PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 34, Fasc. 1 (2014), pp. 97–117

DIMENSION RESULTS RELATED TO THE ST. PETERSBURG GAME*

BY

PETER KERN (DÜSSELDORF) AND LINA WEDRICH (DÜSSELDORF)

Abstract. Let S_n be the total gain in n repeated St. Petersburg games. It is known that $n^{-1}(S_n - n\log_2 n)$ converges in distribution along certain geometrically increasing subsequences and its possible limiting random variables can be parametrized as $Y(t)$ with $t \in \left[\frac{1}{2}, 1\right]$. We determine the Hausdorff and box-counting dimension of the range and the graph for almost all sample paths of the stochastic process $\{Y(t)\}_{t\in[1/2,1]}$. The results are compared to the fractal dimension of the corresponding limiting objects when gains are given by a deterministic sequence initiated by Hugo Steinhaus.

2000 AMS Mathematics Subject Classification: Primary: 60G17; Secondary: 28A78, 28A80, 60G18, 60G22, 60G52.

Key words and phrases: St. Petersburg game, semistable process, sample path, semi-selfsimilarity, range, graph, Hausdorff dimension, box-counting dimension, Steinhaus sequence, iterated function system, self-affine set.

1. INTRODUCTION

The famous St. Petersburg game is easily formulated as a simple coin tossing game. The player's gain $Y = 2^T$ in a single game can be expressed by means of the stopping time $T = \inf\{n \in \mathbb{N} : X_n = 1\}$ of repeated independent tosses $(X_n)_{n\in\mathbb{N}}$ of a fair coin until it first lands heads. For a sequence of gains $(Y_n)_{n\in\mathbb{N}}$ in independent St. Petersburg games the partial sum $S_n = \sum_{k=1}^{n} Y_k$ denotes the total gain in the first n games. To find a fair entrance fee for playing the game is commonly called the *St. Petersburg problem*, frequently raised to the status of a paradox. Since the expectation $\mathbb{E}[Y] = \infty$ is infinite, a fair premium cannot be constructed by the help of the usual law of large numbers. We refer to Jorland [24] and Dutka [12] for the history of the St. Petersburg game and for early solutions of the 300 year old problem.

* This work has been partially supported by Deutsche Forschungsgemeinschaft (DFG) under grant KE1741/6-1.

The first step towards a mathematically satisfactory solution has been achieved by Feller [19], [20] who showed that a premium depending on the number n of games can fulfill a certain weak law of large numbers

$$\frac{S_n}{n\log_2 n} \to 1 \quad \text{in probability,}$$

where $\log_2$ denotes logarithm to the base 2. However, Feller's result does not tell if the game is disadvantageous or advantageous for the player, i.e., if $S_n - n\log_2 n$ is likely to be negative or positive. This question can only be answered by a weak limit theorem and the first theorem of this kind has been shown by Martin-Löf [28] for the subsequence $k(n) = 2^n$:

$$\frac{S_{k(n)} - k(n)\log_2 k(n)}{k(n)} \to X \quad \text{in distribution.} \tag{1.1}$$

The limit X is infinitely divisible with characteristic function $\exp\big(\psi(y)\big)$, where

$$\psi(y) = \int_{0+}^{\infty} e^{iyx} - 1 - iyx \cdot 1_{\{x\leqslant 1\}}\, d\phi(x)$$

and the Lévy measure ϕ is concentrated on $2^{\mathbb{Z}}$ with $\phi(\{2^k\}) = 2^{-k}$ for $k \in \mathbb{Z}$. Hence X is a semistable random variable and the corresponding Lévy process $\{X(t)\}_{t\geqslant 0}$ with $X(1) \stackrel{\mathrm{d}}{=} X$ is a (non-strictly) semistable Lévy process fulfilling the semi-selfsimilarity condition

$$X(2^k t) \stackrel{\mathrm{d}}{=} 2^k\big(X(t) + kt\big) \quad \text{for every } k \in \mathbb{Z} \text{ and } t \geqslant 0. \tag{1.2}$$

For details on semistable random variables and Lévy processes we refer to the monographs [30], [34]. For a semistable setup in general there exists a continuum of possible limit distributions by variation of the subsequence $k(n) \to \infty$ in (1.1). The possible limit distributions for the St. Petersburg game have been characterized by Csörgő and Dodunekova [7] as follows. For $n \in \mathbb{N}$ let us introduce the quantity

$$\gamma_n = n \cdot 2^{-\lceil \log_2 n\rceil} \in \big(\tfrac{1}{2}, 1\big], \tag{1.3}$$

which determines the relative position of $n \in \mathbb{N}$ between its nearest consecutive powers of 2. If $k(n) \to \infty$ is a subsequence such that $\gamma_{k(n)} \to t \in \big[\frac{1}{2}, 1\big]$, Csörgő and Dodunekova [7] have shown that

$$\frac{S_{k(n)} - k(n)\log_2 k(n)}{k(n)} \to Y(t) := t^{-1}\big(X(t) - t\log_2 t\big) \tag{1.4}$$

in distribution, where $Y\big(\frac{1}{2}\big) \stackrel{\mathrm{d}}{=} Y(1) \stackrel{\mathrm{d}}{=} X$ due to (1.2); cf. also [37]. In fact, in Theorem 2.2 of [7] the necessary and sufficient condition for convergence in distribution of the normalized and centralized sums S_n along the subsequence $k(n) \to \infty$

should be stated in terms of the so-called circular convergence of $\gamma_{k(n)}$; for details we refer to page 301 in [5] or page 241 in [6]. It is also possible to interpret (1.4) as convergence in distribution of stochastic processes on the space $D\big[\frac{1}{2},1\big]$ of *càdlàg* functions $\varphi:\big[\frac{1}{2},1\big]\to\mathbb{R}$ equipped with the Skorokhod J_1-topology as follows. Firstly, a direct application of Theorem 14.14 in [26] shows that

$$\left\{\frac{S_{\lfloor 2^n t\rfloor}-\lfloor 2^n t\rfloor n}{2^n}\right\}_{t\in[\frac{1}{2},1]}\to\{X(t)\}_{t\in[\frac{1}{2},1]} \tag{1.5}$$

in distribution on $D\big[\frac{1}{2},1\big]$. Alternatively, one may deduce (1.5) from Theorem 2.1 of Fazekas [18]. Secondly, observe that

$$\frac{S_{\lfloor 2^n t\rfloor}-\lfloor 2^n t\rfloor\log_2\lfloor 2^n t\rfloor}{\lfloor 2^n t\rfloor}=\frac{2^n}{\lfloor 2^n t\rfloor}\left(\frac{S_{\lfloor 2^n t\rfloor}-\lfloor 2^n t\rfloor n}{2^n}-\frac{\lfloor 2^n t\rfloor}{2^n}\log_2\frac{\lfloor 2^n t\rfloor}{2^n}\right),$$

where convergence of the deterministic centerings

$$x_n(t)=-\frac{\lfloor 2^n t\rfloor}{2^n}\log_2\frac{\lfloor 2^n t\rfloor}{2^n}\to -t\log_2 t=x(t)$$

and normalizations

$$y_n(t)=\frac{2^n}{\lfloor 2^n t\rfloor}\to t^{-1}=y(t)$$

can also be interpreted as convergence in $D\big[\frac{1}{2},1\big]$ with continuous limit functions x and y. Since addition and multiplication are continuous at every element of $D\big[\frac{1}{2},1\big]\times D\big[\frac{1}{2},1\big]$ for which the coordinate functions have no common discontinuities (see Theorem 4.1 in [39], respectively Theorem 13.3.2 in [40]), an application of the continuous mapping theorem yields

$$\left\{\frac{S_{\lfloor 2^n t\rfloor}-\lfloor 2^n t\rfloor\log_2\lfloor 2^n t\rfloor}{\lfloor 2^n t\rfloor}\right\}_{t\in[\frac{1}{2},1]}\to\big\{t^{-1}\big(X(t)-t\log_2 t\big)=Y(t)\big\}_{t\in[\frac{1}{2},1]}$$

in distribution on $D\big[\frac{1}{2},1\big]$. Hence we have convergence in distribution of stochastic processes in (1.4) for $k(n)=\lfloor 2^n t\rfloor$ for which circular convergence of $\gamma_{k(n)}$ towards $t\in\big[\frac{1}{2},1\big]$ holds.

The object of our study are local fluctuations of the sample paths of the stochastic process $Y=\{Y(t)\}_{t\in[\frac{1}{2},1]}$. Figure 1 shows typical (approximative) sample paths of $\{Y(t)\}_{t\in[\frac{1}{2},1]}$ generated by $n=2^{16}$ simulated St. Petersburg games. Note that the sample paths do only have upward jumps due to the fact that the Lévy measure ϕ is concentrated on $2^{\mathbb{Z}}$.

The main goal of our paper is to determine the Hausdorff dimension of the range $Y\big(\big[\frac{1}{2},1\big]\big)=\big\{Y(t):\,t\in\big[\frac{1}{2},1\big]\big\}$ and the graph $G_Y\big(\big[\frac{1}{2},1\big]\big)=\big\{\big(t,Y(t)\big):\,t\in\big[\frac{1}{2},1\big]\big\}$ of the stochastic process Y encoding all the possible distributional

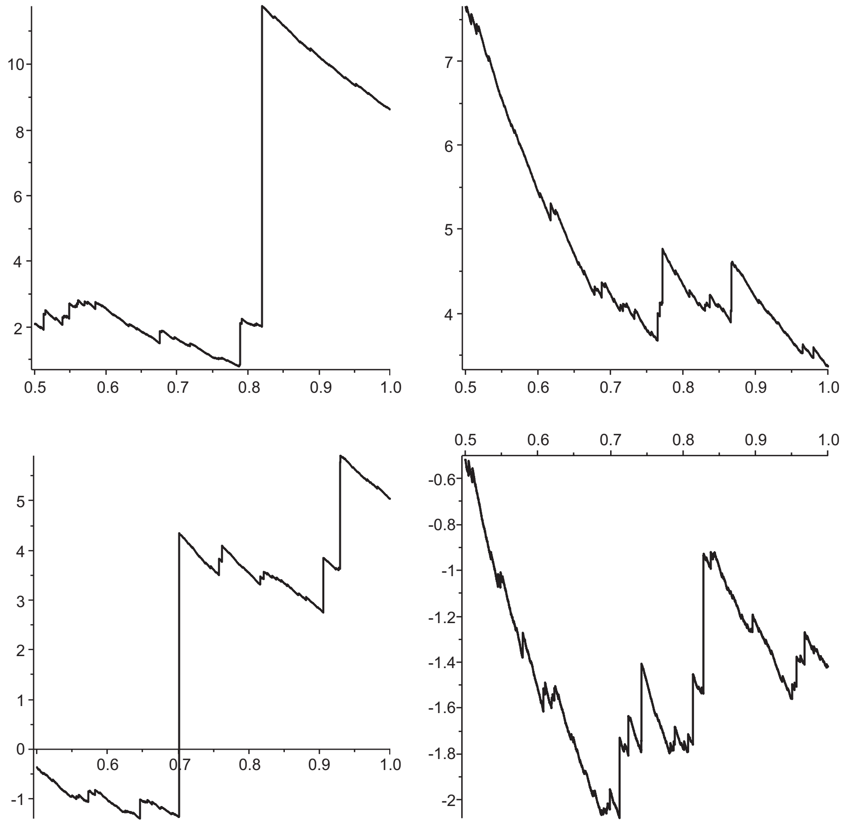

FIGURE 1. Simulation of four approximations to the sample paths of Y. For better visibility the jumps are shown as vertical lines

limits of St. Petersburg games. For an arbitrary subset $F \subseteq \mathbb{R}^d$ the s-dimensional Hausdorff measure is defined as

$$\mathcal{H}^s(F) = \liminf_{\delta \to 0} \Big\{ \sum_{i=1}^{\infty} |F_i|^s : |F_i| \leqslant \delta \text{ and } F \subseteq \bigcup_{i=1}^{\infty} F_i \Big\},$$

where $|F| = \sup\{\|x - y\| : x, y \in F\}$ denotes the diameter of a set $F \subseteq \mathbb{R}^d$ and $\|\cdot\|$ is the Euclidean norm. It can now be shown that there exists a unique value $\dim_{\mathrm{H}} F \geqslant 0$ so that $\mathcal{H}^s(F) = \infty$ for all $0 \leqslant s < \dim_{\mathrm{H}} F$ and $\mathcal{H}^s(F) = 0$ for all $s > \dim_{\mathrm{H}} F$. This critical value is called the *Hausdorff dimension* of F. Specifically, we have

$$\dim_{\mathrm{H}} F = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

For details on the Hausdorff dimension we refer to [15] and [29].

Now let $F \subseteq \mathbb{R}^d$ be a Borel set and denote by $\mathcal{M}^1(F)$ the set of probability measures on F. For $s > 0$ the s-energy of $\mu \in \mathcal{M}^1(F)$ is defined by

$$I_s(\mu) = \int_F \int_F \frac{\mu(dx)\mu(dy)}{\|x-y\|^s}.$$

By Frostman's lemma (see, e.g., [25], [29]) there exists a probability measure $\mu \in \mathcal{M}^1(F)$ with $I_s(\mu) < \infty$ if $\dim_{\mathrm{H}} F > s$. In this case F is said to have *positive s-capacity* $C_s(F)$ given by

$$C_s(F) = \sup\{I_s(\mu)^{-1} : \mu \in \mathcal{M}^1(F)\},$$

and the *capacitary dimension* of F is defined by

$$\dim_{\mathrm{C}} F = \sup\{s > 0 : C_s(F) > 0\} = \inf\{s > 0 : C_s(F) = 0\}.$$

A consequence of Frostman's theorem (see, e.g., [25], [29]) is that for Borel sets $F \subseteq \mathbb{R}^d$ the Hausdorff and capacitary dimension coincide. Therefore, one can prove lower bounds for the Hausdorff dimension with a simple capacity argument: if $I_s(\mu) < \infty$ for some $\mu \in \mathcal{M}^1(F)$ then $\dim_{\mathrm{H}} F = \dim_{\mathrm{C}} F \geqslant s$.

An alternative fractal dimension is the so-called *box-counting dimension* (see, e.g., [15]). For this purpose let $N_\delta(F)$ be the smallest number of closed balls of radius δ that cover the set $F \subseteq \mathbb{R}^d$. The lower and the upper box-counting dimensions of an arbitrary set $F \subseteq \mathbb{R}^d$ are now defined as

$$\underline{\dim}_{\mathrm{B}} F = \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \quad \text{and} \quad \overline{\dim}_{\mathrm{B}} F = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \tag{1.6}$$

and the box-counting dimension of F is given by

$$\dim_{\mathrm{B}} F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

provided that this limit exists. The different fractal dimensions are related as follows:

$$\dim_{\mathrm{H}} F \leqslant \underline{\dim}_{\mathrm{B}} F \leqslant \overline{\dim}_{\mathrm{B}} F \leqslant d. \tag{1.7}$$

Note that there are plenty of sets $F \subseteq \mathbb{R}^d$ where these inequalities are strict.

In Section 2 we will determine the Hausdorff and box-counting dimension of the range $Y\big(\big[\frac{1}{2}, 1\big]\big)$ and the graph $G_Y\big(\big[\frac{1}{2}, 1\big]\big)$ for almost all sample paths of the stochastic process Y. Additionally, in Section 3 we will also consider a deterministic sequence introduced by Steinhaus [35] which is called the *Steinhaus sequence* according to [9]. The Steinhaus sequence $(x_n)_{n\in\mathbb{N}}$ is defined by $x_n = 2^k$ if $n = 2^{k-1} + m \cdot 2^k$ for some $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Alternatively, as in Vardi [38], one can define x_n to be twice the highest power of 2 dividing n. The Steinhaus sequence is explicitly given by

$$2, 4, 2, 8, 2, 4, 2, 16, 2, 4, 2, 8, 2, 4, 2, 32, 2, 4, 2, 8, 2, 4, 2, 16, 2, 4, 2, 8, 2, 4, 2, 64, \ldots$$

and has relative frequencies $\lim_{n\to\infty} n^{-1}\operatorname{card}\{1 \leqslant j \leqslant n : x_j = 2^k\} = 2^{-k}$ for $k \in \mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ has been considered as entrance fees for repeated St. Petersburg games in [35] and [9] and has been proven to be a sequence of nearly asymptotically fair premiums in a certain sense. For details we refer to [9]. In contrast to [35] and [9] we will consider the Steinhaus sequence as a sequence of possible gains in repeated St. Petersburg games. Again, we will determine the Hausdorff and box-counting dimension of the range and the graph of the corresponding limiting object of the centralized and normalized Steinhaus sequence. To do so, we will employ results for iterated function systems as presented in [17].

2. HAUSDORFF DIMENSION OF THE ST. PETERSBURG GAME

2.1. Hausdorff dimension of the range. In this section we evaluate the Hausdorff dimension of the range of the stochastic process $Y = \{Y(t)\}_{t\in[\frac{1}{2},1]}$. We employ common techniques used to calculate Hausdorff dimensions of selfsimilar Lévy processes (see [41], [31], [27]) and adapt them to our situation. Note that the given process Y is neither a Lévy process nor does it have the selfsimilarity property of a semistable process. The result is stated in the theorem below.

THEOREM 2.1. *We have* $\dim_{\mathrm{H}} Y\big([\frac{1}{2},1]\big) = 1$ *almost surely.*

Note that Theorem 2.1 together with (1.7) yields

$$\dim_{\mathrm{H}} Y\big([\tfrac{1}{2},1]\big) = \dim_{\mathrm{B}} Y\big([\tfrac{1}{2},1]\big) = 1$$

almost surely. Since Y is a process on $\mathbb{R}$ it is obvious that $\dim_{\mathrm{H}} Y\big([\frac{1}{2},1]\big) \leqslant 1$ almost surely. Hence for the proof of Theorem 2.1 it is sufficient to prove the following lemma.

LEMMA 2.1. *We have* $\dim_{\mathrm{H}} Y\big([\frac{1}{2},1]\big) \geqslant 1$ *almost surely.*

Proof. As mentioned above we can write

$$Y(t) = t^{-1}\big(X(t) - t\log_2 t\big),$$

where $X = \{X(t)\}_{t\geqslant 0}$ is a semistable Lévy process. To prove the assertion we will apply Frostman's theorem [25], [29] with the probability measure $\sigma = 2\lambda|_{[\frac{1}{2},1]}$, where λ denotes Lebesgue measure. For this purpose let $0 < \gamma < 1$ and note that

σ is an admissible measure for Frostman's lemma, i.e.,

$$\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} \frac{\sigma(ds)\,\sigma(dt)}{|s-t|^{\gamma}} < \infty.$$

By Frostman's theorem it is now sufficient to show that for all $\gamma \in (0,1)$

$$\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} \mathbb{E}[|Y(s)-Y(t)|^{-\gamma}]ds\,dt < \infty.$$

For $r \in (0,1]$ let g_r be the Lebesgue density of $X(r)$ chosen from the class $C^{\infty}(\mathbb{R})$ by Proposition 2.8.1 in [34]. Then we have $M := \sup_{r\in(0,1]}\sup_{x\in\mathbb{R}}|g_r(x)| < \infty$ as in Lemma 3 of [4]; see also Lemma 2.2 in [27]. By symmetry of the integrand we get

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} \mathbb{E}[|Y(s)-Y(t)|^{-\gamma}]ds\,dt\\
&= 2\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{t} \mathbb{E}\Big[\Big|s^{-1}X(s)-\log_2 s - t^{-1}\Big(X(s)+\big(X(t)-X(s)\big)\Big)+\log_2 t\Big|^{-\gamma}\Big]ds\,dt\\
&= 2\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{t}\int_{\mathbb{R}^2} |s^{-1}x - \log_2 s - t^{-1}(x+y)+\log_2 t|^{-\gamma} g_s(x)\,g_{t-s}(y)\,d\lambda^2(x,y)\,ds\,dt\\
&= 2\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{t}\int_{\mathbb{R}^2} \left|\frac{t-s}{st}\,x+\log_2\left(\frac{t}{s}\right)-\frac{y}{t}\right|^{-\gamma} g_s(x)\,g_{t-s}(y)\,d\lambda^2(x,y)\,ds\,dt\\
&= 2\int_{\frac{1}{2}}^{1}\int_{0}^{t-\frac{1}{2}}\int_{\mathbb{R}^2} \left|\frac{w}{t(t-w)}\,x+\log_2\left(\frac{t}{t-w}\right)-\frac{y}{t}\right|^{-\gamma} g_{t-w}(x)\,g_w(y)\,d\lambda^2(x,y)\,dw\,dt,
\end{aligned}$$

where in the last equality we substituted $w = t-s$. Now we write $w \in \big(0,\frac{1}{2}\big]$ as $w = 2^{-m}r$ with $m = m(w) \in \mathbb{N}$ and $r \in \big(\frac{1}{2},1\big]$. This leads us to

$$\begin{aligned}
g_w(y) &= \frac{d}{dy}\mathbb{P}\big(X(w) \leqslant y\big)\\
&= \frac{d}{dy}\mathbb{P}\left(X(2^{-m}r) \leqslant y\right) = \frac{d}{dy}\mathbb{P}\Big(2^{-m}\big(X(r)-mr\big) \leqslant y\Big)\\
&= \frac{d}{dy}\mathbb{P}\big(X(r) \leqslant 2^m y + mr\big) = 2^m g_r(2^m y+mr).
\end{aligned}$$

Using the substitutions

$$v = 2^m y + mr \quad \text{and} \quad u = \frac{t}{2^{-m}}\left(\frac{w}{t(t-w)}x + \log_2\left(\frac{t}{t-w}\right)+\frac{mw}{t}\right),$$

we get

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \left| \frac{w}{t(t-w)}\, x + \log_2\left(\frac{t}{t-w}\right) - \frac{y}{t} \right|^{-\gamma} g_{t-w}(x)\, g_w(y)\, d\lambda^2(x,y) \\
&= 2^m \int_{\mathbb{R}^2} \left| \frac{w}{t(t-w)}\, x + \log_2\left(\frac{t}{t-w}\right) - \frac{y}{t} \right|^{-\gamma} g_{t-w}(x)\, g_r(2^m y + mr)\, d\lambda^2(x,y) \\
&= \int_{\mathbb{R}^2} \left| \frac{w}{t(t-w)}\, x + \log_2\left(\frac{t}{t-w}\right) - \frac{2^{-m}}{t}\, v + \frac{mw}{t} \right|^{-\gamma} g_{t-w}(x) g_r(v)\, d\lambda^2(x,v) \\
&= \frac{t-w}{r} \int_{\mathbb{R}^2} \left| \frac{2^{-m}}{t}(u-v) \right|^{-\gamma} g_{t-w}\big(x(u)\big)\, g_r(v)\, d\lambda^2(u,v) \\
&= \frac{t^\gamma (t-w) 2^{m\gamma}}{r} \Big(\int_A + \int_{A^\complement} \Big)\, |u-v|^{-\gamma}\, g_{t-w}\big(x(u)\big)\, g_r(v)\, d\lambda^2(u,v),
\end{aligned}
$$

where $A = \{(u,v) \in \mathbb{R}^2 : |u-v| \leqslant 1\}$. We now estimate the two integrals separately. Firstly,

$$
\begin{aligned}
&\int_A |u-v|^{-\gamma}\, g_{t-w}\big(x(u)\big)\, g_r(v)\, d\lambda^2(u,v) \\
&\qquad \leqslant M \int_{\mathbb{R}} \Big(\int_{v-1}^{v} (v-u)^{-\gamma} du + \int_{v}^{v+1} (u-v)^{-\gamma} du \Big) g_r(v)\, dv \\
&\qquad = M \int_{\mathbb{R}} \frac{2}{1-\gamma} g_r(v)\, dv = \frac{2M}{1-\gamma}
\end{aligned}
$$

and secondly,

$$
\begin{aligned}
\int_{A^\complement} |u-v|^{-\gamma}\, g_{t-w}\big(x(u)\big)\, g_r(v)\, d\lambda^2(u,v) &\leqslant \int_{A^\complement} g_{t-w}\big(x(u)\big)\, g_r(v)\, d\lambda^2(u,v) \\
&\leqslant \frac{r}{t-w} \int_{\mathbb{R}^2} g_{t-w}(x)\, g_r(v)\, d\lambda^2(x,v) = \frac{r}{t-w}.
\end{aligned}
$$

This leads us to

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \left| \frac{w}{t(t-w)}\, x + \log_2\left(\frac{t}{t-w}\right) - \frac{y}{t} \right|^{-\gamma} g_{t-w}(x)\, g_w(y)\, d\lambda^2(x,y) \\
&\qquad \leqslant t^\gamma 2^{m\gamma} \left(\frac{2M(t-w)}{r(1-\gamma)} + 1 \right) \leqslant t^\gamma 2^{m\gamma} \left(\frac{4M}{1-\gamma} + 1 \right) =: K t^\gamma 2^{m\gamma}.
\end{aligned}
$$

Taken all together, we obtain

$$\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\mathbb{E}[|Y(s)-Y(t)|^{-\gamma}]ds\,dt \leqslant 2K\int_{\frac{1}{2}}^{1}\int_{0}^{t-\frac{1}{2}} t^{\gamma}2^{m(w)\gamma}\,dw\,dt \tag{2.1}$$

$$= 2K\int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}+w}^{1} t^{\gamma}2^{m(w)\gamma}\,dt\,dw = 2K\sum_{m\in\mathbb{N}}\int_{2^{-(m+1)}}^{2^{-m}}\int_{\frac{1}{2}+w}^{1} t^{\gamma}2^{m\gamma}\,dt\,dw$$

$$\leqslant 2K\sum_{m\in\mathbb{N}}\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} t^{\gamma}2^{m\gamma}\,dt\,2^{-m}\,dr = K\sum_{m\in\mathbb{N}}(2^{\gamma-1})^{m}\int_{\frac{1}{2}}^{1} t^{\gamma}\,dt < \infty,$$

since $\gamma - 1 < 0$. This concludes our proof. ∎

2.2. Hausdorff dimension of the graph. In this section we show that the dimension result for the range of the stochastic process Y also holds for its graph $G_Y\big([\frac{1}{2},1]\big)$. We will split the proof into two parts, firstly verifying $\alpha = 1$ as an upper bound and secondly as a lower bound for the Hausdorff dimension of the graph.

We first calculate the upper bound for the Hausdorff dimension of the graph of the semistable Lévy process X and later on transfer the result to the process Y. As X is not strictly semistable, we cannot use the dimension results of [27] without modifying it according to our situation. This parallels investigations for stable Lévy processes, where dimension results for the strictly stable (symmetric) Cauchy process and for an asymmetric (non-strictly) stable Cauchy process have been addressed separately. Nevertheless, the Hausdorff dimension of the range, respectively the graph, coincides for both Cauchy processes; see [2], [22], [36].

PROPOSITION 2.1. *Let* $\big\{Z(t) := \big(t, X(t)\big)\big\}_{t\geqslant 0}$. *Then almost surely*

$$\dim_{\mathrm{H}} Z\big(\big[\tfrac{1}{2},1\big]\big) \leqslant 1.$$

Let $T(a,s) = \int_0^s 1_{B(0,a)}\big(Z(t)\big)\,dt$ denote the sojourn time of the Lévy process Z up to time s in the closed ball $B(0,a) \subseteq \mathbb{R}^2$ with radius a centered at the origin. To prove Proposition 2.1 we need the following lemma.

LEMMA 2.2. *Let Z be the stochastic process defined in Proposition* 2.1. *There exists a positive and finite constant K such that for all* $0 < a \leqslant 1$ *and* $\frac{a}{\sqrt{2}} \leqslant s \leqslant 1$ *we have*

$$\mathbb{E}\,[T(a,s)] \geqslant Ka.$$

P r o o f. Fix $0 < a \leqslant 1$ and let $0 < \delta \leqslant \frac{1}{\sqrt{2}}$, so that

$$2^{-(i_0+1)} < a\delta \leqslant 2^{-i_0} < \frac{a}{\sqrt{2}} < s$$

for some $i_0 \in \mathbb{N}_0$. Furthermore, we choose $0 < \delta \leqslant \frac{1}{\sqrt{2}}$ small enough (i.e. $i_0 \in \mathbb{N}_0$ big enough) so that

$$2i \leqslant \frac{a}{2\sqrt{2}}2^i \quad \text{for all } i > i_0 \tag{2.2}$$

and that additionally

$$\mathbb{P}\Big(\sup_{r\in[1,2)} X(r) < \frac{1}{\delta 2\sqrt{2}}\Big) - \mathbb{P}\Big(\inf_{r\in[1,2)} X(r) \leqslant -\frac{1}{\delta 2\sqrt{2}}\Big) \geqslant \frac{1}{2}. \tag{2.3}$$

Inequality (2.3) holds for $\delta > 0$ small enough since X is a Lévy process and it can be assumed that it has *càdlàg* paths. Thus both $\sup_{r\in[1,2)} X(r)$ and $\inf_{r\in[1,2)} X(r)$ are random variables. We have

$$\begin{aligned}
&\mathbb{E}\left[T(a,s)\right] = \int_0^s \mathbb{P}\big(\|Z(t)\| < a\big)dt \geqslant \int_0^s \mathbb{P}\left(|X(t)| < \frac{a}{\sqrt{2}}, t < \frac{a}{\sqrt{2}}\right)dt \\
&= \int_0^{a/\sqrt{2}} \mathbb{P}\left(|X(t)| < \frac{a}{\sqrt{2}}\right)dt \geqslant \int_0^{2^{-i_0}} \mathbb{P}\left(|X(t)| < \frac{a}{\sqrt{2}}\right)dt \\
&= \sum_{i=i_0+1}^{\infty} \int_{2^{-i}}^{2^{-i+1}} \mathbb{P}\left(|X(t)| < \frac{a}{\sqrt{2}}\right)dt = \sum_{i=i_0+1}^{\infty} 2^{-i}\int_1^2 \mathbb{P}\left(|X(2^{-i}r)| < \frac{a}{\sqrt{2}}\right)dr \\
&= \sum_{i=i_0+1}^{\infty} 2^{-i}\int_1^2 \mathbb{P}\Big(\big|2^{-i}\big(X(r) - ir\big)\big| < \frac{a}{\sqrt{2}}\Big)dr \\
&= \sum_{i=i_0+1}^{\infty} 2^{-i}\int_1^2 \mathbb{P}\left(|X(r) - ir| < \frac{2^i a}{\sqrt{2}}\right)dr.
\end{aligned}$$

By (2.2) and (2.3) the probability above can be estimated from below by

$$\begin{aligned}
\mathbb{P}\left(|X(r) - ir| < \frac{2^i a}{\sqrt{2}}\right) &= \mathbb{P}\left(-\frac{a}{\sqrt{2}}2^i + ir < X(r) < \frac{a}{\sqrt{2}}2^i + ir\right) \\
&\geqslant \mathbb{P}\left(-\frac{a}{\sqrt{2}}2^i + 2i < X(r) < \frac{a}{\sqrt{2}}2^i\right) \\
&= \mathbb{P}\left(X(r) < \frac{a}{\sqrt{2}}2^i\right) - \mathbb{P}\left(X(r) \leqslant -\frac{a}{\sqrt{2}}2^i + 2i\right) \\
&\geqslant \mathbb{P}\Big(\sup_{r\in[1,2)} X(r) < \frac{a}{\sqrt{2}}2^i\Big) - \mathbb{P}\left(\inf_{r\in[1,2)} X(r) \leqslant -\frac{a}{\sqrt{2}}2^i + 2i\right) \\
&\geqslant \mathbb{P}\Big(\sup_{r\in[1,2)} X(r) < \frac{a}{2\sqrt{2}}2^{i_0+1}\Big) - \mathbb{P}\left(\inf_{r\in[1,2)} X(r) \leqslant -\frac{a}{2\sqrt{2}}2^{i_0+1}\right) \\
&\geqslant \mathbb{P}\Big(\sup_{r\in[1,2)} X(r) < \frac{1}{\delta 2\sqrt{2}}\Big) - \mathbb{P}\left(\inf_{r\in[1,2)} X(r) \leqslant -\frac{1}{\delta 2\sqrt{2}}\right) \geqslant \frac{1}{2}.
\end{aligned}$$

Note that δ does not depend on a. It follows that

$$\mathbb{E}\left[T(a,s)\right] \geqslant \sum_{i=i_0+1}^{\infty} 2^{-i} \int_1^2 \frac{1}{2}\,dr = \frac{1}{2} \sum_{i=i_0+1}^{\infty} 2^{-i} = \frac{1}{2} 2^{-i_0} \geqslant \frac{1}{2}\delta a =: Ka,$$

which concludes the proof. ∎

P r o o f o f P r o p o s i t i o n 2.1. Let $K_1 > 0$ be a fixed constant. A family $\Lambda(a)$ of cubes of side $a \in (0,1]$ in $\mathbb{R}^2$ is called K_1-*nested* if no ball of radius a in $\mathbb{R}^2$ can intersect more than K_1 cubes of $\Lambda(a)$. For any $u \geqslant 0$ let $M_u(a,s)$ be the number of these cubes hit by the Lévy process Z at some time $t \in [u, u+s]$. Then a famous covering lemma of Pruitt and Taylor ([33], Lemma 6.1) states that

$$\mathbb{E}[M_u(a,s)] \leqslant 2K_1 s \cdot \left(\mathbb{E}[T(a/3,s)]\right)^{-1}.$$

Lemma 2.2 now enables us to construct a covering of $Z\left(\left[\frac{1}{2},1\right]\right)$ whose expected s-dimensional Hausdorff measure is finite for every $s > 1$. The arguments are analogous to the proof of part (i) of Lemma 3.4 in [27] and thus omitted. ∎

In order to transfer the result of Proposition 2.1 to the process Y we introduce the continuous function $\tau : \left[\frac{1}{2},1\right] \times \mathcal{K}_1 \to \left[\frac{1}{2},1\right] \times \mathcal{K}_2$ with

$$\begin{pmatrix} t \\ x \end{pmatrix} \mapsto \tau(t,x) = \begin{pmatrix} t \\ t^{-1}x - \log_2 t \end{pmatrix}, \tag{2.4}$$

where $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathbb{R}$ are not further specified compact intervals that can vary throughout the paper. It can easily be shown that for a fixed compact interval $\mathcal{K}_1 \subseteq \mathbb{R}$ the function τ is bi-Lipschitz when choosing $\mathcal{K}_2$ such that $\left[\frac{1}{2},1\right] \times \mathcal{K}_2 = \operatorname{Im}(\tau)$. We can now write all elements $\left(t, Y(t)\right)^\top \in G_Y\left(\left[\frac{1}{2},1\right]\right)$ as

$$\begin{pmatrix} t \\ Y(t) \end{pmatrix} = \begin{pmatrix} t \\ t^{-1}X(t) - \log_2 t \end{pmatrix} = \tau\big(t, X(t)\big).$$

Since X is a Lévy process, it can be assumed that all paths are *càdlàg*, and hence that for all fixed $\omega \in \Omega$ there exists a compact interval $\mathcal{K}_1 \subseteq \mathbb{R}$ such that

$$X(t)(\omega) \in \mathcal{K}_1 \quad \text{for all } t \in \left[\tfrac{1}{2},1\right].$$

This means that for $Z = \big(Z(t) = \big(t, X(t)\big)\big)_{t \in [\frac{1}{2},1]}$ and all $\omega \in \Omega$ we have

$$\dim_{\mathrm{H}} Z\left(\left[\tfrac{1}{2},1\right]\right)(\omega) = \dim_{\mathrm{H}} \tau\left(Z\left(\left[\tfrac{1}{2},1\right]\right)\right)(\omega) = \dim_{\mathrm{H}} G_Y\left(\left[\tfrac{1}{2},1\right]\right)(\omega)$$

by Lemma 1.8 in [13]. Since we have shown in Proposition 2.1 that $\dim_{\mathrm{H}} Z\left(\left[\frac{1}{2},1\right]\right) \leqslant 1$ almost surely, we have thus proven the following upper bound.

THEOREM 2.2. *We have* $\dim_{\mathrm{H}} G_Y\left(\left[\frac{1}{2},1\right]\right) \leqslant 1$ *almost surely.*

To prove the lower bound for the Hausdorff dimension of the graph we can use the same technique as for the lower bound in case of the range of Y.

THEOREM 2.3. *We have* $\dim_{\mathrm{H}} G_Y\left(\left[\frac{1}{2},1\right]\right) \geqslant 1$ *almost surely.*

Proof. Let $0<\gamma<1$. By (2.1) we get

$$\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} \mathbb{E}\left[\left\|\big(s,Y(s)\big)^{\top}-\big(t,Y(t)\big)^{\top}\right\|^{-\gamma}\right]ds\,dt$$

$$=\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} \mathbb{E}\Big[\Big((s-t)^2+\big(Y(s)-Y(t)\big)^2\Big)^{-\gamma/2}\Big]ds\,dt$$

$$\leqslant \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} \mathbb{E}\left[|Y(s)-Y(t)|^{-\gamma}\right]ds\,dt<\infty.$$

As in Lemma 2.1 the assertion follows by Frostman's theorem. ■

With similar techniques it is also possible to prove the following dimension result for the box-counting dimension of the graph of the St. Petersburg process Y.

THEOREM 2.4. *We have* $\dim_{\mathrm{B}} G_Y\left(\left[\frac{1}{2},1\right]\right) = 1$ *almost surely.*

Proof. The lower bound follows directly from the almost sure inequalities

$$1 \leqslant \dim_{\mathrm{H}} G_Y\big(\big[\tfrac{1}{2},1\big]\big) \leqslant \underline{\dim}_{\mathrm{B}}\, G_Y\big(\big[\tfrac{1}{2},1\big]\big) \leqslant \overline{\dim}_{\mathrm{B}}\, G_Y\big(\big[\tfrac{1}{2},1\big]\big).$$

For the upper bound it is now sufficient to verify $\overline{\dim}_{\mathrm{B}}\, G_Y\big(\big[\frac{1}{2},1\big]\big) \leqslant 1$ almost surely. Due to the nature of the upper box-counting dimension (see (1.6)) we can again calculate the upper bound for $\overline{\dim}_{\mathrm{B}}\, Z\big(\big[\frac{1}{2},1\big]\big) \leqslant 1$ with the same covering arguments as in the proof of part (i) of Lemma 3.4 in [27]. With the bi-Lipschitz invariance of the upper box-counting dimension (see Section 3.2 in [15]) the proof concludes. ■

REMARK 2.1. *If one prefers to flip an unfair coin, this naturally leads to so-called generalized St. Petersburg games as treated in* [4], [8], [21], [32]. *Let* $p\in(0,1)$ *be the probability of the coin falling heads and let* $q=1-p$. *The single gain in a generalized St. Petersburg game is given by* $q^{-T/\alpha}$ *for some* $\alpha>0$. *We focus on the classical situation* $\alpha=1$ *and slightly modify the gain to* $q^{1-T}p^{-1}$ *for ease of the notation, which results in the limit theorem*

$$\frac{S_{k(n)}-k(n)\log_{1/q}k(n)}{k(n)} \to Y(t)=t^{-1}\big(X(t)-t\log_{1/q}t\big)$$

in distribution, *whenever*

$$q^{\lceil \log_{1/q} k(n) \rceil} k(n) \to t \in [q, 1],$$

where $\{X(t)\}_{t \geqslant 0}$ *is a semistable Lévy process with the semi-selfsimilarity property*

$$X(q^{-k}t) \overset{\mathrm{d}}{=} q^{-k}\big(X(t) + kt\big) \quad \text{for every } k \in \mathbb{Z} \text{ and } t \geqslant 0.$$

We emphasize that with the above techniques our Theorems 2.1–2.4 *also hold for the process* $\{Y(t)\}_{t \in [q,1]}$ *in this generalized situation when replacing the interval by* $[q, 1]$. *Presumably*, *similar results can be shown for general* $\alpha > 0$.

3. HAUSDORFF DIMENSION OF THE STEINHAUS SEQUENCE

Recall the definition of the Steinhaus sequence $(x_n)_{n \in \mathbb{N}}$ given in the Introduction. The asymptotic properties of $(x_n)_{n \in \mathbb{N}}$ have been analyzed in full detail by Csörgő and Simons [9]. Let $s(n) = x_1 + \ldots + x_n$ and $\gamma_n = n \cdot 2^{-\lceil \log_2 n \rceil} \in \left(\frac{1}{2}, 1\right]$ as in (1.3). Then by Theorem 3.3 in [9] we have for any $n \in \mathbb{N}$

$$\frac{s(n) - n \log_2 n}{n} = \xi(\gamma_n), \tag{3.1}$$

where the function $\xi : \left[\frac{1}{2}, 1\right] \to [0, 2]$ is defined by

$$\xi(\gamma) = 2 - \log_2 \gamma - \frac{1}{\gamma} \sum_{k=1}^{\infty} \frac{k\varepsilon_k}{2^k},$$

and the sequence $(\varepsilon_k)_{k \in \mathbb{N}} \subseteq \{0, 1\}$ is given by the dyadic expansion $\gamma = \sum_{k=0}^{\infty} \frac{\varepsilon_k}{2^k}$ of $\gamma \in \left[\frac{1}{2}, 1\right]$ with the convention that $\varepsilon_k = 0$ for infinitely many $k \in \mathbb{N}$. By Theorem 3.1 in [9] the function ξ is *càdlàg* with $\xi\left(\frac{1}{2}\right) = 2 = \xi(1)$ and has jumps precisely at the dyadic rationals in $\left(\frac{1}{2}, 1\right]$. All these jumps are upward and the largest jump occurs from $\xi(1-) = 0$ to $\xi(1) = 2$. The graph of ξ seems to inhere fractal properties as can be seen in Figure 2 below, a replication of Figure 1 in [9]. It follows directly from (3.1) that the sequence $\big(s(n)\big)_{n \in \mathbb{N}}$ of total gains satisfies the asymptotic property of Feller

$$\frac{s(n)}{n \log_2 n} \to 1 \tag{3.2}$$

as $n \to \infty$; see [9]. Note that Feller's law of large numbers does not hold in an almost sure sense. According to classical results in [3], [1], [10] it is known that

$$\limsup_{n \to \infty} \frac{S_n}{n \log_2 n} = \infty \quad \text{and} \quad \liminf_{n \to \infty} \frac{S_n}{n \log_2 n} = 1 \quad \text{almost surely.} \tag{3.3}$$

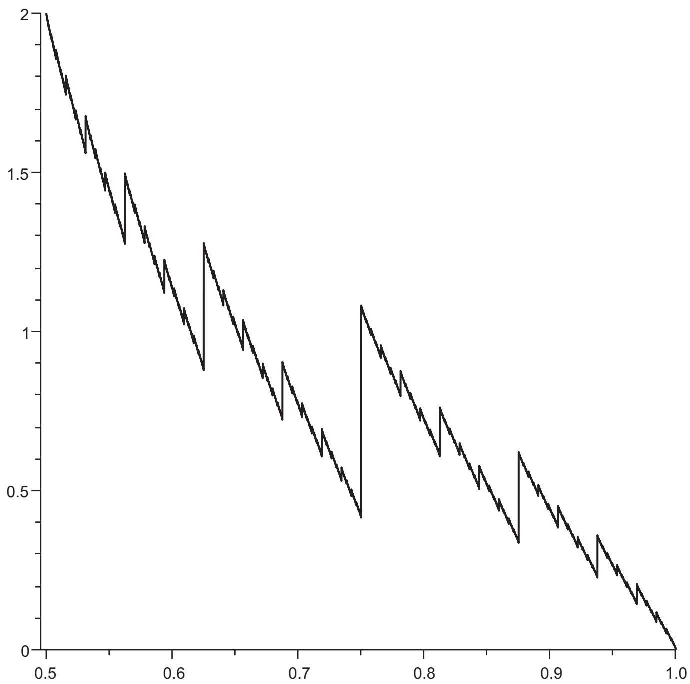

FIGURE 2. Graph of ξ on the interval $\left[\frac{1}{2},1\right)$.
For better visibility the jumps of ξ are shown as vertical lines

More precisely, we have $\mathrm{LIM}\{S_n/(n\log_2 n) : n\in\mathbb{N}\}=[1,\infty]$ almost surely by Corollary 1 in [38], where LIM denotes the set of accumulation points. But there is a version of the strong law of large numbers by Csörgő and Simons [11] when neglecting the largest gain

$$\frac{S_n-\max_{1\leqslant k\leqslant n}X_k}{n\log_2 n}\to 1 \quad \text{almost surely.}$$

A comparison of (3.2) and (3.3) shows that the Steinhaus sequence belongs to an exceptional null set with respect to (3.3) concerning the almost sure limit behavior of the total gain in repeated St. Petersburg games. Moreover, for any sequence $k_n\to\infty$ with $\gamma_{k_n}=k_n\cdot 2^{-\lceil\log_2 k_n\rceil}\to\gamma\in\left[\frac{1}{2},1\right]$ we get from (3.1)

$$\emptyset\neq\mathrm{LIM}\left\{\frac{s(k_n)-k_n\log_2 k_n}{k_n} : n\in\mathbb{N}\right\}\subseteq\{\xi(\gamma),\xi(\gamma-)\}.$$

Hence we may consider the function ξ as the corresponding limiting object when using the same centering and normalization sequences as in (1.4). We will now show that the Steinhaus sequence is not exceptional concerning the local fluctuations of the limit measured by the Hausdorff or box-counting dimension.

It follows directly from the above-stated properties of ξ given in Theorem 3.1 of [9] that the range $\xi\left(\left[\frac{1}{2},1\right]\right)$ equals the interval $(0,2]$ and hence $\dim_{\mathrm{H}}\xi\left(\left[\frac{1}{2},1\right]\right)$

$= 1$ by Theorem 1.12 in [13]. This shows that $\dim_{\mathrm{H}} \xi\big(\big[\frac{1}{2}, 1\big]\big)$ coincides with the Hausdorff dimension of the range of a typical sample path of $\{Y(t)\}_{t\in[\frac{1}{2},1]}$. Clearly, by (1.7) we also have $\dim_{\mathrm{H}} \xi\big(\big[\frac{1}{2}, 1\big]\big) = \dim_{\mathrm{B}} \xi\big(\big[\frac{1}{2}, 1\big]\big) = 1$. A look at Figure 2 suggests that it is merely the graph and not the range of ξ that should inhere fractal properties. In the sequel we will argue that also the graph $G_\xi\big(\big[\frac{1}{2}, 1\big]\big)$ is typical concerning the almost sure dimension properties of the sample graph of $\{Y(t)\}_{t\in[\frac{1}{2},1]}$. To this aim we will apply the inverse τ^{-1} of the bi-Lipschitz function τ defined in (2.4) with $\mathcal{K}_2 = [0, 2] \subseteq \mathbb{R}$. Namely, we now consider the function $\tau^{-1} : \big[\frac{1}{2}, 1\big] \times [0, 2] \to \big[\frac{1}{2}, 1\big] \times \mathcal{K}_1$ with $\tau^{-1}(t, x) = \big(t, t(x + \log_2 t)\big)^\top$, where the compact interval $\mathcal{K}_1 \subseteq \mathbb{R}$ is chosen such that $\big[\frac{1}{2}, 1\big] \times \mathcal{K}_1 = \mathrm{Im}(\tau^{-1})$. Applied to the graph of ξ we get for any $\gamma \in \big[\frac{1}{2}, 1\big]$

$$\tau^{-1}\big(\gamma, \xi(\gamma)\big) = \begin{pmatrix} \gamma \\ \gamma\big(\xi(\gamma) + \log_2 \gamma\big) \end{pmatrix} = \begin{pmatrix} \gamma \\ 2\gamma - \sum\limits_{k=1}^{\infty} (k\varepsilon_k)/2^k \end{pmatrix}$$

and by bi-Lipschitz invariance we have

$$\dim_{\mathrm{H}} G_\xi\big(\big[\tfrac{1}{2}, 1\big]\big) = \dim_{\mathrm{H}} \tau^{-1}\big(G_\xi\big(\big[\tfrac{1}{2}, 1\big]\big)\big). \tag{3.4}$$

The same equality holds for upper and lower box-counting dimensions; see, e.g., [15]. The image $\tau^{-1}\big(G_\xi\big(\big[\frac{1}{2}, 1\big)\big)\big)$ is illustrated in Figure 3 and shows perfect self-similarity. To see this, we may write $\tau^{-1}\big(\gamma, \xi(\gamma)\big) = \big(\gamma, f(\gamma)\big)^\top$ with

$$f(\gamma) = 2\gamma - \sum_{k=1}^{\infty} \frac{k\varepsilon_k}{2^k}.$$

LEMMA 3.1. *For any $\gamma \in \big[\frac{1}{2}, 1\big)$ we have*

$$f\big(\tfrac{1}{2}\gamma + \tfrac{1}{2}\big) = \tfrac{1}{2}\big(1 - \gamma + f(\gamma)\big) = f\big(\tfrac{1}{2}\gamma + \tfrac{1}{4}\big).$$

Proof. For the dyadic expansion $\gamma = \sum_{k=1}^{\infty} \varepsilon_k/2^k$ of $\gamma \in \big[\frac{1}{2}, 1\big)$ we necessarily have $\varepsilon_1 = 1$. Consequently,

$$\tfrac{1}{2}\gamma + \tfrac{1}{2} = \tfrac{1}{2} + \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^{k+1}} = \sum_{k=1}^{\infty} \frac{\varepsilon'_k}{2^k} \quad \text{with } \varepsilon'_k = \begin{cases} 1, & k = 1, \\ \varepsilon_{k-1}, & k \geqslant 2, \end{cases}$$

and

$$\tfrac{1}{2}\gamma + \tfrac{1}{4} = \tfrac{1}{4} + \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^{k+1}} = \sum_{k=1}^{\infty} \frac{\varepsilon''_k}{2^k} \quad \text{with } \varepsilon''_k = \begin{cases} 1, & k = 1, \\ 0, & k = 2, \\ \varepsilon_{k-1}, & k \geqslant 3. \end{cases}$$

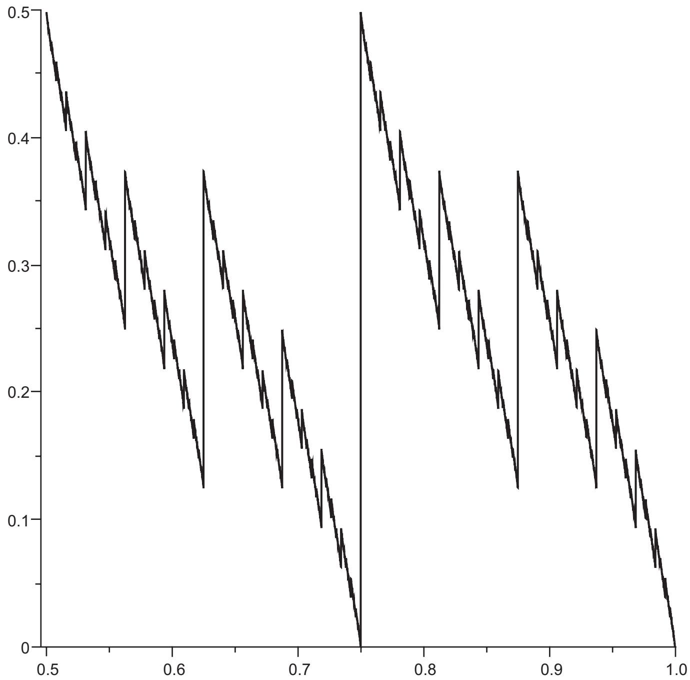

FIGURE 3. Image of $\tau^{-1}\big(G_\xi\big[\frac{1}{2},1\big)\big)$.
For better visibility the jumps are shown as vertical lines

It follows that

$$f\big(\tfrac{1}{2}\gamma+\tfrac{1}{2}\big)=2\big(\tfrac{1}{2}\gamma+\tfrac{1}{2}\big)-\sum_{k=1}^{\infty}\frac{k\varepsilon_k'}{2^k}=\gamma+\tfrac{1}{2}-\sum_{k=2}^{\infty}\frac{k\varepsilon_{k-1}}{2^k}$$

and

$$f\big(\tfrac{1}{2}\gamma+\tfrac{1}{4}\big)=2\big(\tfrac{1}{2}\gamma+\tfrac{1}{4}\big)-\sum_{k=1}^{\infty}\frac{k\varepsilon_k''}{2^k}=\gamma-\sum_{k=3}^{\infty}\frac{k\varepsilon_{k-1}}{2^k}=\gamma+\tfrac{1}{2}-\sum_{k=2}^{\infty}\frac{k\varepsilon_{k-1}}{2^k}.$$

This shows that $f\big(\frac{1}{2}\gamma+\frac{1}{2}\big)=f\big(\frac{1}{2}\gamma+\frac{1}{4}\big)=\gamma+\frac{1}{2}-\sum_{k=2}^{\infty}(k\varepsilon_{k-1})/2^k$, and furthermore we get

$$\begin{aligned}\gamma+\tfrac{1}{2}-\sum_{k=2}^{\infty}\frac{k\varepsilon_{k-1}}{2^k}&=\gamma+\tfrac{1}{2}-\tfrac{1}{2}\sum_{k=1}^{\infty}\frac{(k+1)\varepsilon_k}{2^k}\\&=\gamma+\tfrac{1}{2}-\tfrac{1}{2}\sum_{k=1}^{\infty}\frac{k\varepsilon_k}{2^k}-\tfrac{1}{2}\gamma=\tfrac{1}{2}\big(1-\gamma+f(\gamma)\big),\end{aligned}$$

concluding the proof. ∎

Let $\tau_0, \tau_1 : \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right] \to \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right]$ be the affine contractions given by

$$\tau_0(x,y) = \begin{pmatrix} \frac{1}{2}x + \frac{1}{4} \\ \frac{1}{2}(1 - x + y) \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix},$$

$$\tau_1(x,y) = \begin{pmatrix} \frac{1}{2}x + \frac{1}{2} \\ \frac{1}{2}(1 - x + y) \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Then τ_0 maps $\tau^{-1}\big(G_\xi\big(\big[\frac{1}{2}, 1\big)\big)\big)$ onto its left half and τ_1 onto its right half, i.e., for any $\gamma \in \big[\frac{1}{2}, 1\big)$ we have

$$\tau^{-1}\big(\tfrac{1}{2}\gamma + \tfrac{1}{4}, \xi\big(\tfrac{1}{2}\gamma + \tfrac{1}{4}\big)\big) = \tau_0\big(\gamma, f(\gamma)\big) = \tau_0\big(\tau^{-1}\big(\gamma, \xi(\gamma)\big)\big)$$

and

$$\tau^{-1}\big(\tfrac{1}{2}\gamma + \tfrac{1}{2}, \xi\big(\tfrac{1}{2}\gamma + \tfrac{1}{2}\big)\big) = \tau_1\big(\gamma, f(\gamma)\big) = \tau_1\big(\tau^{-1}\big(\gamma, \xi(\gamma)\big)\big),$$

which follows directly from Lemma 3.1. These contraction properties are illustrated in Figure 4 and show that the image $\tau^{-1}\big(G_\xi\big(\big[\frac{1}{2}, 1\big)\big)\big)$ can be generated by an iterated function system. By Hutchinson [23] there exists a unique non-empty com-

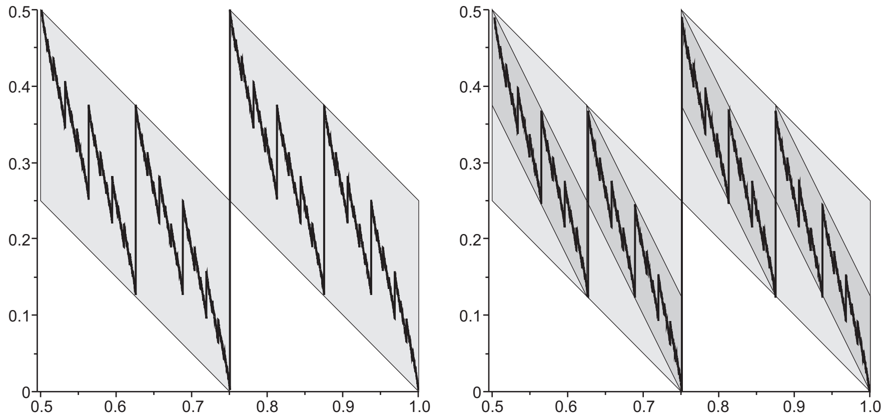

FIGURE 4. Contractions generating the image.
The highlighted parallelograms are $\tau_0(D)$, $\tau_1(D)$ with $D = \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right]$ (left) and their first iterates $\tau_0\big(\tau_0(D)\big)$, $\tau_1\big(\tau_0(D)\big)$, $\tau_0\big(\tau_1(D)\big)$, $\tau_1\big(\tau_1(D)\big)$ (right)

pact set $F \subseteq \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right]$, called the *attractor*, such that $F = \tau_0(F) \cup \tau_1(F)$, which fulfills

$$F = \bigcap_{r=1}^{\infty} \bigcup_{(i_1,\ldots,i_r)\in\{0,1\}^r} \tau_{i_r} \circ \ldots \circ \tau_{i_1}\big(\big[\tfrac{1}{2}, 1\big] \times \big[0, \tfrac{1}{2}\big]\big).$$

In fact, for any $(i_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ the iterated contractions $\tau_{i_r} \circ \ldots \circ \tau_{i_1}$ applied to the square $\left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right]$ converge to a single point of F as $r \to \infty$ and every element of F can be obtained in this way. More precisely, our construction shows that for $\gamma \in \big[\frac{1}{2}, 1\big)$ with dyadic expansion $\gamma = \sum_{k=1}^{\infty} \varepsilon_k / 2^k$ we have $\varepsilon_1 = 1$ and

$$d\Big(\tau_{\varepsilon_r} \circ \ldots \circ \tau_{\varepsilon_2}\big(\big[\tfrac{1}{2}, 1\big] \times \big[0, \tfrac{1}{2}\big]\big), \tau^{-1}\big(\gamma, \xi(\gamma)\big)\Big) \to 0$$

as $r \to \infty$, where $d(A,x) = \inf\{\|y-x\| : y \in A\}$ for $A \subseteq \mathbb{R}^2$ and $x \in \mathbb{R}^2$. Since we required $\varepsilon_k = 0$ for infinitely many $k \in \mathbb{N}$, the only possible limit points of F missing are those with $\tau_{\varepsilon_k} = \tau_1$ for all but finitely many $k \geqslant 2$. For these necessarily $\gamma = \frac{1}{2} + \sum_{k=2}^{\infty} \varepsilon_k / 2^k \in \big(\frac{1}{2}, 1\big]$ is a dyadic rational and we have

$$d\Big(\tau_{\varepsilon_r} \circ \ldots \circ \tau_{\varepsilon_2}\big(\big[\tfrac{1}{2}, 1\big] \times \big[0, \tfrac{1}{2}\big]\big), \tau^{-1}\big(\gamma, \xi(\gamma-)\big)\Big) \to 0$$

as $r \to \infty$. The above arguments show that F is the closure of $\tau^{-1}\big(G_\xi\big(\big[\frac{1}{2}, 1\big)\big)\big)$ and since the dyadic rationals are countable, by elementary properties of the Hausdorff dimension and (3.4) we get

$$\dim_{\mathrm{H}} F = \dim_{\mathrm{H}} \tau^{-1}\big(G_\xi\big(\big[\tfrac{1}{2}, 1\big)\big)\big) = \dim_{\mathrm{H}} G_\xi\big(\big[\tfrac{1}{2}, 1\big]\big). \tag{3.5}$$

The same equality holds for upper and lower box-counting dimensions; see, e.g., page 44 in [15].

A common way to calculate the fractal dimension of the self-affine invariant set F is by means of the singular value function. For an overview of such methods we refer to [17]. The linear part of both affine mappings τ_0 and τ_1 is equal to the linear contraction with associated matrix

$$L = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1/2 \end{pmatrix}.$$

By induction one easily calculates for $r \in \mathbb{N}$

$$L^r = \begin{pmatrix} 1/2^r & 0 \\ -r/2^r & 1/2^r \end{pmatrix}$$

and the singular values of L^r are the positive roots of the eigenvalues of $(L^r)^\top L^r$ which can be calculated as

$$\alpha_1^{(r)} = \frac{1}{2^r}\sqrt{\frac{r^2+2+\sqrt{r^4+4r^2}}{2}} \quad \text{and} \quad \alpha_2^{(r)} = \frac{1}{2^r}\sqrt{\frac{r^2+2-\sqrt{r^4+4r^2}}{2}}.$$

These determine the singular value function of L^r for $r \in \mathbb{N}$ given by

$$\varphi^s(L^r) = \begin{cases} (\alpha_1^{(r)})^s & \text{for } 0 < s \leqslant 1, \\ \alpha_1^{(r)}(\alpha_2^{(r)})^{s-1} & \text{for } 1 < s \leqslant 2. \end{cases} \tag{3.6}$$

Now the affinity dimension of F is defined by

$$\begin{aligned} \dim_{\mathrm{A}} F &= \inf\Big\{s > 0 : \sum_{r=1}^{\infty} \sum_{(i_1,\ldots,i_r)\in\{0,1\}^r} \varphi^s(L_{i_r} \circ \ldots \circ L_{i_1}) < \infty\Big\} \\ &= \inf\Big\{s > 0 : \sum_{r=1}^{\infty} 2^r \varphi^s(L^r) < \infty\Big\}, \end{aligned}$$

where $L_0,\ L_1$ are the linear parts of the affine contractions τ_0, τ_1, respectively, and the last equality holds since $L_0 = L = L_1$ in our situation. The special form of the singular values $\alpha_1^{(r)}$, $\alpha_2^{(r)}$ of L^r together with (3.6) shows that $\dim_{\mathrm{A}} F = 1$.

Since the union $F = \tau_0(F) \cup \tau_1(F)$ is disjoint, by Proposition 2 in [16] we get a lower bound for the Hausdorff dimension of F:

$$\begin{aligned}\dim_{\mathrm{H}} F &\geqslant \inf\Big\{s > 0 : \sum_{r=1}^{\infty} \sum_{(i_1,\ldots,i_r)\in\{0,1\}^r} \big(\varphi^s\big((L_{i_r} \circ \ldots \circ L_{i_1})^{-1}\big)\big)^{-1} < \infty\Big\}\\ &= \inf\Big\{s > 0 : \sum_{r=1}^{\infty} 2^r \big(\varphi^s(L^{-r})\big)^{-1} < \infty\Big\}.\end{aligned}$$

Again, by induction one easily calculates for $r \in \mathbb{N}$

$$L^{-r} = \begin{pmatrix} 2^r & 0 \\ r \cdot 2^r & 2^r \end{pmatrix}$$

and the singular values of L^{-r} are

$$\beta_1^{(r)} = 2^r \sqrt{\frac{r^2 + 2 + \sqrt{r^4 + 4r^2}}{2}} \quad \text{and} \quad \beta_2^{(r)} = 2^r \sqrt{\frac{r^2 + 2 - \sqrt{r^4 + 4r^2}}{2}},$$

which shows that $\dim_{\mathrm{H}} F \geqslant 1$. Since by Proposition 1 in [17] we have

$$\dim_{\mathrm{H}} F \leqslant \underline{\dim}_{\mathrm{B}} F \leqslant \overline{\dim}_{\mathrm{B}} F \leqslant \dim_{\mathrm{A}} F,$$

the above calculations altogether show:

THEOREM 3.1. *We have* $\dim_{\mathrm{H}} G_\xi\big(\big[\frac{1}{2}, 1\big]\big) = 1 = \dim_{\mathrm{B}} G_\xi\big(\big[\frac{1}{2}, 1\big]\big)$.

This shows that the graph of ξ, being the limiting object of the Steinhaus sequence (considered as a possible sequence of total gains in repeated St. Petersburg games), is not exceptional concerning the Hausdorff or box-counting dimension of the sample graph $G_Y\big(\big[\frac{1}{2}, 1\big]\big)$ calculated in Section 2.

Acknowledgments. We are grateful to the referee for several helpful and detailed comments that have led to an improvement of our presentation.

REFERENCES

[1] A. Adler, *Generalized one-sided laws of the iterated logarithm for random variables barely with or without finite mean*, J. Theoret. Probab. 3 (1990), pp. 587–597.
[2] R. M. Blumenthal and R. K. Getoor, *A dimension theorem for sample functions of stable processes*, Illinois J. Math. 4 (1960), pp. 370–375.
[3] Y. S. Chow and H. Robbins, *On sums of independent random variables with infinite moments and "fair" games*, Proc. Natl. Acad. Sci. USA 47 (1961), pp. 330–335.

[4] S. Csörgő, *Rates of merge in generalized St. Petersburg games*, Acta Sci. Math. (Szeged) 68 (2002), pp. 815–847.

[5] S. Csörgő, *Merging asymptotic expansions in generalized St. Petersburg games*, Acta Sci. Math. (Szeged) 73 (2007), pp. 297–331.

[6] S. Csörgő, *Probabilistic approach to limit theorems for the St. Petersburg game*, Acta Sci. Math. (Szeged) 76 (2010), pp. 233–350.

[7] S. Csörgő and R. Dodunekova, *Limit theorems for the Petersburg game*, in: *Sums, Trimmed Sums and Extremes*, M. G. Hahn et al. (Eds.), Progr. Probab., Vol. 23, Birkhäuser, Boston 1991, pp. 285–315.

[8] S. Csörgő and P. Kevei, *Merging asymptotic expansions for cooperative gamblers in generalized St. Petersburg games*, Acta Math. Hungar. 121 (2008), pp. 119–156.

[9] S. Csörgő and G. Simons, *On Steinhaus' resolution of the St. Petersburg paradox*, Probab. Math. Statist. 14 (1993), pp. 157–172.

[10] S. Csörgő and G. Simons, *A strong law of large numbers for trimmed sums, with applications to generalized St. Petersburg games*, Statist. Probab. Lett. 26 (1996), pp. 65–73.

[11] S. Csörgő and G. Simons, *St. Petersburg games with the largest gains withheld*, Statist. Probab. Lett. 77 (2007), pp. 1185–1189.

[12] J. Dutka, *On the St. Petersburg paradox*, Arch. Hist. Exact Sci. 39 (1988), pp. 13–39.

[13] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, Cambridge 1985.

[14] K. J. Falconer, *The Hausdorff dimension of self-affine fractals*, Math. Proc. Cambridge Philos. Soc. 103 (1988), pp. 339–350.

[15] K. J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, Chichester 1990.

[16] K. J. Falconer, *The dimension of self-affine fractals II*, Math. Proc. Cambridge Philos. Soc. 111 (1992), pp. 169–179.

[17] K. J. Falconer, *Dimension of self-affine sets: A survey*, in: *Further Developments in Fractals and Related Fields*, J. Barral and S. Seuret (Eds.), Trends Math., Vol. 13, Birkhäuser, Basel 2013, pp. 115–134.

[18] I. Fazekas, *Merging to semistable processes*, Theory Probab. Appl. 56 (2012), pp. 621–633.

[19] W. Feller, *Note on the law of large numbers and "fair" games*, Ann. Math. Statist. 16 (1945), pp. 301–304.

[20] W. Feller, *An Introduction to Probability Theory and Its Applications, Vol. 1*, third edition, Wiley, New York 1968.

[21] A. Gut, *Limit theorems for a generalized St. Petersburg game*, J. Appl. Probab. 47 (2011), pp. 752–760. Correction (2013), http://www2.math.uu.se/~allan/86correction.pdf.

[22] J. Hawkes, *Measure function properties of the asymmetric Cauchy process*, Mathematika 17 (1970), pp. 68–78.

[23] J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. 30 (1981), pp. 713–747.

[24] G. Jorland, *The Saint Petersburg paradox 1713–1937*, in: *The Probabilistic Revolution, Volume 1: Ideas in History*, L. Krüger et al. (Eds.), MIT Press, Cambridge, MA, 1987, pp. 157–190.

[25] J.-P. Kahane, *Some Random Series of Functions,* second edition, Cambridge University Press, Cambridge 1985.

[26] O. Kallenberg, *Foundations of Modern Probability*, Springer, New York 1997.

[27] P. Kern and L. Wedrich, *The Hausdorff dimension of operator semistable Lévy processes*, J. Theoret. Probab. (2013), to appear.

[28] A. Martin-Löf, *A limit theorem which clarifies the "Petersburg paradox"*, J. Appl. Probab. 22 (1985), pp. 634–643.

[29] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, Cambridge 1995.

[30] M. M. Meerschaert and H.-P. Scheffler, *Limit Distributions for Sums of Independent Random Vectors*, Wiley, New York 2001.

[31] M. M. Meerschaert and Y. Xiao, *Dimension results for sample paths of operator stable Lévy processes*, Stochastic Process. Appl. 115 (2005), pp. 55–75.
[32] G. Pap, *The accuracy of merging approximations in generalized St. Petersburg games*, J. Theoret. Probab. 24 (2011), pp. 240–270.
[33] W. E. Pruitt and S. J. Taylor, *Sample path properties of processes with stable components*, Z. Wahrsch. verw. Gebiete 12 (1969), pp. 267–289.
[34] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge 1999.
[35] H. Steinhaus, *The so-called Petersburg paradox*, Colloq. Math. 2 (1949), pp. 56–58.
[36] S. J. Taylor, *Sample path properties of a transient stable process*, J. Math. Mech. 16 (1967), pp. 1229–1246.
[37] I. Vardi, *The limiting distribution of the St. Petersburg game*, Proc. Amer. Math. Soc. 123 (1995), pp. 2875–2882.
[38] I. Vardi, *The St. Petersburg game and continued fractions*, C. R. Acad. Sci. Paris Sér. I Math. 324 (1997), pp. 913–918.
[39] W. Whitt, *Some useful functions for functional limit theorems*, Math. Oper. Res. 5 (1980), pp. 67–85.
[40] W. Whitt, *Stochastic-Process Limits: An Introduction to Stochastic-Process Limits and Their Application to Queues*, Springer, New York 2002.
[41] Y. Xiao, *Random fractals and Markov processes*, in: *Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot*, M. L. Lapidus and M. van Frankenhuijsen (Eds.), AMS, Providence 2004, pp. 261–338.

Peter Kern
Institute of Mathematics
Heinrich-Heine-Universität Düsseldorf
Universitätsstr. 1
40225 Düsseldorf, Germany
E-mail: kern@math.uni-duesseldorf.de

Lina Wedrich
Institute of Mathematics
Heinrich-Heine-Universität Düsseldorf
Universitätsstr. 1
40225 Düsseldorf, Germany
E-mail: wedrich@math.uni-duesseldorf.de

Received on 29.5.2013;
revised version on 23.11.2013