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EXTREMES OF MOVING AVERAGES AND MOVING MAXIMA ON A REGULAR LATTICE*

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Abstract. We study the extremal behaviour of spatial moving averages and moving maxima on a regular discrete grid. Our main assumption is that these random fields are stationary and regularly varying with the tail index $\alpha > 0$. Using the asymptotic theory for point processes we characterise the limiting behaviour of their extremes over an increasing grid. Our approach builds on the results of Davis and Resnick concerning linear processes.

By analogy to the analysis of time series data, an appropriate Hill estimator of the tail index can be defined. We exhibit a sufficient condition for the consistency of this estimator in a certain class of spatial lattice models. Finally, we show that this condition holds for the models in our title.

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1. INTRODUCTION

Increasing interest in mathematical modelling of environmental catastrophes has caused an incursion of extreme value theory to spatial statistics. This led to a large number of research papers on spatial extremes. For an interesting recent review we refer to Davison et al. [6] and Turkman et al. [16].

The fact that spatial data are rarely stationary is a big obstacle for extreme value theory, since the stationarity assumption underlies most of its methods. It is not uncommon to model nonstationarity in space using covariates and then to use stationary models. One extensively studied class of models are (stationary) max-stable processes. Such an approach to modelling frequently assumes that data can be transformed to a common univariate marginal distribution, which is further typically assumed to be standard Fréchet or Gumbel (see [4] and [9] for examples of

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the former and latter) distribution. Data are nevertheless collected over an arbitrary discrete or continuous set of points in $\mathbb{R}^+$.

The aim of our paper, on the other hand, is to study asymptotic behaviour of extremes for two natural stationary models with a general marginal distribution in the maximum domain of attraction of Fréchet law with index α. These are moving average and moving maxima processes over a regular rectangular grid. More precisely, we consider an array $\{Z_{i,j} : i, j \in \mathbb{Z}\}$ of real-valued iid random variables which for some slowly varying function L and $\alpha > 0$ satisfy

$$P(|Z_{i,j}| > x) = x^{-\alpha} L(x), \tag{1.1}$$

and

$$\frac{P(Z_{i,j} > x)}{P(|Z_{i,j}| > x)} \to p \quad \text{and} \quad \frac{P(Z_{i,j} \leqslant -x)}{P(|Z_{i,j}| > x)} \to q \tag{1.2}$$

as $x \to \infty, 0 \leqslant p \leqslant 1, q = 1 - p$. Under mild conditions (see Section 2) on arrays of real numbers $\{c_{k,l} : k, l \in \mathbb{Z}\}$ and $\{\varphi_{k,l} : k, l \in \mathbb{Z}\}$ one can define the array of spatial moving averages

$$X_{i,j} := \sum_{k,l \in \mathbb{Z}} c_{k,l} Z_{i+k,j+l}, \quad (i, j) \in \mathbb{Z}^2,$$

and the spatial moving maxima

$$Y_{i,j} := \bigvee_{k,l \in \mathbb{Z}} \varphi_{k,l} Z_{i+k,j+l}, \quad (i, j) \in \mathbb{Z}^2.$$

Limiting distribution of maxima over a discrete grid has been studied earlier. The problem seems natural in areas like (satellite) image analysis and geostatistics where the regular grid data are often encountered. Building on results of Leadbetter et al. [11], Turkman [15] considers a rather general stationary random field over a regular grid (spatio-temporal even), imposing technical restrictions on local dependence in each spatial and temporal direction. He is able to calculate an extremal index for certain random fields of this kind, which he illustrates using a (unilateral) moving average process (which is also a special case of moving average processes studied here).

Our main results in Section 2 show that some classical results of Davis and Resnick [5] for univariate regularly varying moving average processes carry over to the spatial moving averages and moving maxima. More precisely, we prove a point process convergence result which will enable us to study the weak limit behaviour of extremes of $\{X_{i,j} : i, j \in \mathbb{Z}\}$ and $\{Y_{i,j} : i, j \in \mathbb{Z}\}$ over a finite square lattice. Moreover, we are able to determine extremal index, and in particular find asymptotic distribution for the scaled maxima in both models. Although we explore a two-dimensional square grid only, our main results extend with proper adjustments to an arbitrary dimension and grid.

Finally, since infinite order moving averages and moving maxima permit an approximation with simpler m-dependent processes, we are able to show that the Hill estimator of the tail index $\alpha > 0$ is consistent in either of two cases (cf. Resnick and Stărică [13], [14]). This gives hope that the tail index estimation, although notoriously difficult even in one dimension, can be performed in spatial setting using standard univariate routines and estimators.

2. ASYMPTOTICS FOR EXTREMES

2.1. Background results on point processes. We recall some basic notions about point processes (see Kallenberg [10] for details). For a state space $(E, \mathcal{E})$, where $\mathcal{E}$ is the σ-algebra generated by open sets, let $M_p(E)$ denote the space of Radon point measures on E, and $\mathcal{M}_p(E)$ be the smallest σ-algebra making the evaluation maps $m \mapsto m(F)$ measurable, where $m \in M_p(E)$, and $F \in \mathcal{E}$ is an arbitrary fixed set. We denote weak convergence of measures by $\Rightarrow$. If we define $C_K^+(E) := \{f : E \to \mathbb{R}_+ :\ f \text{ continuous with compact support}\}$, a sequence of measures $\mu_n \in M_p(E)$, $n > 0$, *converges vaguely* to μ_0 (written $\mu_n \xrightarrow{v} \mu_0$) if $\mu_n(f) \to \mu_0(f)$ for all $f \in C_K^+(E)$.

Recall that a point process on E is a measurable mapping from a probability space to $\big(M_p(E), \mathcal{M}_p(E)\big)$. A Poisson process on $(E, \mathcal{E})$ with mean measure μ (or Poisson random measure with mean measure μ, PRM(μ) for short) is a point process ξ satisfying, for all $F \in \mathcal{E}$,

$$P\big(\xi(F) = k\big) = \begin{cases} e^{-\mu(F)}\big(\mu(F)\big)^k/k! & \text{if } \mu(F) < \infty, \\ 0 & \text{if } \mu(F) = \infty, \end{cases}$$

and such that, for arbitrary mutually disjoint $F_1, \ldots, F_n \in \mathcal{E}, \xi(F_1), \ldots, \xi(F_n)$ are independent. Here, we assume μ is a Radon measure, i.e. a Borel measure that is finite on compact sets.

Let $\{Z_{i,j}\}$ be an iid array with regularly varying tail probabilities satisfying (1.1) and (1.2). Further, let $\{a_n\}$ be a sequence of positive constants such that

$$n^2 P(|Z_{0,0}| > a_n x) \to x^{-\alpha} \quad \text{for all } x > 0. \tag{2.1}$$

We can define a_n as $\inf\{x : P(|Z_{i,j}| > x) \leqslant n^{-2}\}$. Define the measure

$$\lambda(dx) = \alpha p x^{-\alpha-1}\mathbf{1}_{(0,\infty)}(x)dx + \alpha q(-x)^{-\alpha-1}\mathbf{1}_{(-\infty,0)}(x)dx$$

on the space $\mathbb{R} \setminus \{0\}$, and on the set $\mathbb{R}^2 \times \mathbb{R} \setminus \{0\}$ the measure

$$\lambda' = Leb \times Leb \times \lambda,$$

where Leb indicates the Lebesgue measure on $\mathbb{R}$. Under the assumptions (1.1) and (1.2), we have

$$n^2 P(a_n^{-1} Z_{i,j} \in \cdot) \xrightarrow{v} \lambda(\cdot).$$

Adapting the arguments on pp. 154–156 of Resnick [12], it is straightforward to prove the following.

PROPOSITION 2.1. *For each n suppose $\{Z_{n,i,j} : i,j \in \mathbb{Z}\}$ are iid random elements of $(E,\mathcal{E})$ and let λ be a Radon measure on $(E,\mathcal{E})$. Define*

$$\xi_n := \sum_{i,j\in\mathbb{Z}} \delta_{(i/n,j/n,Z_{n,i,j})}$$

and suppose ξ is PRM on $\mathbb{R}^2 \times E$ with mean measure $\lambda' = Leb \times Leb \times \lambda$. Then $\xi_n \Rightarrow \xi$ in $M_p(\mathbb{R}^2 \times E)$ if and only if

$$n^2 P(Z_{n,1,1} \in \cdot) \xrightarrow{v} \lambda(\cdot) \text{ on } E. \tag{2.2}$$

If we set $Z_{n,i,j} := a_n^{-1} Z_{i,j}$, it is easily checked that $\{Z_{n,i,j}\}$ and λ satisfy (2.2) from Proposition 2.1 on the state space $\mathbb{R} \setminus \{0\}$, so

$$\sum_{i,j\in\mathbb{Z}} \delta_{(i/n,j/n,a_n^{-1}Z_{i,j})} \Rightarrow \sum_h \delta_{(t_h^{(1)},t_h^{(2)},w_h)}, \tag{2.3}$$

where $t_h^{(1)}, t_h^{(2)}, w_h$ are such that the sum on the right is a Poisson random measure with mean measure $\lambda' = Leb \times Leb \times \lambda$ on $\mathbb{R}^2 \times (\mathbb{R} \setminus \{0\})$.

For a fixed $m \in \mathbb{N}$ set $Z_{i,j}^{(m)} := (Z_{i-m,j-m}, Z_{i-m+1,j-m}, \dots, Z_{i+m,j+m})$, $Z_{i,j}^{(m)} \in \mathbb{R}^{(2m+1)^2}$. Set

$$I_n = \sum_{i,j\in\mathbb{Z}} \delta_{(i/n,j/n,a_n^{-1}Z_{i,j}^{(m)})} \quad \text{and} \quad I = \sum_h \sum_{k=1}^{(2m+1)^2} \delta_{(t_h^{(1)},t_h^{(2)},w_h\mathbf{e}_k)}, \tag{2.4}$$

where $\mathbf{e}_k$ is the element of the canonical base of $\mathbb{R}^{(2m+1)^2}$ with k-th component equal to one and the rest equal to zero, and $t_h^{(1)}, t_h^{(2)}, w_h$ are defined as above. The state space for the processes $\{I_n\}$ and I is $E = \mathbb{R}^2 \times (\mathbb{R}^{(2m+1)^2} \setminus \{0\})$. The usual topology is modified so that the compact sets of $\mathbb{R}^{(2m+1)^2} \setminus \{0\}$ are those compact sets in $\mathbb{R}^{(2m+1)^2}$ which are bounded away from the origin, and $\mathcal{E}$ is the σ-algebra generated by the new topology.

Let S be the collection of all sets B of the form

$$B = (a_0, a_1] \times (a_2, a_3] \times (b_1, c_1] \times \dots \times (b_{(2m+1)^2}, c_{(2m+1)^2}],$$

where the $(2m+1)^2$-dimensional rectangle $(b_1, c_1] \times \dots \times (b_{(2m+1)^2}, c_{(2m+1)^2}]$ is bounded away from zero and $a_0 < a_1, a_2 < a_3, b_i < c_i, b_i \neq 0, c_i \neq 0$ for $i = 1, \dots, (2m+1)^2$. Let $Z_{i,j}^k$ denote the k-th component of the vector $Z_{i,j}^{(m)}$, i.e.

$$Z_{i,j}^k = Z^{(m)}_{(i-m,j-m)+((k-1) \pmod m),\lfloor (k-1)/(2m+1)\rfloor)}.$$

Here, for $a, b \in \mathbb{Z}$, $a \pmod b$ denotes the remainder in the integer division of a by b.

The following lemma is proved in Davis and Resnick [5] for the one-dimensional lattice, but the proof remains also valid in our setting.

LEMMA 2.1. *Let* $\widetilde{I}_n = \sum_{i,j\in\mathbb{Z}} \sum_{k=1}^{(2m+1)^2} \delta_{(i/n, j/n, a_n^{-1} Z_{i,j}^k \cdot \mathbf{e}_k)}$. *Then* $I_n(B) - \widetilde{I}_n(B) \to 0$ *in probability for all* $B \in S$.

Now we can prove the following result.

THEOREM 2.1. *Let* $\{Z_{i,j}\}$ *be iid satisfying* (1.1) *and* (1.2) *with* $\{a_n\}$ *satisfying* (2.1). *Then for each fixed positive integer* m

$$\sum_{i,j\in\mathbb{Z}} \delta_{(i/n, j/n, a_n^{-1} Z_{i,j}^{(m)})} \Rightarrow \sum_h \sum_{k=1}^{(2m+1)^2} \delta_{(t_h^{(1)}, t_h^{(2)}, w_h \mathbf{e}_k)}$$

in $M_p\big(\mathbb{R}^2 \times (\mathbb{R}^{(2m+1)^2} \setminus \{0\})\big)$ *as* $n \to \infty$, *where* $\mathbf{e}_k \in \mathbb{R}^{(2m+1)^2}$ *and* $t_h^{(1)}, t_h^{(2)}, w_h$ *are defined as in* (2.4).

Proof. The class S is a DC-semiring (see Kallenberg [10], Theorem 4.2), so it suffices to show $\big(I_n(B_1), \ldots, I_n(B_j)\big) \Rightarrow \big(I(B_1), \ldots, I(B_j)\big)$ for any $j \geqslant 1$ and sets $B_1, \ldots, B_j \in S$. However, by Lemma 2.1 it suffices to show

$$\big(\widetilde{I}_n(B_1), \ldots, \widetilde{I}_n(B_j)\big) \Rightarrow \big(I(B_1), \ldots, I(B_j)\big)$$

or, equivalently, $\widetilde{I}_n \Rightarrow I$.

Note that

$$\begin{aligned}\sum_{i,j\in\mathbb{Z}} \delta_{(u_i, u_j, v_{ij})} &\mapsto \Big(\sum_{i,j\in\mathbb{Z}} \delta_{(u_i, u_j, v_{ij}\cdot \mathbf{e}_1)}, \ldots, \sum_{i,j\in\mathbb{Z}} \delta_{(u_i, u_j, v_{ij}\cdot \mathbf{e}_{(2m+1)^2})}\Big) \\ &\mapsto \sum_{i,j\in\mathbb{Z}} \sum_{k=1}^{(2m+1)^2} \delta_{(u_i, u_j, v_{ij}\cdot \mathbf{e}_k)}\end{aligned}$$

is a continuous mapping, so by (2.3) and the continuous mapping theorem we obtain $\widetilde{I}_n \Rightarrow I$ as desired. ■

2.2. Spatial moving averages. Let $\{Z_{i,j}\}$ be iid satisfying (1.1) and (1.2). Define

$$X_{i,j} := \sum_{k,l\in\mathbb{Z}} c_{k,l} Z_{i+k, j+l}. \tag{2.5}$$

It can be shown (Cline [3], Theorem 2.1) that $\{X_{i,j} : i, j \in \mathbb{Z}\}$ are well defined if $\{c_{k,l}\}$ satisfy

$$\sum_{k,l\in\mathbb{Z}} |c_{k,l}|^\delta < \infty \quad \text{for some } \delta < \alpha,\ \delta \leqslant 1, \tag{2.6}$$

and in this case

$$\lim_{t\to\infty} \frac{P\big(\big|\sum_{k,l\in\mathbb{Z}} c_{k,l} Z_{k,l}\big| > t\big)}{P(|Z_{0,0}| > t)} = \sum_{k,l\in\mathbb{Z}} |c_{k,l}|^\alpha. \tag{2.7}$$

First we prove the following lemma.

LEMMA 2.2. *If* $\{c_{k,l}\}$ *satisfies* (2.6) *then for any* $\gamma > 0$

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\Big[a_n^{-1} \bigvee_{i,j=-n}^{n} \Big| \sum_{k,l=-m}^{m} c_{k,l} Z_{i+k,j+l} - X_{i,j}\Big| > \gamma\Big] = 0. \tag{2.8}$$

P r o o f. Note that

$$\begin{aligned} P\Big[a_n^{-1} \bigvee_{i,j=-n}^{n} \Big| \sum_{k,l=-m}^{m} c_{k,l} Z_{i+k,j+l} - X_{i,j}\Big| > \gamma\Big] \\ = P\Big[a_n^{-1} \bigvee_{i,j=-n}^{n} \Big| \sum_{|k|,|l|>m} c_{k,l} Z_{i+k,j+l}\Big| > \gamma\Big]. \end{aligned}$$

Since $\big\{\sum_{|k|,|l|>m} c_{k,l} Z_{i+k,j+l} : i,j \in \{-n,\dots,n\}\big\}$ is stationary, the expression above is bounded by

$$\begin{aligned} &(2n+1)^2 P\Big[a_n^{-1}\Big|\sum_{|k|,|l|>m} c_{k,l} Z_{k,l}\Big| > \gamma\Big] \\ &= \frac{(2n+1)^2}{n^2} \cdot \frac{P\big[a_n^{-1}\big|\sum_{|k|,|l|>m} c_{k,l} Z_{i+k,j+l}\big| > \gamma\big]}{P[|Z_{0,0}| > a_n\gamma]} \cdot n^2 P[|Z_{0,0}| > a_n\gamma]. \end{aligned}$$

By (2.1) and (2.7), as $n \to \infty$ this converges to $4\sum_{|k|,|l|>m} |c_{k,l}|^\alpha \gamma^{-\alpha}$, which converges to zero as $m \to \infty$. ■

We can now state and prove a convergence result for point processes based on $\{X_{i,j}\}$.

THEOREM 2.2. *Suppose that* $\{Z_{i,j}\}, \{a_n\}, \{c_{k,l}\}$ *satisfy* (1.1), (1.2), (2.1), *and* (2.6), *and* $\{X_{i,j}\}$ *is given by* (2.5). *Let* $\{(t_h^{(1)}, t_h^{(2)}, w_h)\}$ *be the points of PRM*(λ') *on* $\mathbb{R}^2 \times (\mathbb{R}\setminus\{0\})$. *Then*

$$\sum_{i,j\in\mathbb{Z}} \delta_{(i/n,j/n,a_n^{-1}X_{i,j})} \Rightarrow \sum_h \sum_{k,l\in\mathbb{Z}} \delta_{(t_h^{(1)},t_h^{(2)},w_h c_{k,l})}$$

in $M_p\big(\mathbb{R}^2 \times (\mathbb{R}\setminus\{0\})\big)$ *as* $n \to \infty$.

P r o o f. By Theorem 2.1 we have for any positive integer m

$$\sum_{i,j\in\mathbb{Z}} \delta_{(i/n,j/n,a_n^{-1}Z_{i,j}^{(m)})} \Rightarrow \sum_h \sum_{k=1}^{(2m+1)^2} \delta_{(t_h^{(1)},t_h^{(2)},w_h\mathbf{e}_k)}. \tag{2.9}$$

The mapping

$$(z_{i-m,j-m},\ldots,z_{i+m,j+m}) \mapsto \sum_{k,l=-m}^{m} c_{k,l} z_{i+k,j+l}$$

induces a continuous mapping from $M_p(E)$ to $M_p\big(\mathbb{R}^2\times(\mathbb{R}\setminus 0)\big)$, and so by the continuous mapping theorem we have

$$\sum_{i,j\in\mathbb{Z}} \delta_{(i/n,j/n,a_n^{-1}\sum_{k,l=-m}^{m} c_{k,l}Z_{i+k,j+l})} \Rightarrow \sum_h \sum_{k,l=-m}^{m} \delta_{(t_h^{(1)},t_h^{(2)},w_h c_{k,l})}.$$

As $m\to\infty$

$$\sum_h \sum_{k,l=-m}^{m} \delta_{(t_h^{(1)},t_h^{(2)},w_h c_{k,l})} \to \sum_h \sum_{k,l\in\mathbb{Z}} \delta_{(t_h^{(1)},t_h^{(2)},w_h c_{k,l})} \tag{2.10}$$

pointwise in the vague metric. Therefore, by Theorem 4.2 in Billingsley [1] it suffices to show for any $\eta>0$

$$\lim_{m\to\infty}\limsup_{n\to\infty} P\Big[\rho\Big(\sum_{i,j\in\mathbb{Z}} \delta_{(i/n,j/n,a_n^{-1}\sum_{k,l=-m}^{m} c_{k,l}Z_{i+k,j+l})}, \sum_{i,j\in\mathbb{Z}} \delta_{(i/n,j/n,a_n^{-1}X_{i,j})}\Big)>\eta\Big] = 0,$$

where ρ is the metric inducing the vague topology on $M_p(E)$. To accomplish this it is enough to prove (see Kallenberg [10]) that for all $f\in C_K^+\big(\mathbb{R}^2\times(\mathbb{R}\setminus 0)\big)$

$$\lim_{m\to\infty}\limsup_{n\to\infty} P\Bigg[\Bigg|\sum_{i,j\in\mathbb{Z}} f\Bigg(\frac{i}{n},\frac{j}{n},a_n^{-1}\sum_{k,l=-m}^{m} c_{k,l}Z_{i+k,j+l}\Bigg) - \sum_{i,j\in\mathbb{Z}} f\Bigg(\frac{i}{n},\frac{j}{n},a_n^{-1}X_{i,j}\Bigg)\Bigg| > \eta\Bigg] = 0. \tag{2.11}$$

Suppose the support of f is contained in $[a,b]\times[c,d]\times([-K+\gamma_0,-K^{-1}-\gamma_0]\cup[K^{-1}+\gamma_0,K-\gamma_0])$ for some K,γ_0, where $(K+K^{-1})/2>\gamma_0>0$, $0<a<b$, and $0<c<d$. Set

$$\omega(\gamma)=\sup\{|f(t_1,t_2,x)-f(t_1,t_2,y)| : x\cdot y>0, |x-y|\leqslant\gamma,\ t_1,t_2\in\mathbb{R}\}.$$

Since f has compact support, it is also uniformly continuous, and so $\omega(\gamma) \to 0$ as $\gamma \to 0$. Define

$$A_{m,n} := \big\{\omega : a_n^{-1} \bigvee_{i,j=-n}^{n} \big| \sum_{k,l=-m}^{m} c_{k,l} Z_{i+k,j+l}(\omega) - X_{i,j}(\omega)\big| \leqslant \gamma\big\},$$

$$B(K) := [-K, -K^{-1}] \cup [K^{-1}, K],$$

and let $\gamma < \min\{\gamma_0, K\}$. Denote the event on the left of (2.11) by $\Delta_{m,n}$. Then

$$\begin{aligned}\lim_{m\to\infty} \limsup_{n\to\infty} P(\Delta_{m,n}) &= \lim_{m\to\infty} \limsup_{n\to\infty} P[(\Delta_{m,n} \cap A_{m,n}) \cup (\Delta_{m,n} \cap A_{m,n}^c)] \\ &= \lim_{m\to\infty} \limsup_{n\to\infty} P(\Delta_{m,n} \cap A_{m,n}),\end{aligned}$$

since by (2.8) we have $\lim_{m\to\infty} \limsup_{n\to\infty} P(\Delta_{m,n} \cap A_{m,n}^c) = 0$. Note that on $A_{m,n}$ it follows that $a_n^{-1} \sum_{k,l=-m}^{m} c_{k,l} Z_{i+k,j+l} \in B(K)$ implies

$$\left| f\left(\frac{i}{n}, \frac{j}{n}, a_n^{-1} \sum_{k,l=-m}^{m} c_{k,l} Z_{i+k,j+l}\right) - f\left(\frac{i}{n}, \frac{j}{n}, a_n^{-1} X_{i,j}\right) \right| \leqslant \omega(\gamma),$$

and $a_n^{-1} \sum_{k,l=-m}^{m} c_{k,l} Z_{i+k,j+l} \notin B(K)$ implies

$$f\left(\frac{i}{n}, \frac{j}{n}, a_n^{-1} \sum_{k,l=-m}^{m} c_{k,l} Z_{i+k,j+l}\right) = f\left(\frac{i}{n}, \frac{j}{n}, a_n^{-1} X_{i,j}\right) = 0.$$

Therefore,

$$\begin{aligned}&\lim_{m\to\infty} \limsup_{n\to\infty} P(\Delta_{m,n}) = \lim_{m\to\infty} \limsup_{n\to\infty} P[\Delta_{m,n} \cap A_{m,n} \cap B(K)] \\ &\leqslant \lim_{m\to\infty} \limsup_{n\to\infty} P\big[\omega(\gamma) \\ &\quad \times \sum_{i,j\in\mathbb{Z}} \delta_{(i/n, j/n, a_n^{-1} \sum_{k,l=-m}^{m} c_{k,l} Z_{i+k,j+l})}\big([a,b] \times [c,d] \times B(K)\big) > \eta\big] \\ &= \lim_{m\to\infty} P\big[\omega(\gamma) \sum_{k,l=-m}^{m} \sum_{h} \delta_{(t_h^{(1)}, t_h^{(2)}, w_h c_{k,l})}\big([a,b] \times [c,d] \times B(K)\big) > \eta\big] \\ &= P\big[\omega(\gamma) \sum_{k,l\in\mathbb{Z}} \sum_{h} \delta_{(t_h^{(1)}, t_h^{(2)}, w_h c_{k,l})}\big([a,b] \times [c,d] \times B(K)\big) > \eta\big],\end{aligned}$$

where the last equality is given by Theorem 2.1 and (2.10). Since

$$\sum_{h} \sum_{k,l\in\mathbb{Z}} \delta_{(t_h^{(1)}, t_h^{(2)}, w_h c_{k,l})}\big([a,b] \times [c,d] \times B(K)\big) < \infty \text{ a.s.},$$

the proof is concluded by letting $\gamma \to 0$. ■

From Theorem 2.2 we obtain the limiting distribution for maxima of $\{X_{i,j}\}$. Let us put

$$M_n = \max_{0\leqslant i,j\leqslant n} X_{i,j}, \quad c_+ = \max_{k,l}\{c_{k,l}, 0\}, \quad c_- = \max_{k,l}\{-c_{k,l}, 0\}.$$

Using Theorem 2.2 we have

$$\begin{aligned}
&P(a_n^{-1}M_n \leqslant x) = P\big[\sum_{i,j=0}^{n} \delta_{(i/n,j/n,a_n^{-1}X_{i,j})}\big([0,1]\times[0,1]\times(x,\infty)\big) = 0\big]\\
&\to P\big[\sum_h \sum_{k,l\in\mathbb{Z}} \delta_{(t_h^{(1)},t_h^{(2)},w_h c_{k,l})}\big([0,1]\times[0,1]\times(x,\infty)\big) = 0\big]\\
&= P\big[\sum_h \delta_{w_h}\big((x/c_+,\infty)\big) = 0, \sum_h \delta_{w_h}\big((-\infty,-x/c_-)\big) = 0\big]\\
&= \exp\{-p(x/c_+)^{-\alpha}\}\cdot\exp\{-q(x/c_-)^{-\alpha}\} = \exp\{-(pc_+^\alpha + qc_-^\alpha)x^{-\alpha}\}.
\end{aligned}$$

We can summarize this in the following result.

COROLLARY 2.1. *Under the conditions above, the partial maxima of moving average process* $\{X_{i,j}\}$ *satisfy*

$$P(a_n^{-1}M_n \leqslant x) \to \exp\{-(pc_+^\alpha + qc_-^\alpha)x^{-\alpha}\} \quad \textit{as } n\to\infty$$

for all $x > 0$.

Observe that the extremal index for arrays of random variables can be defined as in the case of stationary random sequences (see, e.g., Leadbetter et al. [11]; see also Ferreira and Pereira [7], and Jakubowski and Soja-Kukiela [8]). In particular, let $\{\widetilde{X}_{i,j}\}$ be iid copies of $\{X_{i,j}\}$ and $\widetilde{M}_n = \max_{0\leqslant i,j\leqslant n}\widetilde{X}_{i,j}$. If we assume $c_{k,l}\geqslant 0$ and $Z_{i,j}\geqslant 0$ (therefore $p=1$), from (2.1) and (2.7) we can calculate that

$$P(a_n^{-1}\widetilde{M}_n \leqslant x) \to \exp\big\{-\sum_{k,l\in\mathbb{Z}} |c_{k,l}|^\alpha x^{-\alpha}\big\}.$$

Therefore, from Corollary 2.1 it follows that the extremal index of the array $\{X_{i,j}\}$ equals

$$\frac{c_+^\alpha}{\sum_{k,l\in\mathbb{Z}}|c_{k,l}|^\alpha}.$$

2.3. Spatial moving maxima. Again, let $\{Z_{i,j}\}$ be iid satisfying (1.1) and (1.2). Define

$$Y_{i,j} := \bigvee_{k,l\in\mathbb{Z}} \varphi_{k,l} Z_{i+k,j+l}. \tag{2.12}$$

If $\{\varphi_{k,l}\}$ satisfy

$$\sum_{k,l\in\mathbb{Z}} |\varphi_{k,l}|^\delta < \infty \quad \text{for some } \delta<\alpha,\ \delta\leqslant 1, \tag{2.13}$$

then, as above, the sum $\sum_{k,l\in\mathbb{Z}} \varphi_{k,l} Z_{k,l}$ is finite a.s., and obviously we have $P(Y_{i,j} < \infty) = 1$. In this case, one can also show that $(Y_{i,j})$ is a stationary random field and (see Cline [3], Theorem 2.3) has the following regular variation property:

$$\lim_{t\to\infty} \frac{P\big(\big|\bigvee_{k,l\in\mathbb{Z}} \varphi_{k,l} Z_{k,l}\big| > t\big)}{P(|Z_{0,0}| > t)} = \sum_{k,l\in\mathbb{Z}} |\varphi_{k,l}|^{\alpha}. \tag{2.14}$$

As in Subsection 2.2, one can show the following lemma.

LEMMA 2.3. *If* $\{\varphi_{k,l}\}$ *satisfies* (2.13) *then for any* $\gamma > 0$

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\Big[a_n^{-1} \bigvee_{i,j=-n}^{n} \Big| \bigvee_{k,l=-m}^{m} \varphi_{k,l} Z_{i+k,j+l} - Y_{i,j}\Big| > \gamma\Big] = 0. \tag{2.15}$$

For fixed $p, q \in \mathbb{N}$ we define random vectors

$$Y_{i,j}^{(p,q)} := (Y_{i-p,j-q}, Y_{i-p+1,j-q}, \ldots, Y_{i+p,j+q-1}, Y_{i+p,j+q}), \quad i, j \in \mathbb{Z},$$

and analogously we define a vector of real numbers $\varphi_{i,j}^{(p,q)}$, $i, j \in \mathbb{Z}$.

Finally, we can state a convergence result for point processes based on $\{Y_{i,j}\}$.

THEOREM 2.3. *Suppose that* $\{Z_{i,j}\}, \{a_n\}, \{\varphi_{k,l}\}$ *satisfy* (1.1), (1.2), (2.1), *and* (2.13), *and* $\{Y_{i,j}\}$ *is given by* (2.12). *Let* $\{(t_h^{(1)}, t_h^{(2)}, w_h)\}$ *be the points of PRM*(λ') *on* $\mathbb{R}^2 \times (\mathbb{R} \setminus \{0\})$. *Then*

$$\sum_{i,j\in\mathbb{Z}} \delta_{(i/n, j/n, a_n^{-1} Y_{i,j})} \Rightarrow \bigvee_{k,l\in\mathbb{Z}} \sum_{h} \delta_{(t_h^{(1)}, t_h^{(2)}, w_h \varphi_{k,l})}$$

in $M_p\big(\mathbb{R}^2 \times (\mathbb{R} \setminus \{0\})\big)$ *as* $n \to \infty$.

P r o o f. The proof is analogous to the proof of Theorem 2.2, we use only a different continuous mapping from $M_p(E)$ to $M_p\big(\mathbb{R}^2 \times (\mathbb{R} \setminus 0)\big)$, namely

$$(z_{i-m,j-m}, \ldots, z_{i+m,j+m}) \mapsto \bigvee_{k,l=-m}^{m} \varphi_{k,l} z_{i+k,j+l},$$

and then apply the same argument but using Lemma 2.3 instead of Lemma 2.2. ■

Finally, if we put $M'_n = \max_{0\leqslant i,j\leqslant n} Y_{i,j}$, and $\varphi_+ = \max_{k,l}\{\varphi_{k,l}, 0\}$, $\varphi_- = \max_{k,l}\{-\varphi_{k,l}, 0\}$, we have

COROLLARY 2.2. *Under the conditions above, the partial maxima of a moving maxima process* $\{Y_{i,j}\}$ *satisfy*

$$P(a_n^{-1} M'_n \leqslant y) \to \exp\{-(p\varphi_+^{\alpha} + q\varphi_-^{\alpha}) y^{-\alpha}\} \quad \text{as } n \to \infty$$

for all $y > 0$.

Therefore, from (2.14) and Corollary 2.2, if $\varphi_{k,l} \geqslant 0$ and $Z_{i,j} \geqslant 0$, the extremal index for the array $\{Y_{i,j}\}$ equals

$$\frac{\varphi_+^\alpha}{\sum_{k,l\in\mathbb{Z}}|\varphi_{k,l}|^\alpha}.$$

3. CONSISTENCY OF THE HILL ESTIMATOR

Let $\{W_{i,j}\}$ be an arbitrary array of (possibly dependent) identically distributed regularly varying random variables with tail index α indexed over square lattice $\mathbb{Z}^2$. Let $E := \langle 0, \infty]$ be the one-point uncompactification of $[0, \infty]$ so that the compact sets of E are of the form U^c, where U is an open set in $[0, \infty\rangle$ containing zero. Suppose $\mathcal{E}$ is the Borel σ-field on E. Define the measure μ on $(E, \mathcal{E})$ by

$$\mu(\langle x, \infty]) = x^{-\alpha}, \qquad x > 0. \tag{3.1}$$

Let $M_+(E)$ and $C_K^+(E)$ be spaces defined as above. Let (k_n) be an increasing sequence of positive integers such that $k_n \to \infty$ and $n/k_n \to \infty$ as $n \to \infty$. For $\{W_{i,j}\}$, we define

$$b(t) := \inf\{x : P(|W_{0,0}| > x) \leqslant t^{-1}\}.$$

It is shown in Resnick and Stărică [14] that the condition

$$\mu_n := \frac{1}{k_n^2} \sum_{i,j=-n}^{n} \delta_{|W_{i,j}|/b_n} \xrightarrow{P} \mu \tag{3.2}$$

is sufficient for the consistency of the Hill estimator. In other words, if we denote the k-th largest order statistics of the sample $\{|W_{i,j}|\colon i, j = -n, \ldots, n\}$ by $|W|_{(k)}$, the convergence of measures in (3.2) implies that the Hill estimator H_n, defined by

$$H_n := \frac{1}{k_n^2} \sum_{i=1}^{k_n^2} \log|W|_{(i)} - \log|W|_{(k_n^2+1)},$$

converges in probability to α^{-1}.

3.1. Sufficient condition for consistency. We will find a useful way of proving condition (3.2) for different types of arrays.

PROPOSITION 3.1. *Suppose for each $n \in \mathbb{N}$ that $\{X_{n,i,j}\colon i, j \in \mathbb{Z}\}$ is a stationary array of random elements of E. Let $\{k_n\}$ be a sequence such that $k_n \to \infty$ and $n/k_n \to \infty$. Suppose that*

$$\frac{(2n)^2}{k_n^2} P(X_{n,1,1} \in \cdot) \xrightarrow{v} \mu, \tag{3.3}$$

where $\mu(\{x\}) = 0$ *for all* $x \in (0, \infty)$. *Assume further that* $\{X_{n,i,j}\}$ *satisfies the following two conditions*:

(i) *For any sequence* l_n *such that* $l_n \to \infty$ *and* $l_n/k_n \to 0$ *and squares*

$$I_{a,b} = [(a-1)k_n + 1, ak_n - l_n] \times [(b-1)k_n + 1, bk_n - l_n],$$

$a, b = -\lfloor n/k_n \rfloor + 1, \ldots, -1, 0, 1, \ldots, \lfloor n/k_n \rfloor$, *we have*, *for* $f \in C_K^+(E)$,

$$\lim_{n\to\infty} E\Bigg(\prod_{a,b=-\lfloor n/k_n \rfloor+1}^{\lfloor n/k_n \rfloor} \exp\Bigg\{ \frac{-1}{k_n^2} \sum_{i,j\in I_{a,b}} f(X_{n,i,j}) \Bigg\} \Bigg) - \\ - \prod_{a,b=-\lfloor n/k_n \rfloor+1}^{\lfloor n/k_n \rfloor} E \exp\Bigg\{ \frac{-1}{k_n^2} \sum_{i,j\in I_{a,b}} f(X_{n,i,j}) \Bigg\} = 0. \tag{3.4}$$

(ii) *Define* $I' := [1, k_n] \times [1, k_n] \setminus \{(1,1)\}$. *For any* $f \in C_K^+(E)$

$$\lim_{n\to\infty} \frac{n^2}{k_n^4} \sum_{(i,j)\in I'} E\big(f(X_{n,1,1}) f(X_{n,i,j})\big) = 0. \tag{3.5}$$

Then

$$\mu_n := \frac{1}{k_n^2} \sum_{i,j=-n}^{n} \delta_{X_{n,i,j}} \Rightarrow \mu \text{ in } M_+(E). \tag{3.6}$$

REMARK 3.1. *For* $f \in C_K^+(E)$ *assume that* $[c, \infty]$ *contains the support of* f *and set* $\|f\| = \sup_E f(x)$. *Then* $f \leqslant \|f\| \mathbf{1}_{[c,\infty]}$ *and*

$$E\big(f(X_{n,1,1}) f(X_{n,i,j})\big) \leqslant \|f\|^2 P(X_{n,1,1} > c, X_{n,i,j} > c).$$

Therefore, *condition* (3.5) *is implied by*

$$\lim_{n\to\infty} \frac{n^2}{k_n^4} \sum_{(i,j)\in I'} P(X_{n,0,0} > x, X_{n,i,j} > x) = 0 \quad \text{for all } x > 0. \tag{3.7}$$

Proof. For simplicity, note that k_n in (3.3) can be chosen so that n/k_n is an integer and let us put

$$p = p_n = \lfloor n/k_n \rfloor = \frac{n}{k_n} \quad \text{and} \quad N_{n,a,b} = N_{a,b} = \sum_{(i,j)\in I_{a,b}} \delta_{X_{n,i,j}}.$$

Next, we consider random measures

$$m_n = \frac{1}{k^2} \sum_{a,b=-p+1}^{p} \widetilde{N}_{a,b},$$

where $\widetilde{N}_{a,b}$ are iid copies of $N_{a,b}$ introduced above. From (3.4) it follows that

$$m_n \Rightarrow \mu \quad \text{if and only if} \quad \mu_n \Rightarrow \mu \text{ in } M_+(E). \tag{3.8}$$

Hence it suffices to prove the former statement. Let us consider the family $\mathcal{B}$ of relatively compact sets on $\mathcal{E}$. Notice that

$$E\big(m_n(B)\big) = (2p)^2 \frac{(k_n - l_n)^2}{k_n^2} P(X_{n,1,1} \in B),$$

so from (3.3) it follows that $m_n(B) \to 0$ for all $B \in \mathcal{B}$ such that $\mu(B) = 0$. For $\mu(B) > 0$ define

$$\begin{aligned}
T_n(B) &= \frac{m_n(B)}{\big((2n)/k_n\big)^2 P(X_{n,1,1} \in B)} - 1 \\
&= \left(\frac{k_n}{2n}\right)^2 \sum_{a,b=-p+1}^{p} \left(\frac{\widetilde{N}_{a,b}}{k_n^2 P(X_{n,1,1} \in B)} - 1\right).
\end{aligned}$$

Note that $T_n(B)$ is a sum of iid terms whose expectation tends to zero as n tends to infinity, since

$$\frac{E\big(\widetilde{N}_{a,b}(B)\big)}{k_n^2 P(X_{n,1,1} \in B)} = \frac{(k_n - l_n)^2}{k_n^2} \to 1.$$

Therefore, it suffices to prove that $\operatorname{Var}\big(T_n(B)\big) \to 0$ as $n \to \infty$ to conclude that $T_n(B) \xrightarrow{P} 0$, which will prove the claim. We have

$$\begin{aligned}
&\operatorname{Var}\big(T_n(B)\big) \\
&= \operatorname{Var}\left(\sum_{a,b=-p+1}^{p} \left(\frac{1}{(2n)^2} \frac{\widetilde{N}_{a,b}}{P(X_{n,1,1} \in B)} - \frac{k_n^2}{(2n)^2}\right)\right) \\
&= \frac{(2p)^2}{(2n)^4 P(X_{n,1,1} \in B)^2} E\big(\widetilde{N}_{1,1}(B) - (k_n - l_n)^2 P(X_{n,1,1} \in B)\big)^2 \\
&= \frac{(2p)^2}{(2n)^4 P(X_{n,1,1} \in B)^2} E\Big(\sum_{(i,j) \in I_{1,1}} \big(\delta_{X_{n,i,j}}(B) - P(X_{n,i,j} \in B)\big)\Big)^2 \\
&= \frac{(2p)^2}{(2n)^4 P(X_{n,1,1} \in B)^2} \Big[\sum_{(i,j) \in I_{1,1}} E\big(\delta_{X_{n,i,j}}(B) - P(X_{n,i,j} \in B)\big)^2 \\
&\quad + \sum_{(i,j) \neq (k,l) \in I_{1,1}} E\big(\delta_{X_{n,i,j}}(B) - P(X_{n,i,j} \in B)\big)\big(\delta_{X_{n,k,l}}(B) - P(X_{n,k,l} \in B)\big)\Big] \\
&= \frac{(2p)^2}{(2n)^4 P(X_{n,1,1} \in B)^2} \big[(k_n - l_n)^2 \operatorname{Var}\big(\delta_{X_{n,1,1}}(B)\big) \\
&\quad + \sum_{(i,j) \neq (k,l) \in I_{1,1}} E\big(\delta_{X_{n,i,j}}(B) \delta_{X_{n,k,l}}(B)\big) \\
&\quad + (k_n - l_n)^2 [(k_n - l_n)^2 - 1] P(X_{n,1,1} \in B)^2\big] \\
&=: I_{n,1} + I_{n,2} + I_{n,3}.
\end{aligned}$$

We consider each of the three terms separately. We have

$$\begin{aligned}
I_{n,1} &= \frac{(2p)^2}{(2n)^4 P(X_{n,1,1}\in B)^2}(k_n-l_n)^2 \operatorname{Var}\big(\delta_{X,n,1,1}(B)\big)\\
&= \frac{1}{\big((2n)/k_n\big)^4 P(X_{n,1,1}\in B)^2}\frac{(k_n-l_n)^2}{k_n^2}\frac{(2n)^2}{k_n^4}\operatorname{Var}\big(\delta_{X_{n,1,1}}(B)\big)\to 0
\end{aligned}$$

as $n\to\infty$ because k_n can be chosen in such a way that $n/k_n^2\to 0$ as $n\to\infty$. Next,

$$\begin{aligned}
I_{3,n} &= \frac{(2p)^2}{(2n)^4 P(X_{n,1,1}\in B)^2}(k_n-l_n)^2[(k_n-l_n)^2-1]P(X_{n,1,1}\in B)^2\\
&= \frac{(2n)^2}{k_n^2(2n)^4}(k_n-l_n)^2[(k_n-l_n)^2-1]\leqslant \frac{k_n^2}{(2n)^2}\to 0.
\end{aligned}$$

Note that $B\in\mathcal{B}$ implies that B is bounded away from zero, i.e. there exists $\delta>0$ such that $B\subset[\delta,\infty]$. Define

$$C := \frac{(2p)^2}{(2n)^4 P(X_{n,1,1}\in B)^2}.$$

We have

$$\begin{aligned}
&I_{2,n} =\\
&= C\sum_{(i,j)\neq(k,l)\in I_{1,1}} E\big(\delta_{X_{n,i,j}}(B)\delta_{X_{n,k,l}}(B)\big)\\
&\leqslant C\sum_{(i,j)\neq(k,l)\in I_{1,1}} P(X_{n,i,j}>\delta, X_{n,k,l}>\delta)\\
&= C\sum_{(i,j)\neq(1,1)\in I_{1,1}} (k_n-l_n-i+1)(k_n-l_n-j+1)P(X_{n,1,1}>\delta, X_{n,i,j}>\delta)\\
&\leqslant \frac{1}{\mu(B)^2}\frac{(2p)^2}{k_n^4}\sum_{(i,j)\in I'}(k_n-l_n-i+1)(k_n-l_n-j+1)P(X_{n,1,1}>\delta, X_{n,i,j}>\delta)\\
&\leqslant \frac{4n^2k_n^2}{\mu(B)^2k_n^6}\sum_{(i,j)\in I'}P(X_{n,1,1}>\delta, X_{n,i,j}>\delta),
\end{aligned}$$

which goes to zero by (3.5) and (3.7). Thus we can conclude that $T_n(B)\xrightarrow{P}0$ and $m_n\Rightarrow\mu$, which completes the proof. ∎

We shall say that a stationary array $\{W_{i,j}\}$ of random elements is *pairwise m-dependent* if for all $i,j,k,l\in\mathbb{Z}$, $W_{i,j}$ and $W_{i+k,j+l}$ are independent whenever $|k|,|l|>m$. The following proposition shows that if an array can be approximated by an m-dependent array then it also satisfies (3.6), which makes the Hill estimator consistent in this case as well (cf. Resnick and Stărică [14]).

PROPOSITION 3.2. *Suppose for each* $n \geqslant 1, m \geqslant 1$, $\{X^{(m)}_{n,i,j} : i, j \in \mathbb{Z}\}$ *is a stationary array of* m*-dependent random elements of* E, *and* $\{X_{n,i,j} : i, j \in \mathbb{Z}\}$ *is a stationary array of random elements of* E *for each* $n \geqslant 1$. *Suppose there exist Radon measures* $\mu^{(m)}$ *on* E *and a sequence* $\{k_n\}$, $k_n \to \infty$, $n/k_n \to \infty$ *such that for any fixed* m

$$\frac{(2n)^2}{k_n^2} P(X^{(m)}_{n,1,1} \in \cdot) \xrightarrow{v} \mu^{(m)}(\cdot) \quad \text{as } n \to \infty. \tag{3.9}$$

Suppose further that $\mu^{(m)} \xrightarrow{v} \mu$ *as* $m \to \infty$. *Finally, assume that*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{n^2}{k_n^2} P(|X^{(m)}_{n,1,1} - X_{n,1,1}| > \varepsilon) = 0 \quad \text{for all } \varepsilon > 0. \tag{3.10}$$

Then

$$\frac{1}{k_n^2} \sum_{i,j=-n}^{n} \delta_{X_{n,i,j}} \Rightarrow \mu \text{ in } M_+(E). \tag{3.11}$$

P r o o f. First we apply Proposition 3.1 to $\{X^{(m)}_{n,i,j}\}$. As in the proof of Proposition 3.1, we put $p = p_n = \lfloor n/k_n \rfloor$. Condition (3.3) is given, and condition (3.4) holds trivially since for $l_n > m$

$$\Big\{ \sum_{i,j\in I_{a,b}} f(X^{(m)}_{n,i,j}), a, b = -p+1, \ldots, p \Big\}$$

are independent.

For simplicity, we put

$$p_{i,j}(x, y) = P(X^{(m)}_{n,1,1} > x, X^{(m)}_{n,i,j} > y) \quad \text{and} \quad I'_m = [1, m] \times [1, m] \setminus \{(1, 1)\}.$$

To check the condition (3.5) we apply (3.7). Note that

$$\begin{aligned}
&\frac{n^2}{k_n^4} \sum_{i,j\in I'} p_{i,j}(x, y) \leqslant \\
&\leqslant \frac{n^2}{k_n^4} \Big(\sum_{(i,j)\in I'_m} p_{i,j}(x, y) + \sum_{(i,j)\in I\setminus I'_m} p_{i,j}(x, y) \Big) \\
&= \frac{n^2}{k_n^4} \Big(\sum_{(i,j)\in I'_m} p_{i,j}(x, y) + \sum_{(i,j)\in I\setminus I'_m} P(X^{(m)}_{n,1,1} > x) P(X^{(m)}_{n,i,j} > y) \Big) \\
&\leqslant \frac{n^2}{k_n^4} \sum_{(i,j)\in I'_m} P(X^{(m)}_{n,i,j} > y) + \frac{n^2}{k_n^4} (k_n - m)^2 P(X^{(m)}_{n,1,1} > x) P(X^{(m)}_{n,1,1} > y) \\
&\leqslant \frac{m^2}{4k_n^2} \frac{(2n)^2}{k_n^2} P(X^{(m)}_{n,i,j} > y) + \frac{k_n^2}{16n^2} \frac{(2n)^2}{k_n^2} P(X^{(m)}_{n,1,1} > x) \frac{(2n)^2}{k_n^2} P(X^{(m)}_{n,1,1} > y) \\
&= \frac{m^2}{4k_n^2} \big(\mu^{(m)}(\langle y, \infty\rangle) + o(1) \big) + \frac{k_n^2}{16n^2} \big(\mu^{(m)}(\langle y, \infty\rangle) + o(1) \big)^2.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \frac{n^2}{k_n^4} \sum_{(i,j)\in I'} P(X^{(m)}_{n,1,1} > x, X^{(m)}_{n,i,j} > y) \to 0.$$

By Proposition 3.1 we can conclude

$$\frac{1}{k_n^2} \sum_{i,j=-n}^{n} \delta_{X^{(m)}_{n,i,j}} \Rightarrow \mu^{(m)} \text{ in } M_+(E). \tag{3.12}$$

To complete the proof we will again use the converging together argument. It suffices to show that for any $f \in C_K^+(E)$

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left(\left| \frac{1}{k_n^2} \sum_{i,j=-n}^{n} f(X^m_{n,i,j}) - \frac{1}{k_n^2} \sum_{i,j=-n}^{n} f(X_{n,i,j}) \right| > \varepsilon \right) = 0. \tag{3.13}$$

Since f is continuous with compact support, it is uniformly continuous, and therefore

$$\omega_f(\theta) := \sup_{|x-y|\leqslant\theta, x,y\in E} |f(x) - f(y)| \to 0 \quad \text{as } \theta \to 0.$$

Suppose that the support of f is contained in $[a, \infty]$ for some $a > 0$. Let $\delta < a/2$ and define the following sets:

$$\begin{aligned} A_{i,j} &= \{|X^{(m)}_{n,i,j} - X_{n,i,j}| \leqslant \delta, X^{(m)}_{n,i,j} \geqslant a - \delta\}, \\ B_{i,j} &= \{|X^{(m)}_{n,i,j} - X_{n,i,j}| \leqslant \delta, X^{(m)}_{n,i,j} < a - \delta\}, \\ C_{i,j} &= \{|X^{(m)}_{n,i,j} - X_{n,i,j}| > \delta\} \end{aligned}$$

for $i, j = -n, \ldots, n$. The observed probability is now

$$\begin{aligned} P&\left(\left| \frac{1}{k_n^2} \sum_{i,j=-n}^{n} f(X^{(m)}_{n,i,j}) - \frac{1}{k_n^2} \sum_{i,j=-n}^{n} f\left(X_{n,i,j}\right) \right| > \varepsilon \right) \\ &\leqslant P\left(\frac{1}{k_n^2} \sum_{i,j=-n}^{n} |f(X^{(m)}_{n,i,j}) - f\left(X_{n,i,j}\right)| \mathbf{1}_{A_{i,j}} > \varepsilon/3 \right) \\ &\quad + P\left(\frac{1}{k_n^2} \sum_{i,j=-n}^{n} |f(X^{(m)}_{n,i,j}) - f\left(X_{n,i,j}\right)| \mathbf{1}_{B_{i,j}} > \varepsilon/3 \right) \\ &\quad + P\left(\frac{1}{k_n^2} \sum_{i,j=-n}^{n} |f(X^{(m)}_{n,i,j}) - f\left(X_{n,i,j}\right)| \mathbf{1}_{C_{i,j}} > \varepsilon/3 \right) \\ &\leqslant P\left(\omega_f(\delta) \frac{1}{k_n^2} \sum_{i,j=-n}^{n} \delta_{X^{(m)}_{n,i,j}}([a-\delta, \infty]) > \varepsilon/3 \right) + 0 \\ &\quad + \frac{3(2n+1)^2}{k_n^2 \varepsilon} E\big(|f(X^{(m)}_{n,i,j}) - f(X_{n,i,j})| \mathbf{1}_{C_{i,j}} \big), \end{aligned}$$

where the second term is zero because $X^{(m)}_{n,i,j}$ and $X_{n,i,j}$ from $B_{i,j}$ are not in the support of f. From (3.12) we have

$$\frac{1}{k_n^2}\sum_{i,j=-n}^{n}\delta_{X^{(m)}_{n,i,j}}([a-\delta,\infty]) \Rightarrow \mu^{(m)}([a-\delta,\infty]) \Rightarrow (a-\delta)^{-\alpha},$$

so for δ and $\omega_f(\delta)$ sufficiently small we get

$$\limsup_{n\to\infty} P\left(\omega_f(\delta)\frac{1}{k_n^2}\sum_{i,j=-n}^{n}\delta_{X^{(m)}_{n,i,j}}([a-\delta,\infty]) > \varepsilon/3\right) = 0.$$

Finally, define $M := \sup_{x\in E} f(x) < \infty$ and observe that

$$\begin{aligned}\frac{3(2n+1)^2}{k_n^2\varepsilon}&E\big(|f(X^{(m)}_{n,i,j}) - f(X_{n,i,j})|\mathbf{1}_{C_{i,j}}\big)\\ &\leqslant \frac{6M(2n+1)^2}{k_n^2\varepsilon}P(|X^{(m)}_{n,1,1} - X_{n,1,1}| > \delta),\end{aligned}$$

which goes to zero as $n\to\infty$ by (3.10). Now (3.11) follows from (3.12), and (3.13) from Theorem 4.2 in Billingsley [1]. ■

We now apply Proposition 3.2 to the two models of Section 2.

3.2. Consistency of Hill estimator for moving averages and moving maxima. As before, assume that $\{Z_{i,j}\}$ and $\{c_{k,l}\}$ satisfy (1.1), (1.2), and (2.6), and that $\{X_{i,j}\}$ is given by (2.5). Let $\{k_n\}$ be a sequence such that $k_n\to\infty$, $n/k_n\to\infty$ (for instance, take $k_n = \sqrt{n}$).

Recall that b is a quantile function for $\{X_{i,j}\}$,

$$b(t) := \inf\{x : P(|X_{0,0}| > x) \leqslant t^{-1}\},$$

and let b_n mean $b\big((2n)^2/k_n^2\big)$. Also, we define

$$X^{(m)}_{i,j} := \sum_{k,l=-m}^{m} c_{k,l} Z_{i+k,j+l}.$$

Recall that the measure μ is defined by (3.1) and define

$$\mu^{(m)} = \frac{\sum_{k,l=-m}^{m}|c_{k,l}|^\alpha}{\sum_{k,l\in\mathbb{Z}}|c_{k,l}|^\alpha}\mu.$$

For

$$X_{n,i,j} := \frac{X_{i,j}}{b_n} \quad\text{and}\quad X^{(m)}_{n,i,j} := \frac{X^{(m)}_{i,j}}{b_n},$$

it is not difficult to show that they satisfy the conditions of Proposition 3.2. First note that

$$\frac{n^2}{k_n^2} P(|X_{n,1,1}^{(m)} - X_{n,1,1}| > \varepsilon) = \frac{n^2}{k_n^2} P\big(\big|\sum_{|k|,|l|>m} c_{k,l} Z_{i+k,j+l}\big| > b_n \varepsilon\big).$$

Applying (2.7), we see that this is asymptotically (as $n \to +\infty$) equivalent to

$$\frac{n^2}{k_n^2} \cdot \sum_{|k|,|l|>m} |c_{k,l}|^\alpha \big(\sum_{k,l\in\mathbb{Z}} |c_{k,l}|^\alpha\big)^{-1} \varepsilon^{-\alpha} \cdot \frac{k_n^2}{n^2}$$
$$= \sum_{|k|,|l|>m} |c_{k,l}|^\alpha \big(\sum_{k,l\in\mathbb{Z}} |c_{k,l}|^\alpha\big)^{-1} \varepsilon^{-\alpha},$$

which goes to zero as $m \to \infty$ for all $\varepsilon > 0$, so condition (3.10) holds.

From Cline [3] we can conclude that

$$\frac{(2n)^2}{k_n^2} P\Big(\frac{X_{i,j}^{(m)}}{b_n} > x\Big) \to \frac{\sum_{k,l=-m}^m |c_{k,l}|^\alpha}{\sum_{k,l\in\mathbb{Z}} |c_{k,l}|^\alpha} x^{-\alpha} \quad \text{as } n \to \infty,$$

which satisfies condition (3.9). Therefore, the Hill estimator is consistent in the spatial moving average case. Analogously, one can prove that condition (3.2) holds for the array $\{Y_{i,j}\}$ of spatial moving maxima given by (2.12).

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