in G).$$

If $n\in\mathbb{N}$, $x_1,\dots,x_n\in G$, and $c_1,\dots,c_n\in\mathbb{C}$, then Lemma 2.1 implies that

$$\int_G\Big\|\sum_{i=1}^n c_iR(z^{-1}x_i)\Big\|^2h(gz)\mathrm{d}\mu(z)\leqslant c(g)\int_G\Big\|\sum_{i=1}^n c_iR(z^{-1}x_i)\Big\|^2h(z)\mathrm{d}\mu(z)$$

for $g\in G$, where

$$c(g)=e^{c\cdot \mathrm{d}(g,e)}\qquad (g\in G).$$

It follows that

$$\Big\|\sum_{i=1}^n c_iR'(gx_i)\Big\|_2^2\leqslant c(g)\Big\|\sum_{i=1}^n c_iR'(x_i)\Big\|_2^2\qquad (g\in G),$$

from which we conclude that there exists a unique representation $(\pi_R, \mathscr{K}_R)$ of G such that

$$\pi_R(g)R'(x) = R'(gx) \quad (g, x \in G).$$

Furthermore,

$$\|\pi_R(g)\| \leqslant \sqrt{c(g)} \quad (g \in G).$$

We proceed to show that the representation is strongly continuous. Since $\operatorname{span}\{R'(x) : x \in G\}$ is total in $\mathscr{K}_R$ and $\|\pi_R(g)\| \leqslant \sqrt{c(g)}$, where $g \mapsto \sqrt{c(g)}$ is a continuous function, it is enough to show that

$$\lim_{n\to\infty} \pi_R(g_n)R'(x) = R'(x)$$

for $x \in G$, when $(g_n)_{n\in\mathbb{N}}$ is a sequence converging to the identity $e \in G$ (since G is second countable, we do not have to consider nets). But $\pi_R(g_n)R'(x) = R'(g_n x)$, so this follows simply from continuity of R'. ∎

Proof of Theorem 2.1. Let us assume that φ is a continuous Hermitian Herz–Schur multiplier and use Proposition 1.1 to find a Hilbert space $\mathscr{H}$ and bounded, continuous maps $P, Q : G \to \mathscr{H}$ such that

$$\varphi(y^{-1}x) = \langle P(x), Q(y)\rangle \quad (x, y \in G)$$

and

$$\|P\|_\infty = \|Q\|_\infty = \|\varphi\|_{M_0A(G)}^{1/2}.$$

Define

$$a_\pm(x, y) = \frac{1}{4}\langle P(x) \pm Q(x), P(y) \pm Q(y)\rangle \quad (x, y \in G).$$

This gives rise to two positive definite, bounded kernels on $G \times G$ satisfying

$$a_+(x, y) - a_-(x, y) = \frac{1}{2}\varphi(y^{-1}x) + \frac{1}{2}\varphi^*(y^{-1}x) = \varphi(y^{-1}x) \quad (x, y \in G)$$

and

$$a_+(x, x) + a_-(x, x) = \frac{1}{2}\|P(x)\|^2 + \frac{1}{2}\|Q(x)\|^2 \leqslant \|\varphi\|_{M_0A(G)} \quad (x \in G).$$

Let

$$h(g) = \frac{e^{-c\cdot \mathrm{d}(g,e)}}{\int_G e^{-c\cdot \mathrm{d}(x,e)}\mathrm{d}\mu(x)} \quad (g \in G)$$

for some $c > b$, when d is a proper, left invariant metric on G satisfying (2.1) (cf. Lemma 2.1). Define $(P \pm Q)' : G \to L^2(G, \mathscr{H}, \mu)$ by

$$(P \pm Q)'(x)(z) = \sqrt{h(z)}(P \pm Q)(z^{-1}x) \quad (x, z \in G).$$

By Lemma 2.2 there exist strongly continuous representations $(\pi_{P\pm Q}, \mathscr{K}_{P\pm Q})$, where $\mathscr{K}_{P\pm Q} = \overline{\operatorname{span}}\{(P' \pm Q')(x) : x \in G\}$ and $\pi_{P\pm Q}(g)(P \pm Q)'(x) = (P \pm Q)'(gx)$ for $g, x \in G$. Furthermore, these representations satisfy

$$\|\pi_{P\pm Q}(g)\| \leqslant e^{\frac{c}{2}\cdot \mathrm{d}(g,e)} \quad (g \in G). \tag{2.2}$$

Put

$$A_\pm(x,y) = \langle (P \pm Q)'(x), (P \pm Q)'(y)\rangle_{\mathscr{K}_{P\pm Q}} \quad (x, y \in G).$$

Then $A_\pm$ are positive definite, bounded kernels on $G \times G$ satisfying

$$A_+(x,y) - A_-(x,y) = \varphi(y^{-1}x) \quad (x, y \in G) \tag{2.3}$$

and

$$A_+(x,x) + A_-(x,x) \leqslant \|\varphi\|_{M_0A(G)} \quad (x \in G). \tag{2.4}$$

To make the notation less cumbersome, let $\pi_\pm = \pi_{P\pm Q}$ and $\mathscr{K}_\pm = \mathscr{K}_{P\pm Q}$ and define $\xi_\pm = (P \pm Q)'(e)$. Notice that

$$\langle \pi_\pm(x)\xi_\pm, \pi_\pm(y)\xi_\pm\rangle_{\mathscr{K}_\pm} = A_\pm(x,y) \quad (x, y \in G),$$

and that (2.2) now reads

$$\|\pi_\pm(g)\| \leqslant e^{\frac{c}{2}\cdot \mathrm{d}(g,e)} \quad (g \in G).$$

Put

$$\mathscr{K} = \mathscr{K}_+ \oplus \mathscr{K}_-, \quad \xi = \xi_+ \oplus \xi_-, \quad \eta = \xi_+ \oplus -\xi_-, \quad \text{and} \quad \pi = \pi_+ \oplus \pi_-.$$

Observe that π is a strongly continuous representation such that

$$\|\pi(g)\| \leqslant e^{\frac{c}{2}\cdot \mathrm{d}(g,e)} \quad (g \in G) \tag{2.5}$$

and

$$\langle \pi(x)\xi, \pi(y)\eta\rangle_{\mathscr{K}} = \varphi(y^{-1}x) \quad (x, y \in G).$$

Finally, observe that

$$\|\pi(x)\xi\|^2 = \|\pi_+(x)\xi_+\|^2 + \|\pi_-(x)\xi_-\|^2 = A_+(x,x) + A_-(x,x) \leqslant \|\varphi\|_{M_0A(G)}$$

for $x \in G$, and similarly

$$\|\pi(y)\eta\|^2 \leqslant \|\varphi\|_{M_0A(G)}$$

for $y \in G$. This completes the proof. ∎

COROLLARY 2.1. *If φ is a continuous Herz–Schur multiplier on a second countable, locally compact group G, and d is a proper, left invariant metric on G satisfying* (2.1) *for some $a, b > 0$ (which exist according to* [5]*), then there exist a strongly continuous representation $(\pi, \mathscr{H})$ and vectors $\xi, \eta \in \mathscr{H}$ such that*

$$\varphi(y^{-1}x) = \langle \pi(x)\xi, \pi(y)\eta \rangle \quad (x, y \in G)$$

with

$$\sup_{x \in G} \|\pi(x)\xi\| \leqslant \sqrt{2}\|\varphi\|_{M_0A(G)}^{1/2} \quad \textit{and} \quad \sup_{y \in G} \|\pi(y)\eta\| \leqslant \sqrt{2}\|\varphi\|_{M_0A(G)}^{1/2}.$$

Moreover, for every fixed $c > b$, $(\pi, \mathscr{H})$ can be chosen such that

$$\|\pi(g)\| \leqslant e^{\frac{c}{2}\cdot \mathrm{d}(g,e)} \quad (g \in G).$$

P r o o f. This follows from Theorem 2.1 since

$$\varphi = \Re(\varphi) + i\Im(\varphi),$$

where

$$\Re(\varphi) = \frac{\varphi + \varphi^*}{2} \quad \text{and} \quad \Im(\varphi) = \frac{\varphi - \varphi^*}{2i}$$

are continuous Hermitian Herz–Schur multipliers with

$$\|\Re(\varphi)\|_{M_0A(G)} \leqslant \|\varphi\|_{M_0A(G)} \quad \text{and} \quad \|\Im(\varphi)\|_{M_0A(G)} \leqslant \|\varphi\|_{M_0A(G)}.$$

Thus the proof is complete. ∎

P r o o f o f T h e o r e m 1.1. Assume that φ is a continuous Herz–Schur multiplier and use Proposition 1.1 to find a Hilbert space $\mathscr{H}$ and bounded, continuous maps $P, Q : G \to \mathscr{H}$ such that

$$\varphi(y^{-1}x) = \langle P(x), Q(y) \rangle \quad (x, y \in G)$$

and

$$\|P\|_\infty = \|Q\|_\infty = \|\varphi\|_{M_0A(G)}^{1/2}.$$

Let

$$h(g) = \frac{e^{-c\cdot \mathrm{d}(g,e)}}{\int_G e^{-c\cdot \mathrm{d}(x,e)} \mathrm{d}\mu(x)} \quad (g \in G)$$

for some $c > b$, when d is a proper, left invariant metric on G satisfying (2.1) (cf. Lemma 2.1). Define $P', Q' : G \to L^2(G, \mathscr{H}, \mu)$ by

$$P'(x)(z) = \sqrt{h(z)}P(z^{-1}x) \quad \text{and} \quad Q'(y)(z) = \sqrt{h(z)}Q(z^{-1}y) \quad (z \in G)$$

for $x, y \in G$. According to Lemma 2.2 there exists a strongly continuous representation $(\pi_P, \mathscr{K}_P)$, where $\mathscr{K}_P = \overline{\operatorname{span}}\{P'(x) \,:\, x \in G\}$ and $\pi_P(g)P'(x) = P'(gx)$ for $g, x \in G$. Furthermore, this representation satisfies

$$\|\pi_P(g)\| \leqslant e^{\frac{c}{2}\cdot \mathrm{d}(g,e)} \qquad (g \in G).$$

Observe that

$$\|P'(x)\|_2^2, \|Q'(y)\|_2^2 \leqslant \|\varphi\|_{M_0A(G)}$$

and

$$\langle P'(x), Q'(y)\rangle_{L^2(G,\mathscr{H},\mu)} = \int_G h(z)\langle P(z^{-1}x), Q(z^{-1}y)\rangle_{\mathscr{H}} \mathrm{d}\mu(z) = \varphi(y^{-1}x)$$

for $x, y \in G$. Put $\xi = P'(e)$ and $\eta = P_{\mathscr{K}_P}Q'(e)$, where $P_{\mathscr{K}_P}$ is the orthogonal projection on $\mathscr{K}_P$. Note that $\xi, \eta \in \mathscr{K}_P$ and

$$\varphi(y^{-1}x) = \langle \pi_P(y^{-1}x)\xi, \eta\rangle_{\mathscr{K}_P} = \langle \pi_P(x)\xi, \pi_P(y^{-1})^*\eta\rangle_{\mathscr{K}_P} \qquad (x, y \in G).$$

It is clear that $\|\pi_P(x)\xi\|_{\mathscr{K}_P}^2 = \|P'(x)\|_2^2 \leqslant \|\varphi\|_{M_0A(G)}$. The corresponding result for $\|\pi_P(y^{-1})^*\eta\|_{\mathscr{K}_P}^2$ requires more work. For $x \in G$ arbitrary we find that

$$\begin{aligned}\langle \pi_P(y^{-1})P'(x), P_{\mathscr{K}_P}Q'(e)\rangle_{\mathscr{K}_P} &= \langle P'(y^{-1}x), P_{\mathscr{K}_P}Q'(e)\rangle_{\mathscr{K}_P} \\ &= \langle P'(y^{-1}x), Q'(e)\rangle_{\mathscr{H}} = \varphi(y^{-1}x) \\ &= \langle P'(x), Q'(y)\rangle_{\mathscr{H}} = \langle P'(x), P_{\mathscr{K}_P}Q'(y)\rangle_{\mathscr{K}_P}\end{aligned}$$

from which we conclude that $\pi_P(y^{-1})^*P_{\mathscr{K}_P}Q'(e) = P_{\mathscr{K}_P}Q'(y)$, and therefore

$$\|\pi_P(y^{-1})^*\eta\|_{\mathscr{K}_P}^2 = \|P_{\mathscr{K}_P}Q'(y)\|_{\mathscr{K}_P}^2 \leqslant \|Q'(y)\|_2^2 \leqslant \|\varphi\|_{M_0A(G)}.$$

Thus the proof is complete. ■

REMARK 2.1. *For the free group on N generators $(2 \leqslant N < \infty)$ the constants a, b in* (2.1) *may be chosen as*

$$a = \frac{N}{(N-1)(2N-1)} \qquad \textit{and} \qquad b = \ln(2N-1).$$

This implies that for $r > \sqrt{2N-1}$ the representations $(\pi, \mathscr{H})$ from Theorems 1.1 *and* 2.1 *and Corollary* 2.1 *may be chosen to satisfy $\|\pi(g)\| \leqslant r^{\mathrm{d}(g,e)}$ for all $g \in G$.*

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