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REFINED SOLUTIONS OF TIME INHOMOGENEOUS OPTIMAL STOPPING PROBLEM AND ZERO-SUM GAME VIA DIRICHLET FORM

BY

YIPENG YANG (HOUSTON, TEXAS)

Abstract. The properties of value functions of time inhomogeneous optimal stopping problem and zero-sum game (Dynkin game) are studied through time dependent Dirichlet form. Under the absolute continuity condition on the transition function of the underlying process and some other assumptions, the refined solutions without exceptional starting points are proved to exist, and the value functions of the optimal stopping problem and zero-sum game, which belong to certain functional spaces, are characterized as the solutions of some variational inequalities, respectively.

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1. INTRODUCTION

Let $\mathbf{M} = (X_t, P_{(s,x)})$ be a right continuous Markov process with left limit on a locally compact separable metric space $\mathbb{X}$, and assume that $\mathbf{M}$ is not necessarily time homogeneous. For two finely continuous functions g, h on $[0, \infty) \times \mathbb{X}$ and a constant $\alpha > 0$, define the return function of a stopping problem:

(1.1) $$J_{(s,x)}(\sigma) = E_{(s,x)}\big(e^{-\alpha\sigma} g(s+\sigma, X_{s+\sigma})\big),$$

and the return function of a zero-sum game:

(1.2) $$J_{(s,x)}(\tau, \sigma) = E_{(s,x)}\big[e^{-\alpha(\tau\wedge\sigma)}\big(g(s+\sigma, X_{s+\sigma})I_{\tau>\sigma} + h(s+\tau, X_{s+\tau})I_{\tau\leqslant\sigma}\big)\big].$$

The values of the above stopping problem and zero-sum game are defined as $\tilde{e}_g = \sup_\sigma J_{(s,x)}(\sigma)$ and $\tilde{\tilde{w}} = \sup_\sigma \inf_\tau J_{(s,x)}(\tau, \sigma)$, respectively. This kind of optimal stopping problems has been continually developed due to its broad applications in finance, resource control and production management.

In the time homogeneous case, where $\mathbf{M}, g, h$ are all time homogeneous, it is well known that except on an appropriate properly exceptional set N of x, $\tilde{e}_g(x)$

is a quasi-continuous version of the solution of a variational inequality problem formulated in terms of the Dirichlet form, see Nagai [7]. Furthermore, $\tilde{e}_g(x) = E_x[e^{-\alpha\hat{\sigma}} g(X_{\hat{\sigma}})]$, $x \notin N$, where $\hat{\sigma} = \inf\{t > 0 : \tilde{e}_g(X_t) = g(X_t)\}$. This result was successfully extended by Zabczyk [15] to Dynkin game (zero-sum game) where $\tilde{\tilde{w}}(x), x \notin N$, was shown to be the quasi-continuous version of the solution of a certain variational inequality problem, and $\tilde{\tilde{w}}(x) = J_x(\hat{\tau}, \hat{\sigma})$, $x \notin N$, where $\hat{\sigma} = \inf\{t > 0 : \tilde{\tilde{w}}(X_t) = g(X_t)\}$, $\hat{\tau} = \inf\{t > 0 : \tilde{\tilde{w}}(X_t) = h(X_t)\}$. In their work, there always exists an exceptional set N of x. In 2006, Fukushima and Menda [2] showed that if the transition function of $\mathbf{M}$ is absolutely continuous with respect to the underlying measure $\mathbf{m}$, then the above-mentioned results hold for all initial points x, and $\tilde{e}_g$ and $\tilde{\tilde{w}}$ being finely continuous solve the respect variational inequality problem. Furthermore, as a characteristic of regularity, $\tilde{e}_g$ and $\tilde{\tilde{w}}$ were shown belonging to certain functional spaces, respectively.

However, more work is needed to extend these results to the time inhomogeneous case, especially the characteristics of the value functions. Using the time dependent Dirichlet form (generalized Dirichlet form) with $\mathbf{M}$ being a diffusion, Oshima [9] showed that under some conditions, $\tilde{e}_g$ (also $\tilde{\tilde{w}}$) is finely and cofinely continuous with quasi-every starting point (s, x) of $\mathbf{M}$, and except on an exceptional set N, $\tilde{e}_g$ (also $\tilde{\tilde{w}}$) is characterized as a version of the solution of a variational inequality problem, while a similar regularity result was obtained, see, e.g., Theorem 1.2 in [9].

Another approach to study the properties of the value functions of optimal stopping problem and Dynkin game is the penalty method, see, e.g., [13] and [14]. Recently, Palczewski and Stettner [10], [11] used the penalty method to characterize the continuity of the value functions of a time inhomogeneous optimal stopping problem as well as the Dynkin game, although no further regularity result other than continuity was investigated. In their works [10], [11], the underlying process $\mathbf{M}$ is assumed to satisfy the weak Feller continuity property. Lamberton [5] derived the continuity property of the value function of a one-dimensional optimal stopping problem, and the value function was characterized as the unique solution of a variational inequality in the sense of distributions. However, that result was difficult to be extended to multidimensional diffusions.

In this paper, through the time dependent Dirichlet form, it is shown that under the absolute continuity condition on the transition probability function p_t and some other assumptions, the value functions of the optimal stopping problem and the zero-sum game, with any starting point (s, x), are finely continuous and characterized as solutions of certain variational inequality problems, and the value functions do belong to, respectively, the functional spaces $\mathscr{W}$ as in (2.1), and $\mathscr{J}$ as in (3.17). Further continuity properties are also discussed. This result is then applied in Section 4 to the time inhomogeneous optimal stopping problem and zero-sum game where the underlying process is a multidimensional time inhomogeneous Itô diffusion.

2. TIME DEPENDENT DIRICHLET FORM

In this section we define the settings for the time dependent Dirichlet form that are similar to those in [9], although some results from [12], whose notions are different, will be used later. Let $\mathbb{X}$ be a locally compact separable metric space and $\mathbf{m}$ be a positive Radon measure on $\mathbb{X}$ with full support. For each $t \geqslant 0$, define $(E^{(t)}, F)$ as an $\mathbf{m}$-symmetric Dirichlet form on $H = L^2(\mathbb{X}; \mathbf{m})$ with a sector constant independent of t, and for any $u \in F$ we assume that $E^{(t)}(u,u)$ is a measurable function of t and satisfies

$$\lambda^{-1}\|u\|_F^2 \leqslant E_1^{(t)}(u,u) \leqslant \lambda\|u\|_F^2$$

for some constant $\lambda > 0$, where $E_\alpha^{(t)}(u,v) = E^{(t)}(u,v) + \alpha(u,v)_{\mathbf{m}},\ \alpha > 0$, and the F-norm is defined as $\|u\|_F^2 = E_1^{(0)}(u,u)$. We also assume that F is regular in the usual sense [3].

Define F' as the dual space of F. Then it can be seen that $F \subset H = H' \subset F'$. For each t, there exists an operator $L^{(t)}$ from F to F' such that

$$-(L^{(t)}u, v) = E^{(t)}(u,v), \quad u, v \in F.$$

Further, the F'-norm is defined as

$$\|v\|_{F'} = \sup_{\|u\|_F = 1} \{(v,u)\},$$

where (v,u) denotes the canonical coupling between $v \in F'$ and $u \in F$.

Define the spaces

$$\mathscr{H} = \{\varphi(t,\cdot) \in H : \|\varphi\|_{\mathscr{H}} < \infty\},$$

where

$$\|\varphi\|_{\mathscr{H}}^2 = \int_{\mathbb{R}} \|\varphi(t,\cdot)\|_H^2 dt,$$

and

$$\mathscr{F} = \{\varphi(t,\cdot) \in F : \|\varphi\|_{\mathscr{F}} < \infty\},$$

where

$$\|\varphi\|_{\mathscr{F}}^2 = \int_{\mathbb{R}} \|\varphi(t,\cdot)\|_F^2 dt.$$

Clearly, $\mathscr{F} \subset \mathscr{H} = \mathscr{H}' \subset \mathscr{F}'$ densely and continuously, where $\mathscr{H}', \mathscr{F}'$ are the dual spaces of $\mathscr{H}, \mathscr{F}$, respectively.

For any $\varphi \in \mathscr{F}$, considering φ to be a function of $t \in \mathbb{R}$ with values in F, the distribution derivative $\partial\varphi/\partial t$ is considered to be a function of $t \in \mathbb{R}$ with values in F' such that

$$\int_{\mathbb{R}} \frac{\partial\varphi}{\partial t}(t,\cdot)\xi(t)dt = -\int_{\mathbb{R}} \varphi(t,\cdot)\xi'(t)dt$$

for any $\xi \in C_0^\infty(\mathbb{R})$. Then we can define the space $\mathscr{W}$ as

$$\mathscr{W} = \left\{ \varphi(t,x) \in \mathscr{F} : \frac{\partial \varphi}{\partial t} \in \mathscr{F}', \|\varphi\|_{\mathscr{W}} < \infty \right\}, \tag{2.1}$$

where

$$\|\varphi\|_{\mathscr{W}}^2 = \left\| \frac{\partial \varphi}{\partial t} \right\|_{\mathscr{F}'}^2 + \|\varphi\|_{\mathscr{F}}^2.$$

Since $\mathscr{F}$ and $\mathscr{F}'$ are Banach spaces, it is easy to see that $\mathscr{W}$ is also a Banach space. Further, $\mathscr{W}$ is dense in $\mathscr{F}$.

We further define the bilinear form $\mathcal{E}$ by

$$\mathcal{E}(\varphi,\psi) = \begin{cases} -\left\langle \dfrac{\partial \varphi}{\partial t}, \psi \right\rangle + \int\limits_{\mathbb{R}} E^{(t)}\big(\varphi(t,\cdot),\psi(t,\cdot)\big)dt, & \varphi \in \mathscr{W}, \psi \in \mathscr{F}, \\ \left\langle \dfrac{\partial \psi}{\partial t}, \varphi \right\rangle + \int\limits_{\mathbb{R}} E^{(t)}\big(\varphi(t,\cdot),\psi(t,\cdot)\big)dt, & \varphi \in \mathscr{F}, \psi \in \mathscr{W}, \end{cases} \tag{2.2}$$

where $\langle \partial\varphi/\partial t, \psi\rangle = \int_{\mathbb{R}} (\partial\varphi/\partial t, \psi)dt$. We call $(\mathcal{E}, \mathscr{F})$ a *time dependent Dirichlet form* on $\mathscr{H}$ (see [9]).

As in [9] we may introduce the time space process $Z_t = \big(\tau(t), X_t\big)$ on the domain $\mathbb{Z} = \mathbb{R} \times \mathbb{X}$ with uniform motion $\tau(t)$. Then the resolvent $R_\alpha f$ of Z_t defined by

$$R_\alpha f(s,x) = E_{(s,x)}\Big(\int\limits_0^\infty e^{-\alpha t} f(s+t, X_{s+t})dt\Big), \quad (s,x) = z, \ \ f \in \mathscr{H}, \tag{2.3}$$

satisfies

$$\left(\alpha - \frac{\partial}{\partial t} - L^{(t)}\right) R_\alpha f(t,x) = f(t,x), \quad \forall t \geqslant 0. \tag{2.4}$$

Furthermore, $R_\alpha f$ is considered to be a version of $G_\alpha f \in \mathscr{W}$, where G_α is the resolvent associated with the form $\mathcal{E}_\alpha(,) = \mathcal{E}(,) + \alpha(,)_\nu$ and it satisfies

$$\mathcal{E}_\alpha(G_\alpha f, \varphi) = (f,\varphi)_\nu, \quad \forall \varphi \in \mathscr{F}, \tag{2.5}$$

where $d\nu(t,x) = dt d\mathbf{m}(x)$. We may write $(\cdot,\cdot)_\nu$ as $(\cdot,\cdot)_{\mathscr{H}}$ to indicate it as the inner product in $\mathscr{H}$.

We now define $\mathcal{A}$ as a bilinear form on $\mathscr{F} \times \mathscr{F}$ by

$$\mathcal{A}(\varphi,\psi) = \int\limits_{\mathbb{R}} E^{(t)}\big(\varphi(t,\cdot),\psi(t,\cdot)\big)dt;$$

then $(\mathcal{A}, \mathscr{F})$ is a coercive closed form on $\mathscr{H}$ with a sector constant, see, e.g., page 27 of [12]. Let $\mathcal{A}_\alpha(\varphi,\psi) = \mathcal{A}(\varphi,\psi) + \alpha(\varphi,\psi)_{\mathscr{H}}$. Then it can be seen that

$\mathcal{E}_\alpha(\varphi,\psi) = -\langle \partial\varphi/\partial t, \psi\rangle + \mathcal{A}_\alpha(\varphi,\psi)$ if $\varphi \in \mathscr{W}, \psi \in \mathscr{F}$, and $\mathcal{E}_\alpha(\varphi,\psi) = \langle \partial\psi/\partial t, \varphi\rangle + \mathcal{A}_\alpha(\varphi,\psi)$ if $\varphi \in \mathscr{F}, \psi \in \mathscr{W}$. Also notice that if $\varphi \in \mathscr{W}$, $\langle \partial\varphi/\partial t, \varphi\rangle = 0$, which implies $\mathcal{E}_\alpha(\varphi,\varphi) = \mathcal{A}_\alpha(\varphi,\varphi)$ in this case, see Corollary 1.1 in [8].

A function $\varphi \in \mathscr{F}$ is called an *α-potential* if $\mathcal{E}_\alpha(\varphi,\psi) \geqslant 0$ for any nonnegative function $\psi \in \mathscr{W}$. Denote by $\mathscr{P}_\alpha$ the family of all α-potential functions. A function $\varphi \in \mathscr{F}$ is called *α-excessive* if and only if $\varphi \geqslant 0$ and $nG_{n+\alpha}\varphi \leqslant \varphi$ a.e. for any $n \geqslant 0$. For any α-potential $\varphi \in \mathscr{F}$, define its α-excessive regularization as

$$\tilde{\varphi} = \lim_{n\to\infty} nR_{n+\alpha}\varphi.$$

For any function $g \in \mathscr{H}$, let

$$\mathscr{L}_g = \{\varphi \in \mathscr{F} : \varphi \geqslant g \ \ \nu \text{ a.e.}\};$$

then the following result holds (see Proposition 1.6 in [12] with a complete proof):

LEMMA 2.1. *For any $\epsilon > 0$ and $\alpha > 0$, there exists a unique function $g_\epsilon^\alpha \in \mathscr{W}$ associated with $g \in \mathscr{H}$ such that*

$$\mathcal{E}_\alpha(g_\epsilon^\alpha,\psi) = \frac{1}{\epsilon}\big((g_\epsilon^\alpha - g)^-,\psi\big)_{\mathscr{H}}, \quad \forall \psi \in \mathscr{F}, \tag{2.6}$$

where $(g_\epsilon^\alpha - g)^- = (g - g_\epsilon^\alpha) \vee 0$.

By Theorem 1.2 in [9] (also Proposition 1.7 in [12]), $e_g = \lim_{\epsilon\to 0} g_\epsilon^\alpha$ converges increasingly, strongly in $\mathscr{H}$ and weakly in $\mathscr{F}$, and furthermore, e_g is the minimal function of $\mathscr{P}_\alpha \cap \mathscr{L}_g$ satisfying

$$\mathcal{A}_\alpha(e_g,e_g) \leqslant \mathcal{E}_\alpha(e_g,\psi), \quad \forall \psi \in \mathscr{L}_g \cap \mathscr{W}. \tag{2.7}$$

Given any open set $A \in \mathbb{Z}$, the capacity of A is defined by

$$\mathrm{Cap}(A) = \mathcal{E}_\alpha(e_{I_A},\psi), \quad \psi \in \mathscr{W}, \psi = 1 \text{ a.e. on } A.$$

If $\varphi \in \mathscr{F}$ is an α-potential, then there exists a positive Radon measure μ_φ^α on $\mathbb{Z}$ (see eq. 1.12 in [9] or Lemma 3.2 in [8]) such that

$$\mathcal{E}_\alpha(\varphi,\psi) = \int_{\mathbb{Z}} \psi(z)d\mu_\varphi^\alpha(z) \quad \text{for any } \psi \in C_0(\mathbb{Z}) \cap \mathscr{W}.$$

By Lemma 1.4 in [9], μ_φ^α does not charge any Borel set of zero capacity. Put $e_A = e_{I_A}$ and $\mu_A^\alpha = \mu_{e_A}^\alpha$. Then the capacity of the set A can also be defined by

$$\mathrm{Cap}(A) = \mu_A^\alpha(\bar{A}).$$

The notion of the capacity is extended to any Borel set by the usual manner. A set is called *exceptional* if it is of zero capacity. If a statement holds everywhere except on an exceptional set N, we say the statement holds *quasi-everywhere* (q.e.).

3. REFINED SOLUTIONS OF TIME INHOMOGENEOUS OPTIMAL STOPPING PROBLEM AND ZERO-SUM GAME

In this section we will characterize the properties of the value functions $\tilde{e}_g(s,x) = \sup_\sigma J_{(s,x)}(\sigma)$ and $\tilde{\tilde{w}}(s,x) = \sup_\sigma \inf_\tau J_{(s,x)}(\tau,\sigma)$ of the time inhomogeneous optimal stopping problem and zero-sum game, respectively. We first assume that the transition probability function p_t of the process X_t satisfies the *absolute continuity* condition:

$$p_t(x,\cdot) \ll \mathbf{m}(\cdot), \quad \forall t. \tag{3.1}$$

In fact, the strong Feller property implies the absolute continuity condition on p_t, see, e.g., page 165 of [3].

3.1. The time inhomogeneous optimal stopping problem. Consider $\tilde{e}_g(z) = \sup_\sigma J_z(\sigma), z = (s,x)$, where

$$J_z(\sigma) = J_{(s,x)}(\sigma) = E_{(s,x)}\big(e^{-\alpha\sigma} g(s+\sigma, X_{s+\sigma})\big). \tag{3.2}$$

Oshima showed (see Theorem 3.1 in [9]) that if $g \in \mathscr{F}$ is quasi-continuous and $\mathscr{L}_g \cap \mathscr{W} \neq \emptyset$, then $\tilde{e}_g \in \mathscr{F}$ while e_g solves the variational inequality (2.7), and $\tilde{e}_g(z) = \sup_\sigma J_z(\sigma)$ for all $z \in \mathbb{Z}/N$. In what follows we give conditions under which $\tilde{e}_g \in \mathscr{W}$ and Oshima's result holds for all initial points z.

It is assumed that $g \in \mathscr{W}$ is a finely continuous function on $\mathbb{Z}$ such that

$$|g(t,x)| \leqslant \varphi(t,x) \tag{3.3}$$

for some finite α-excessive function $\varphi \in \mathscr{W}$ on $\mathbb{Z}$. We also assume that there exists a constant K such that

$$\sup_{\epsilon>0} \frac{1}{\epsilon} \|(g_\epsilon^\alpha - g)^-\|_{\mathscr{H}} \leqslant K\|g\|_{\mathscr{H}}, \tag{3.4}$$

where g_ϵ^α solves (2.6). In the rest of this section, the notion K_i for some index i denotes a constant.

LEMMA 3.1. *Under the assumptions* (3.3) *and* (3.4), $e_g \in \mathscr{W}$.

P r o o f. It has been proved that $e_g \in \mathcal{L}_g \cap P_\alpha$, and $e_g \in \mathscr{F}$, see Theorem 1.2 of [9] or Proposition 1.7 of [12]. Furthermore,

$$\sup_\epsilon \|g_\epsilon^\alpha - \varphi\|_{\mathscr{F}} \leqslant K_1\|\varphi\|_{\mathscr{W}},$$

and

$$\sup_\epsilon \|g_\epsilon^\alpha\|_{\mathscr{F}} \leqslant \sup_\epsilon \|g_\epsilon^\alpha - \varphi\|_{\mathscr{F}} + \|\varphi\|_{\mathscr{F}} \leqslant K_1\|\varphi\|_{\mathscr{W}} + \|\varphi\|_{\mathscr{F}},$$

where $\varphi \in \mathscr{L}_g \cap \mathscr{W}$. Now, since g_ϵ^α satisfies

$$\left\langle -\frac{\partial g_\epsilon^\alpha}{\partial t}, \psi \right\rangle + \mathcal{A}_\alpha(g_\epsilon^\alpha, \psi) = \frac{1}{\epsilon}\left((g_\epsilon^\alpha - g)^-, \psi\right)_{\mathscr{H}}, \quad \forall \psi \in \mathscr{F},$$

we have

$$\begin{aligned} \left\| \frac{\partial g_\epsilon^\alpha}{\partial t} \right\|_{\mathscr{F}'} &= \sup_{\|\psi\|_{\mathscr{F}}=1} \left\langle \frac{\partial g_\epsilon^\alpha}{\partial t}, \psi \right\rangle \\ &= \sup_{\|\psi\|_{\mathscr{F}}=1} \left(\mathcal{A}_\alpha(g_\epsilon^\alpha, \psi) - \frac{1}{\epsilon}\left((g_\epsilon^\alpha - g)^-, \psi\right)_{\mathscr{H}} \right) \\ &\leqslant \sup_{\|\psi\|_{\mathscr{F}}=1} \mathcal{A}_\alpha(g_\epsilon^\alpha, \psi) + \sup_{\|\psi\|_{\mathscr{F}}=1} \frac{1}{\epsilon}\left((g_\epsilon^\alpha - g)^-, \psi\right)_{\mathscr{H}}. \end{aligned} \tag{3.5}$$

By the sector condition, there exists a constant K_2 such that

$$|\mathcal{A}_\alpha(g_\epsilon^\alpha, \psi)| \leqslant K_2 \|g_\epsilon^\alpha\|_{\mathscr{F}} \|\psi\|_{\mathscr{F}},$$

and hence

$$\sup_{\|\psi\|_{\mathscr{F}}=1} \mathcal{A}_\alpha(g_\epsilon^\alpha, \psi) \leqslant K_2 \|g_\epsilon^\alpha\|_{\mathscr{F}}.$$

On the other hand, by the Cauchy–Schwarz inequality, we have

$$\frac{1}{\epsilon}\left((g_\epsilon^\alpha - g)^-, \psi\right)_{\mathscr{H}} \leqslant \frac{1}{\epsilon} \|(g_\epsilon^\alpha - g)^-\|_{\mathscr{H}} \|\psi\|_{\mathscr{H}},$$

and

$$\|\psi\|_{\mathscr{H}} \leqslant K_3 \|\psi\|_{\mathscr{F}},$$

whence

$$\sup_{\|\psi\|_{\mathscr{F}}=1} \frac{1}{\epsilon}\left((g_\epsilon^\alpha - g)^-, \psi\right)_{\mathscr{H}} \leqslant \frac{1}{\epsilon} K_3 \|(g_\epsilon^\alpha - g)^-\|_{\mathscr{H}}.$$

Now, by taking $\sup_\epsilon$ of (3.5) and by (3.4), we get

$$\sup_\epsilon \left\| \frac{\partial g_\epsilon^\alpha}{\partial t} \right\|_{\mathscr{F}'} \leqslant K_2 K_1 \|\varphi\|_{\mathscr{W}} + K_2 \|\varphi\|_{\mathscr{F}} + K K_3 \|g\|_{\mathscr{H}} < \infty.$$

Therefore,

$$\begin{aligned} \sup_\epsilon \|g_\epsilon^\alpha\|_{\mathscr{W}} &= \sup_\epsilon \left(\left\| \frac{\partial g_\epsilon^\alpha}{\partial t} \right\|_{\mathscr{F}'} + \|g_\epsilon^\alpha\|_{\mathscr{F}} \right) \\ &\leqslant K_2 K_1 \|\varphi\|_{\mathscr{W}} + K_2 \|\varphi\|_{\mathscr{F}} + K K_3 \|g\|_{\mathscr{H}} + K_1 \|\varphi\|_{\mathscr{W}} + \|\varphi\|_{\mathscr{F}}, \end{aligned} \tag{3.6}$$

which indicates the uniform boundedness. As a consequence, there is $\eta \in \mathscr{W}$ and a subsequence of g_ϵ^α such that $\lim_{\epsilon\to 0} g_\epsilon^\alpha = \eta$ weakly in $\mathscr{W}$ by the Banach–Alaoglu theorem. Then, by the Banach–Saks theorem, the Cesàro mean of this subsequence of g_ϵ^α converges to η in $\mathscr{H}$, and since it has been shown that $\lim_{\epsilon\to 0} g_\epsilon^\alpha = e_g$ in $\mathscr{H}$, we obtain $\eta = e_g$ (see also the proof of Lemma I.2.12 in [6]). ∎

COROLLARY 3.1. *There exist constants K_4, K_5 such that $\|e_g\|_{\mathscr{W}} \leqslant K_4\|\varphi\|_{\mathscr{W}} + K_5\|g\|_{\mathscr{H}}$.*

P r o o f. This can be seen by (3.6) in the proof of Lemma 3.1 and the fact that $\|\varphi\|_{\mathscr{F}} \leqslant \|\varphi\|_{\mathscr{W}}$. ∎

It has been shown that $e_g = \lim_{\epsilon\to 0} g_\epsilon^\alpha$ converges increasingly, strongly in $\mathscr{H}$ and weakly in $\mathscr{F}$, and e_g is an α-potential dominating g that satisfies

$$\mathcal{A}_\alpha(e_g, e_g) \leqslant \mathcal{E}_\alpha(e_g, \psi), \quad \forall \psi \in \mathscr{L}_g \cap \mathscr{W},$$

see Theorem 1.2 in [9]. Furthermore, Theorem 3.1 in [9] says

$$\tilde{e}_g(z) = \sup_\sigma J_z(\sigma) = E_z\big(e^{-\alpha\sigma_B} g(Z_{s+\sigma_B})\big) \text{ q.e.,}$$

where the supremum is taken over all stopping times σ and $B = \{z : \tilde{e}_g(z) = g(z)\}$.

We can revise Theorem 1.2 of [9] and get the following result:

COROLLARY 3.2. *Under the assumptions* (3.3) *and* (3.4), *$e_g = \lim_{\epsilon\to 0} g_\epsilon^\alpha$ converges increasingly, strongly in $\mathscr{H}$, and weakly in both $\mathscr{F}$ and $\mathscr{W}$. Furthermore, e_g is the minimal function of $\mathscr{P}_\alpha \cap \mathscr{L}_g \cap \mathscr{W}$ satisfying*

$$\mathcal{E}_\alpha(e_g, e_g) \leqslant \mathcal{E}_\alpha(e_g, \psi), \quad \forall \psi \in \mathscr{L}_g \cap \mathscr{W}. \tag{3.7}$$

P r o o f. Now, since $e_g \in \mathscr{W}$, we have $\langle \partial e_g/\partial t, e_g\rangle = 0$ (see Lemma 1.1 of [8]). Hence $\mathcal{A}_\alpha(e_g, e_g) = \mathcal{E}_\alpha(e_g, e_g)$. The rest of the proof is the same as in [9]. ∎

Now we may revise Theorem 3.1 of [9] and combine with Theorem 1 of [2] to get the following result (refined solutions):

THEOREM 3.1. *Let $g(z) = g(t,x)$ be a finely continuous function satisfying* (3.3). *Assume* (3.4) *and the absolute continuity condition* (3.1). *Let $e_g \in \mathscr{L}_g \cap \mathscr{W}$ be the solution of* (3.7), *and $\tilde{e}_g$ be its α-excessive regularization. Then*

$$\tilde{e}_g(z) = \sup_\sigma J_z(\sigma), \quad \forall z = (s,x) \in \mathbb{Z}, \tag{3.8}$$

where $J_z(\sigma) = J_{(s,x)}(\sigma) = E_{(s,x)}\big(e^{-\alpha\sigma} g(s+\sigma, X_{s+\sigma})\big)$. Furthermore, let the set $B = \{z \in \mathbb{Z} : \tilde{e}_g(z) = g(z)\}$ and let σ_B be the first hitting time of B defined by $\sigma_B = \inf\{t > 0 : \tilde{e}_g(Z_{s+t}) = g(Z_{s+t})\}$. Then

$$\tilde{e}_g(z) = E_z[e^{-\alpha\sigma_B} g(Z_{s+\sigma_B})], \quad \forall z = (s,x) \in \mathbb{Z}. \tag{3.9}$$

P r o o f. Notice that $\varphi \wedge \tilde{e}_g$ is an α-potential dominating g, and $\tilde{e}_g$ is the smallest α-potential dominating g. We get $\tilde{e}_g \leqslant \varphi \wedge \tilde{e}_g \leqslant \varphi$ ν-a.e. and this implies the finiteness of $\tilde{e}_g$.

Now, because $e_g \geqslant g$ ν a.e., we have $nR_{n+\alpha}e_g(z) \geqslant nR_{n+\alpha}g(z)$ for all $z \in \mathbb{Z}$, $n > 0$, and the fact $\tilde{e}_g$ being α-excessive implies

$$\tilde{e}_g(z) \geqslant \lim_{n\to\infty} nR_{n+\alpha}g(z), \quad \forall z \in \mathbb{Z}.$$

By the absolute continuity condition and finiteness of g, we get

$$\lim_{n\to\infty} nR_{n+\alpha}g(z) = g(z), \quad \forall z \in \mathbb{Z},$$

and therefore $\tilde{e}_g(z) \geqslant g(z)$ for all $z \in \mathbb{Z}$. Then we have

$$\tilde{e}_g(z) \geqslant E_z\left(e^{-\alpha\sigma}\tilde{e}_g(Z_{s+\sigma})\right) \geqslant E_z\left(e^{-\alpha\sigma}g(Z_{s+\sigma})\right) \tag{3.10}$$

for any stopping time σ, which implies $\tilde{e}_g(z) \geqslant J_z(\sigma)$ for all $z \in \mathbb{Z}$. Hence $\tilde{e}_g(z) \geqslant \sup_\sigma J_z(\sigma)$ for all $z \in \mathbb{Z}$.

Since $e_g \in \mathscr{W}$ is an α-potential, by Theorem 4.2 in [8], there exists a positive Radon measure μ^α of finite energy such that

$$\mathcal{E}_\alpha(e_g, w) = \int_{\mathbb{Z}} w(z)\mu^\alpha(dz), \quad \forall w \in C_0(\mathbb{Z}) \cap \mathscr{W}, \tag{3.11}$$

and $\tilde{e}_g(z) = R_\alpha\mu^\alpha(z)$ which is defined by $R_\alpha\mu^\alpha(z) = \int_{\mathbb{Z}} r_\alpha(z,y)\mu^\alpha(dy)$, where $r_\alpha(z,y)$ is a suitable resolvent density. (The definition of finite energy integrals can also be found in Section 2.2 of [3].)

By the finiteness of e_g, $\|R_\alpha\mu^\alpha\|_\infty < \infty$. Considering Z_t the underlying process which satisfies the absolute continuity condition, we apply Theorem 3.1 in [1] in concluding that there exists a positive continuous additive functional A_t in the strict sense such that

$$\tilde{e}_g(z) = E_z\Big(\int_0^\infty e^{-\alpha t} dA_t\Big), \quad \forall z \in \mathbb{Z}.$$

Set $B = \{z \in \mathbb{Z} : \tilde{e}_g(z) = g(z)\}$. Then

$$\int_{B^c} \big(\tilde{e}_g(z) - g(z)\big)\mu^\alpha(dz) = \int_{\mathbb{Z}} \big(\tilde{e}_g(z) - g(z)\big)\mu^\alpha(dz) = \mathcal{E}_\alpha(e_g, e_g - g).$$

Since e_g is an α-potential, and $e_g - g$ is nonnegative, $\mathcal{E}_\alpha(e_g, e_g - g) \geqslant 0$, which implies $\mathcal{E}_\alpha(e_g, e_g) - \mathcal{E}_\alpha(e_g, g) \geqslant 0$. On the other hand, e_g satisfies (3.7), which implies $\mathcal{E}_\alpha(e_g, e_g) - \mathcal{E}_\alpha(e_g, g) \leqslant 0$. Now, it can be concluded that $\mathcal{E}_\alpha(e_g, e_g) - \mathcal{E}_\alpha(e_g, g) = 0$, and consequently $\mu^\alpha(B^c) = 0$. Further, we get

$$E_z\Big(\int_0^\infty e^{-\alpha t} I_{B^c}(Z_{s+t}) dA_t\Big) = R_\alpha(I_{B^c}\mu)(z) = 0, \quad \forall z \in \mathbb{Z}.$$

By the strong Markov property, we have for any stopping time $\sigma \leqslant \sigma_B$

$$\tilde{e}_g(z) = E_z\big[\int_0^\sigma e^{-\alpha t} dA_t\big] + E_z[e^{-\alpha\sigma}\tilde{e}_g(Z_{s+\sigma})], \tag{3.12}$$

and because

$$0 \leqslant E_z\big[\int_0^\sigma e^{-\alpha t} dA_t\big] \leqslant E_z\big(\int_0^\infty e^{-\alpha t} I_{B^c}(Z_{s+t}) dA_t\big) = 0,$$

we have $\tilde{e}_g(z) = E_z[e^{-\alpha\sigma}\tilde{e}_g(Z_{s+\sigma})]$, $\sigma \leqslant \sigma_B$. Replacing σ by σ_B and replacing $\tilde{e}_g(Z_{s+\sigma})$ by $g(Z_{s+\sigma_B})$, we get $\tilde{e}_g(z) = E_z[e^{-\alpha\sigma_B} g(Z_{s+\sigma_B})]$, which also implies the fine continuity of $\tilde{e}_g$, and this together with (3.10) completes the proof. ∎

COROLLARY 3.3. *If* $\mathbf{M}$ *is a diffusion process, then under the conditions in Theorem* 3.1, $\tilde{e}_g(z)$ *for all* $z \in \mathbb{Z}$ *is continuous along the sample paths.*

Proof. Oshima has shown that if $\mathbf{M}$ is a diffusion process, $\tilde{e}_g(z)$ is finely and cofinely continuous for q.e. z, which further implies that $\tilde{e}_g(z)$ is continuous along the sample paths, see page 573 of [9]. While under the conditions in Theorem 3.1, we showed that there does not exist the exceptional set of z, and (3.9) implies the continuity of $\tilde{e}_g$ along the sample paths for all $z \in \mathbb{Z}$. ∎

REMARK 3.1. *If* X_t *is a non-degenerate Itô diffusion,* $\tilde{e}_g(z)$ *becomes a continuous function. This gives an alternate proof of the continuity of the value function, while Palczewski and Stettner* [10] *used a penalty method with weak Feller assumption to prove it. Although the continuity of the value function* $\tilde{e}_g(z)$ *is not a surprising result, little is known about the regularity of* $\tilde{e}_g(z)$ *beyond continuity. In this paper some conditions are found to show that* $\tilde{e}_g(z) \in \mathscr{W}$. *That does not necessarily mean that* $\tilde{e}_g(z)$ *are differentiable, and in many cases they are not, but we do provide a path for the search of further smoothness result.*

In Palczewski and Stettner's works [10], [11], the optimal policy is to stop the game at the stopping time $\dot{\sigma}_B = \inf\{t \geqslant 0 : \tilde{e}_g(Z_{s+t}) \leqslant g(Z_{s+t})\}$ or, equivalently, $\dot{\sigma}_B = \inf\{t \geqslant 0 : \tilde{e}_g(Z_{s+t}) = g(Z_{s+t})\}$. Oshima showed that (see page 571 in [9])

$$E_z[e^{-\alpha\dot{\sigma}_B}\tilde{e}_g(Z_{s+\dot{\sigma}_B})] = E_z[e^{-\alpha\sigma_B}\tilde{e}_g(Z_{s+\sigma_B})] \text{ q.e.},$$

and concluded that the set of irregular points of the set B is exceptional.

Notice that $\dot{\sigma}_B \leqslant \sigma_B$. By Theorem 3.1 we can see that

$$\begin{aligned}\tilde{e}_g(z) &\geqslant E_z[e^{-\alpha\dot{\sigma}_B}\tilde{e}_g(Z_{s+\dot{\sigma}_B})] \geqslant E_z[e^{-\alpha\sigma_B}\tilde{e}_g(Z_{s+\sigma_B})] \\ &\geqslant E_z[e^{-\alpha\sigma_B} g(Z_{s+\sigma_B})] = \tilde{e}_g(z), \quad \forall z,\end{aligned}$$

which implies

$$E_z[e^{-\alpha\dot{\sigma}_B}\tilde{e}_g(Z_{s+\dot{\sigma}_B})] = E_z[e^{-\alpha\sigma_B}\tilde{e}_g(Z_{s+\sigma_B})], \quad \forall z,$$

and, as a byproduct, we get the following result:

COROLLARY 3.4. *Under the conditions in Theorem* 3.1, *there does not exist the exceptional set of irregular boundary points of B.*

Therefore, it is feasible to replace σ_B by $\dot{\sigma}_B$ in the results in the rest of this paper.

REMARK 3.2. *In condition* (3.4) *which is used to characterize the regularities of the value function of optimal stopping*, g_ϵ^α *solves a PDE which involves the generator of the stochastic process, and the part* $(g_\epsilon^\alpha - g)^-$ *involves the reward function* g. *Therefore*, *it links both the underlying process* $\mathbf{M}$ *and the reward function* g. *The regularity of the value function* $\tilde{e}_g(z)$ *certainly depends on both* $\mathbf{M}$ *and* g, *and it is natural to expect conditions on further smoothness result to be connected to both* $\mathbf{M}$ *and* g *in some way.*

3.2. The time inhomogeneous zero-sum game. In this section we will refine the solution of the two-obstacle problem (zero-sum game) in [9].

Let $g(t,x), h(t,x) \in \mathscr{W}$ be finely continuous functions satisfying

(3.13)
$$g(t,x) \leqslant h(t,x), \quad |g(t,x)| \leqslant \varphi(t,x), \quad |h(t,x)| \leqslant \psi(t,x), \quad \forall (t,x) \in \mathbb{Z},$$

where $\varphi, \psi \in \mathscr{W}$ are two bounded α-excessive functions. We also assume that g, h satisfy the condition (3.4). Suppose there exist bounded α-excessive functions $v_1(t,x), v_2(t,x) \in \mathscr{W}$ such that

$$g(t,x) \leqslant v_1(t,x) - v_2(t,x) \leqslant h(t,x), \quad \forall (t,x) \in \mathbb{Z}, \tag{3.14}$$

in which case we say g and h satisfy the *separability condition* [2].

Define the sequences of α-excessive functions $\{\varphi_n\}$ and $\{\psi_n\}$ inductively by

$$\varphi_0 = \psi_0 = 0, \psi_n = e_{\varphi_{n-1}-h}, \varphi_n = e_{\psi_n+g}, \quad n \geqslant 1.$$

Then the following holds:

LEMMA 3.2. *Assume that* (3.14) *is satisfied. Then* φ_n, ψ_n *are well defined and* $\lim_{n\to\infty}\varphi_n = \bar{\varphi}$, $\lim_{n\to\infty}\psi_n = \bar{\psi}$ *converge increasingly*, *strongly in* $\mathscr{H}$ *and weakly in both* $\mathscr{F}$ *and* $\mathscr{W}$.

Proof. We only need to show the convergence in $\mathscr{W}$ and the rest of this lemma is just Lemma 2.1 in [9]. Firstly, $\varphi_0 = 0 \leqslant v_1$ and $\varphi_0 \in \mathscr{W}$. Suppose $\varphi_{n-1} \in \mathscr{W}$ is well defined and satisfies $\varphi_{n-1} \leqslant v_1$. Then $\varphi_{n-1} - h \leqslant v_1 - h \leqslant v_2$.

Hence $\psi_n = e_{\varphi_{n-1}-h} \in \mathscr{W}$ is well defined by Lemma 3.1, and we also have $\psi_n \leqslant v_2$ since $e_{\varphi_{n-1}-h}$ is the smallest α-potential dominating $\varphi_{n-1} - h$. Now, since $\psi_n + g \leqslant v_2 + g \leqslant v_1$, it follows that $\varphi_n = e_{\psi_n+g} \in \mathscr{W}$ is well defined and is dominated by v_1.

Notice that $\varphi_0 \leqslant \varphi_1$. Suppose $\varphi_{n-1} \leqslant \varphi_n$. Then $\psi_n = e_{\varphi_{n-1}-h} \leqslant e_{\varphi_n-h} = \psi_{n+1}$, and consequently $\varphi_n = e_{\psi_n+g} \leqslant e_{\psi_{n+1}+g} = \varphi_{n+1}$. Also by Lemma 3.1 we get

$$\|\varphi_n\|_{\mathscr{W}} = \|e_{\psi_n+g}\|_{\mathscr{W}} \leqslant K_4\|v_1\|_{\mathscr{W}} + K_5\|\psi_n + g\|_{\mathscr{H}}.$$

Notice that $g \leqslant \psi_n + g \leqslant v_1$, which implies that $\|\psi_n + g\|_{\mathscr{H}}$ is uniformly bounded in n, and as a consequence, $\|\varphi_n\|_{\mathscr{W}}$ is uniformly bounded in n. In a similar manner we can show that $\|\psi_n\|_{\mathscr{W}}$ is uniformly bounded. The convergence of φ_n, ψ_n in $\mathscr{W}$ follows as in the proof of Lemma 3.1. ∎

COROLLARY 3.5. *Under the separability condition*, $\bar\varphi = e_{\bar\psi+g}$, $\bar\psi = e_{\bar\varphi-h}$, *and they satisfy*

$$\begin{aligned} &\mathcal{E}_\alpha(\bar\varphi, \bar\varphi) \leqslant \mathcal{E}_\alpha(\bar\varphi, w), \quad \forall w \in \mathscr{L}_{\bar\psi+g} \cap \mathscr{W}, \\ &\mathcal{E}_\alpha(\bar\psi, \bar\psi) \leqslant \mathcal{E}_\alpha(\bar\psi, w), \quad \forall w \in \mathscr{L}_{\bar\varphi-h} \cap \mathscr{W}. \end{aligned} \tag{3.15}$$

Proof. Since $\bar\varphi$ is an α-potential dominating $\bar\psi + g$, we get $e_{\bar\psi+g} \leqslant \bar\varphi$. On the other hand, $\bar\varphi = \lim_{n\to\infty} \varphi_n = \lim_{n\to\infty} e_{\psi_n+g} \leqslant e_{\bar\psi+g}$, and hence $\bar\varphi = e_{\bar\psi+g}$. Similarly, $\bar\psi = e_{\bar\varphi-h}$. The proof of (3.15) is immediate by Corollary 3.2. ∎

COROLLARY 3.6. *If a pair of α-excessive functions (V_1, V_2) satisfy $g \leqslant V_1 - V_2 \leqslant h$, then $\bar\varphi \leqslant V_1$, $\bar\psi \leqslant V_2$, and $\bar w := \bar\varphi - \bar\psi$ is the unique function in $\mathscr{J}$ satisfying*

$$\mathcal{E}_\alpha(\bar w, \bar w) \leqslant \mathcal{E}_\alpha(\bar w, w), \quad \forall w \in \mathscr{J},\ g \leqslant w \leqslant h, \tag{3.16}$$

where

$$\mathscr{J} = \{w = \varphi_1 - \varphi_2 + v : \varphi_1, \varphi_2 \in \mathscr{W} \text{ are } \alpha\text{-potentials}, v \in \mathscr{W}\}. \tag{3.17}$$

Proof. Clearly, $\varphi_{n-1} - h \leqslant \psi_n$ and $\psi_n + g \leqslant \bar\varphi_n$; hence $g \leqslant \bar\varphi - \bar\psi \leqslant h$. If g, h satisfy the separability condition with respect to V_1, V_2, then we would have $\varphi_n \leqslant V_1$ and $\psi_n \leqslant V_2$, and as a consequence $\bar\varphi \leqslant V_1$, $\bar\psi \leqslant V_2$.

Now (3.16) is equivalent to

$$\mathcal{E}_\alpha(\bar\varphi, \bar\varphi) + \mathcal{E}_\alpha(\bar\psi, \bar\psi) \leqslant \mathcal{E}_\alpha(\bar\varphi, w + \bar\psi) + \mathcal{E}_\alpha(\bar\psi, \bar\varphi - w), \qquad g \leqslant w \leqslant h,$$

which holds by (3.15). Suppose there are two solutions $\bar w_1, \bar w_2 \in \mathscr{J}$ satisfying (3.16). Notice that $\bar w_1 - \bar w_2 \in \mathscr{W}$ and $\left\langle \frac{\partial(\bar w_1 - \bar w_2)}{\partial t}, \bar w_1 - \bar w_2 \right\rangle = 0$, which implies

$$\left\langle \frac{\partial \bar w_1}{\partial t}, \bar w_2 \right\rangle + \left\langle \frac{\partial \bar w_2}{\partial t}, \bar w_1 \right\rangle = 0,$$

and consequently

$$\mathcal{A}_\alpha(\bar{w}_1, \bar{w}_2) + \mathcal{A}_\alpha(\bar{w}_2, \bar{w}_1) = \mathcal{E}_\alpha(\bar{w}_1, \bar{w}_2) + \mathcal{E}_\alpha(\bar{w}_2, \bar{w}_1).$$

Therefore,

$$\begin{aligned}\mathcal{A}_\alpha(\bar{w}_1 - \bar{w}_2, \bar{w}_1 - \bar{w}_2)& \\ &= \mathcal{A}_\alpha(\bar{w}_1, \bar{w}_1) + \mathcal{A}_\alpha(\bar{w}_2, \bar{w}_2) - \mathcal{A}_\alpha(\bar{w}_1, \bar{w}_2) - \mathcal{A}_\alpha(\bar{w}_2, \bar{w}_1)\\ &= \mathcal{A}_\alpha(\bar{w}_1, \bar{w}_1) + \mathcal{A}_\alpha(\bar{w}_2, \bar{w}_2) - \mathcal{E}_\alpha(\bar{w}_1, \bar{w}_2) - \mathcal{E}_\alpha(\bar{w}_2, \bar{w}_1) \leqslant 0,\end{aligned}$$

which implies that $\bar{w}_1 = \bar{w}_2$ a.e. ∎

Let $\tilde{\bar{\varphi}}, \tilde{\bar{\psi}}, \tilde{\bar{w}}$ be the α-excessive modifications of $\bar{\varphi}, \bar{\psi}, \bar{w}$, respectively. We further define for arbitrary pair of stopping times τ, σ the payoff function $J_z(\tau, \sigma)$ as follows:

$$J_z(\tau, \sigma) = E_z\big[e^{-\alpha(\tau\wedge\sigma)}\big(g(Z_{s+\sigma})I_{\tau>\sigma} + h(Z_{s+\tau})I_{\tau\leqslant\sigma}\big)\big], \quad z \in \mathbb{Z}. \tag{3.18}$$

Then we have the following result:

THEOREM 3.2. *Assume that the separability condition on g, h and conditions* (3.1) *and* (3.4) *hold true. Then there exists a finite finely continuous function $\tilde{\bar{w}}(z) \in \mathscr{J}$ satisfying* (3.16) *and the identity*

$$\tilde{\bar{w}}(z) = \sup_\sigma \inf_\tau J_z(\tau, \sigma) = \inf_\tau \sup_\sigma J_z(\tau, \sigma), \quad \forall z = (s, x) \in \mathbb{Z}, \tag{3.19}$$

where σ, τ range over all stopping times. Moreover, the pair $\hat{\tau}, \hat{\sigma}$ defined by

$$\hat{\tau} = \inf\{t > 0 : \bar{w}(Z_{s+t}) = h(Z_{s+t})\}, \quad \hat{\sigma} = \inf\{t > 0 : \bar{w}(Z_{s+t}) = g(Z_{s+t})\}$$

is the saddle point of the game in the sense that

$$J_z(\hat{\tau}, \sigma) \leqslant J_z(\hat{\tau}, \hat{\sigma}) \leqslant J_z(\tau, \hat{\sigma}), \quad z \in \mathbb{Z},$$

for all stopping times τ, σ.

P r o o f. We only need to prove (3.19). By Theorem 3.1, for any $z \in \mathbb{Z}$ we have

$$\begin{aligned}\tilde{\bar{\varphi}}(z) &= \sup_\sigma E_z[e^{-\alpha\sigma}(\tilde{\bar{\psi}} + g)(Z_{s+\sigma})] = E_z[e^{-\alpha\hat{\sigma}}(\tilde{\bar{\psi}} + g)(Z_{s+\hat{\sigma}})],\\ \tilde{\bar{\psi}}(z) &= \sup_\tau E_z[e^{-\alpha\tau}(\tilde{\bar{\varphi}} - H)(Z_{s+\tau})] = E_z[e^{-\alpha\hat{\tau}}(\tilde{\bar{\varphi}} - h)(Z_{s+\hat{\tau}})],\end{aligned} \tag{3.20}$$

and for any stopping times $\sigma \leqslant \hat{\sigma}$, $\tau \leqslant \hat{\sigma}$,

$$\tilde{\bar{\varphi}}(z) = E_z[e^{-\alpha\sigma}\tilde{\bar{\varphi}}(Z_{s+\sigma})], \quad \tilde{\bar{\psi}}(z) = E_z[e^{-\alpha\tau}\tilde{\bar{\psi}}(Z_{s+\tau})], \quad \forall z = (s, x) \in \mathbb{Z}.$$

By (3.12), we could take $\{e^{-\alpha t}\bar{\varphi}(Z_{s+t})\}$ and $\{e^{-\alpha t}\bar{v}(Z_{s+t})\}$ as nonnegative P_z-supermartingales. Therefore, for any $z \in \mathbb{Z}$ and any stopping times τ, σ, we have

$$\tilde{\bar{\varphi}}(z) \geqslant E_z[e^{-\alpha\sigma}\tilde{\bar{\varphi}}(Z_{s+\sigma})], \quad \tilde{\bar{\psi}}(z) \geqslant E_z[e^{-\alpha\tau}\tilde{\bar{\psi}}(Z_{s+\tau})].$$

Consequently, for any $z \in \mathbb{Z}$,

$$\begin{aligned}\tilde{\bar{w}}(z) &= \tilde{\bar{\varphi}}(z) - \tilde{\bar{\psi}}(z) \leqslant E_z[e^{-\alpha(\hat{\sigma}\wedge\tau)}\tilde{\bar{\varphi}}(Z_{s+\hat{\sigma}\wedge\tau})] - E_z[e^{-\alpha(\hat{\sigma}\wedge\tau)}\tilde{\bar{\psi}}(Z_{s+\hat{\sigma}\wedge\tau})] \\ &= E_z[e^{-\alpha(\hat{\sigma}\wedge\tau)}\tilde{\bar{w}}(Z_{s+\hat{\sigma}\wedge\tau})] \leqslant E_z\big[e^{-\alpha(\tau\wedge\hat{\sigma})}\big(g(Z_{s+\sigma})I_{\tau>\hat{\sigma}} + h(Z_{s+\tau})I_{\tau\leqslant\hat{\sigma}}\big)\big] \\ &= J_z(\tau, \hat{\sigma}),\end{aligned}$$

where the last inequality is due to the fact that $g(z) \leqslant \tilde{\bar{w}}(z) \leqslant h(z)$ for all $z \in \mathbb{Z}$ and due to (3.20). In a similar manner, we can prove that $\tilde{\bar{w}} \geqslant J_z(\hat{\tau}, \sigma)$, and this completes the proof. ■

If further $\mathbf{M}$ is a diffusion process, then it can be concluded that $\tilde{\bar{w}}(z)$ is continuous along the sample paths.

3.3. Time inhomogeneous optimal stopping problem and zero-sum game with holding cost. Usually the optimal stopping problem and zero-sum game involve a holding cost function $f \in \mathscr{H}$ (see, e.g., [4]) and the return functions become

$$J^f_{(s,x)}(\sigma) = E_{(s,x)}\Big(\int_0^\sigma e^{-\alpha t} f(s+t, X_{s+t})dt + e^{-\alpha\sigma} g(s+\sigma, X_{s+\sigma})\Big), \tag{3.21}$$

and

$$\begin{aligned}J^f_{(s,x)}(\sigma,\tau) = {} & E_{(s,x)}\Big(\int_0^{\sigma\wedge\tau} e^{-\alpha t} f(s+t, X_{s+t})dt\Big) \\ & + E_{(s,x)}\Big(e^{-\alpha(\sigma\wedge\tau)}\big(g(s+\sigma, X_{s+\sigma})I_{\sigma<\tau} + h(s+\tau, X_{s+\tau})I_{\tau\leqslant\sigma}\big)\Big),\end{aligned} \tag{3.22}$$

but this model can be essentially reduced to the classical cases by taking $\hat{g} = g - R_\alpha f$ and $\hat{h} = h - R_\alpha f$ instead of g and h respectively, where R_α is the resolvent and $R_\alpha f$ is considered as a version of $G_\alpha f \in \mathscr{W}$. We assume that conditions (3.3) and (3.4) also apply to $\hat{g}$ for the optimal stopping problem (and similarly conditions (3.13) and (3.14) apply to $\hat{g}, \hat{h}$ for the zero-sum game).

THEOREM 3.3. *Let g be a finely continuous function satisfying* (3.3). *Assume* (3.4) *on g and the absolute continuity condition* (3.1) *on p_t. Let $e^f_g \in \mathscr{W}$ be the solution of*

$$\mathcal{E}_\alpha(e^f_g, \psi - e^f_g) \geqslant (f, \psi - e^f_g)_{\mathscr{H}}, \quad \forall \psi \in \mathscr{L}_g \cap \mathscr{W}, \tag{3.23}$$

and let $\tilde{e}_g^f$ be its α-excessive regularization. Then

$$\tilde{e}_g^f(z) = \sup_\sigma J_z^f(\sigma), \quad \forall z = (s,x) \in \mathbb{Z}, \tag{3.24}$$

where $J_z^f(\sigma)$ is defined as in (3.21), *and $\tilde{e}_g^f(z)$ is finely continuous. Furthermore, let the set $B = \{z \in \mathbb{Z} : \tilde{e}_g^f(z) = g(z)\}$ and let σ_B be the first hitting time of B defined by $\sigma_B = \inf\{t > 0 : \tilde{e}_g^f(Z_{s+t}) = g(Z_{s+t})\}$. Then*

$$\tilde{e}_g^f(z) = E_z[e^{-\alpha\sigma_B} g(Z_{s+\sigma_B})]. \tag{3.25}$$

P r o o f. Define the function

$$J_z^{f0}(\sigma) = E_z\big(e^{-\alpha\sigma}\hat{g}(s+\sigma, X_{s+\sigma})\big),$$

where $\hat{g} = g - R_\alpha f$, and let $\tilde{e}_{\hat{g}}^f = \sup_\sigma J_z^{f0}(\sigma)$. Then, by Theorem 3.1, $e_{\hat{g}}^f$ solves

$$\mathcal{E}_\alpha(e_{\hat{g}}^f, \hat{\psi} - e_{\hat{g}}^f) \geqslant 0, \quad \forall \hat{\psi} \in \mathscr{L}_{\hat{g}} \cap \mathscr{W}, \tag{3.26}$$

and the optimal stopping time is defined by $\sigma_B = \inf\{t > 0 : \tilde{e}_{\hat{g}}^f(Z_{s+t}) = \hat{g}(Z_{s+t})\}$.

By Dynkin's formula,

$$E_{(s,x)}\Big(\int_0^\sigma e^{-\alpha t} f(s+t, X_{s+t})dt\Big) = R_\alpha f(s,x) - E_{(s,x)}\big(e^{-\alpha\sigma} R_\alpha f(s+\sigma, X_{s+\sigma})\big),$$

which leads to

$$J_z^f(\sigma) = J_z^{f0}(\sigma) + R_\alpha f(z),$$

and consequently $e_g^f(z) = e_{\hat{g}}^f(z) + R_\alpha f(z)$.

Now, let $e_{\hat{g}}^f(z) = e_g^f(z) - R_\alpha f(z)$, $\hat{\psi} = \psi - R_\alpha f$ in (3.26). Then we get

$$\mathcal{E}_\alpha(e_g^f - G_\alpha f, \psi - e_g^f) \geqslant 0. \tag{3.27}$$

Since $\mathcal{E}_\alpha(G_\alpha f, \psi - e_{\hat{g}}^f) = (f, \psi - e_{\hat{g}}^f)_{\mathscr{H}}$, this proves (3.23). Also notice that the optimal stopping time can be written as $\sigma_B = \inf\{t > 0 : \tilde{e}_g^f(Z_{s+t}) = g(Z_{s+t})\}$, and this completes the proof. ∎

Similarly, we can modify Theorem 3.2 and get the following result:

THEOREM 3.4. *Let g, h be finely continuous functions satisfying* (3.13) *and* (3.14). *Assume* (3.4) *on g, h and the absolute continuity condition* (3.1) *on p_t. Then there exists a finite finely continuous function $\tilde{\bar{w}}^f \in \mathscr{J}$, $g(z) \leqslant \tilde{\bar{w}}^f(z) \leqslant h(z)$, such that*

$$\mathcal{E}_\alpha(\bar{w}^f, w - \bar{w}^f) \geqslant (f, w - \bar{w}^f)_{\mathscr{H}}, \quad \forall w \in \mathscr{J},\ g \leqslant w \leqslant h, \tag{3.28}$$

and

$$\tilde{\bar{w}}^f(z) = \sup_\sigma \inf_\tau J_z^f(\tau,\sigma) = \inf_\tau \sup_\sigma J_z^f(\tau,\sigma), \quad \forall z = (s,x) \in \mathbb{Z}, \tag{3.29}$$

where $J_z^f(\tau,\sigma)$ *was given by* (3.22) *and* σ, τ *range over all stopping times. Moreover, the pair* $\hat{\tau}, \hat{\sigma}$ *defined by*

$$\hat{\tau} = \inf\{t > 0 : \bar{w}^f(Z_{s+t}) = h(Z_{s+t})\}, \quad \hat{\sigma} = \inf\{t > 0 : \bar{w}^f(Z_{s+t}) = g(Z_{s+t})\}$$

is the saddle point of the game in the sense that

$$J_z^f(\hat{\tau},\sigma) \leqslant J_z^f(\hat{\tau},\hat{\sigma}) \leqslant J_z^f(\tau,\hat{\sigma}), \quad z \in \mathbb{Z},$$

for all stopping times τ, σ.

As an extension of Corollary 3.6, we have the following:

COROLLARY 3.7. *The variational inequality* (3.28) *has a unique solution.*

Proof. The case where $f = 0$ was proved in Corollary 3.6. For a general $f \in \mathscr{H}$, notice again that $(f, w - \bar{w}^f)_{\mathscr{H}} = \mathcal{E}_\alpha(G_\alpha f, w - \bar{w}^f)$. Then we get

$$\mathcal{E}_\alpha\big(\bar{w}^f - G_\alpha f, (w - G_\alpha f) - (\bar{w}^f - G_\alpha f)\big) \geqslant 0, \quad \forall w \in \mathscr{J},\ g \leqslant w \leqslant h.$$

Let $\hat{\bar{w}}^f = \bar{w}^f - G_\alpha f$, $\hat{w} = w - G_\alpha f$, $\hat{g} = g - G_\alpha f$, $\hat{h} = h - G_\alpha f$. Then we obtain

$$\mathcal{E}_\alpha(\hat{\bar{w}}^f, \hat{w} - \hat{\bar{w}}^f) \geqslant 0, \quad \forall \hat{w} \in \mathscr{J},\ \hat{g} \leqslant \hat{w} \leqslant \hat{h},$$

which has a unique solution in view of Corollary 3.6. ∎

4. TIME INHOMOGENEOUS OPTIMAL STOPPING PROBLEM AND ZERO-SUM GAME OF ITÔ DIFFUSION

In this section we are concerned with a multidimensional time inhomogeneous Itô diffusion:

$$dX_t = b(t, X_t)dt + a(t, X_t)dB_t, \qquad X_s = x, \tag{4.1}$$

where, for $m \geqslant n$,

$$X_t = \begin{pmatrix} X_{1t} \\ \vdots \\ X_{nt} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad a = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}, \quad B_t = \begin{pmatrix} B_{1t} \\ \vdots \\ B_{mt} \end{pmatrix},$$

and $a_{i,j}, b_i, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$, are continuous functions of t and X_t. Define the square matrix $[A_{i,j}] = \mathbf{A} = \frac{1}{2}aa^T$. We assume $\mathbf{A}$ is uniformly non-degenerate, and a, b satisfy the usual Lipschitz conditions so that (4.1) has a unique strong solution. B_t in (4.1) is assumed to be the standard multidimensional Brownian motion. Thus we are given a system $(\Omega, \mathcal{F}, \mathcal{F}_t, X, \theta_t, P_x)$, where $(\Omega, \mathcal{F})$ is a measurable space, $X = X(\omega)$ is a mapping of Ω into $C(\mathbb{R}^n)$, $\mathcal{F}_t$ is the sigma algebra generated by X_s $(s \leqslant t)$, and θ_t is a shift operator in Ω such that $X_s(\theta_t\omega) = X_{s+t}(\omega)$. Here P_x $(x \in \mathbb{R})$ is a family of measures under which $\{X_t, t \geqslant 0\}$ is a diffusion with initial state x.

At each time t, define the infinitesimal generator $L^{(t)}$ as

$$L^{(t)}u(x) = \sum_{i=1}^{n} b_i(t,x)\frac{\partial u}{\partial x_i} + \sum_{i,j} A_{i,j}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j}. \tag{4.2}$$

Let the positive Radon measure $\mathbf{m}(dx) = \rho^{(t)}(x)dx$, where $\rho^{(t)}$ satisfies

$$\mathbf{A}\nabla\rho^{(t)} = \rho^{(t)}\mu, \quad \forall t, \tag{4.3}$$

and $\mu_i = b_i - \sum_{j=1}^{n} \partial A_{ji}/\partial x_j, i = 1, 2, \ldots, n$. Notice that when a and b in (4.1) are constants, $\rho^{(t)}$ reduces to

$$\rho^{(t)}(x) = e^{(\mathbf{A}^{-1}b)\cdot x}.$$

Thus the associated Dirichlet form $(E^{(t)}, F)$ densely embedded in $H = L^2(\mathbb{R}^n; \mathbf{m})$ is then given by

$$E^{(t)}(u,v) = \int_{\mathbb{R}^n} \nabla u(x) \cdot \mathbf{A}\nabla v(x)\mathbf{m}(dx), \quad u, v \in F, \tag{4.4}$$

where

$$F = \{u \in H : \ u \text{ is continuous}, \ \|u\|_F^2 = E_1^{(0)}(u,u) < \infty\}.$$

Now we can define the sets $\mathscr{F}, \mathscr{H}, \mathscr{W}$ in the same way as in Section 2, and define the time inhomogeneous Dirichlet form $\mathcal{E}, \mathcal{E}_\alpha$ as well. Furthermore, $(\mathcal{E}, \mathscr{F})$ has the local property, see Theorem 5.3 in [8].

Since X_t is a non-degenerate Itô diffusion, the absolute continuity condition on its transition function automatically holds, and for the same reason, the fine and cofine continuity notion can be changed to the usual continuity.

Let $f \in \mathscr{H}, g \in \mathscr{W}$ be continuous functions satisfying the conditions as in Section 3.3, and define the return function $J_z^f(\sigma)$ as in (3.21). Then we have the following result:

THEOREM 4.1. *Assume* (3.3) *and* (3.4) *on g and the absolute continuity condition* (3.1) *on p_t. Let $e_g^f \in \mathscr{W}$ be the solution of*

$$\mathcal{E}_\alpha(e_g^f, \psi - e_g^f) \geqslant (f, \psi - e_g^f)_{\mathscr{H}}, \quad \forall \psi \in \mathscr{L}_g \cap \mathscr{W}, \tag{4.5}$$

and let $\tilde{e}_g^f$ *be its* α*-excessive regularization. Then*

$$\tilde{e}_g^f(z) = \sup_{\sigma} J_z^f(\sigma), \quad \forall z = (s, x) \in \mathbb{R}^{n+1}, \tag{4.6}$$

where $J_z^f(\sigma)$ *is defined as in* (3.21), *and* $\tilde{e}_g^f(z)$ *is continuous. Furthermore*, *let the set* $B = \{z \in \mathbb{Z} : \tilde{e}_g^f(z) = g(z)\}$ *and let* σ_B *be the first hitting time of* B *defined by* $\sigma_B = \inf\{t > 0 : \tilde{e}_g^f(Z_{s+t}) = g(Z_{s+t})\}$. *Then*

$$\tilde{e}_g^f(z) = E_z[e^{-\alpha\sigma_B} g(Z_{s+\sigma_B})]. \tag{4.7}$$

For the zero-sum game of Itô diffusion with the return function $J_z^f(\sigma, \tau)$ as defined in (3.22), we have the following result:

THEOREM 4.2. *Let* g, h *be continuous functions satisfying* (3.13) *and* (3.14). *Assume* (3.4) *on* g, h *and the absolute continuity condition* (3.1) *on* p_t. *Then there exists a finite and continuous function* $\tilde{\bar{w}}^f \in \mathscr{J}$, $g(z) \leqslant \tilde{\bar{w}}^f(z) \leqslant h(z)$, *such that*

$$\mathcal{E}_\alpha(\bar{w}^f, w - \bar{w}^f) \geqslant (f, w - \bar{w}^f)_{\mathscr{H}}, \quad \forall w \in \mathscr{J}, \ g \leqslant w \leqslant h, \tag{4.8}$$

and

$$\tilde{\bar{w}}^f(z) = \sup_{\sigma} \inf_{\tau} J_z^f(\tau, \sigma) = \inf_{\tau} \sup_{\sigma} J_z^f(\tau, \sigma), \quad \forall z = (s, x) \in \mathbb{Z}, \tag{4.9}$$

where σ, τ *range over all stopping times and* $J_z^f(\sigma, \tau)$ *is defined in* (3.22). *Moreover*, *the pair* $\hat{\tau}, \hat{\sigma}$ *defined by*

$$\hat{\tau} = \inf\{t > 0 : \bar{w}^f(Z_{s+t}) = h(Z_{s+t})\}, \quad \hat{\sigma} = \inf\{t > 0 : \bar{w}^f(Z_{s+t}) = g(Z_{s+t})\}$$

is the saddle point of the game in the sense that

$$J_z^f(\hat{\tau}, \sigma) \leqslant J_z^f(\hat{\tau}, \hat{\sigma}) \leqslant J_z^f(\tau, \hat{\sigma}), \quad z \in \mathbb{Z},$$

for all stopping times τ, σ.

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REFERENCES

[1] M. Fukushima, *On two classes of smooth measures for symmetric Markov processes*, in: *Stochastic Analysis*, M. Métivier and S. Watanabe (Eds.), Lecture Notes in Math., Vol. 1322, Springer, 1988, pp. 17–27.

[2] M. Fukushima and K. Menda, *Refined solutions of optimal stopping games for symmetric Markov processes*, Technology Reports of Kansai University, 48 (2006), pp. 101–110.

[3] M. Fukushima, Y. Ōshima, and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, second edition, de Gruyter, Berlin–New York 2011.

[4] M. Fukushima and M. Taksar, *Dynkin games via Dirichlet froms and singular control of one-dimensional diffusions*, SIAM J. Control Optim. 41 (3) (2002), pp. 682–699.

[5] D. Lamberton, *Optimal stopping with irregular reward functions*, Stochastic Process. Appl. 119 (2009) pp. 3253–3284.

[6] Z.-M. Ma and M. Röckner, *An Introduction to the Theory of (Non-symmetric) Dirichlet Forms*, Springer, Berlin 1992.

[7] H. Nagai, *On an optimal stopping problem and a variational inequality*, J. Math. Soc. Japan 30 (1978), pp. 303–312.

[8] Y. Oshima, *Time dependent Dirichlet forms and related stochastic calculus*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 7 (2004), pp. 281–316.

[9] Y. Oshima, *On an optimal stopping problem of time inhomogeneous diffusion processes*, SIAM J. Control Optim. 45 (2) (2006), pp. 565–579.

[10] J. Palczewski and L. Stettner, *Finite horizon optimal stopping of time-discontinuous functionals with applications to impulse control with delay*, SIAM J. Control Optim. 48 (8) (2010), pp. 4874–4909.

[11] J. Palczewski and L. Stettner, *Stopping of functionals with discontinuity at the boundary of an open set*, Stochastic Process. Appl. 121 (2011), pp. 2361–2392.

[12] W. Stannat, *The Theory of Generalized Dirichlet Forms and Its Applications in Analysis and Stochastics*, Mem. Amer. Math. Soc., Vol. 142 (678), American Mathematical Stociety, 1999.

[13] L. Stettner, *Zero-sum Markov games with stopping and impulsive strategies*, Appl. Math. Optim. 9 (1982), pp. 1–24.

[14] L. Stettner, *Penalty method for finite horizon stopping problems*, SIAM J. Control Optim. 49 (3) (2011), pp. 1078–1099.

[15] J. Zabczyk, *Stopping games for symmetric Markov processes*, Probab. Math. Statist. 4 (2) (1984), pp. 185–196.

Yipeng Yang
Department of Mathematics
University of Houston–Clear Lake
Houston, Texas, 77058
E-mail: yangy@uhcl.edu

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