PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 35, Fasc. 1 (2015), pp. 15–30

INNOVATION AND FACTORIZATION OF THE DENSITY OF A REGULAR PC SEQUENCE*

BY

ANDRZEJ MAKAGON (HAMPTON)

Abstract. In this paper we study an innovation representation of a periodically correlated (PC) sequence and describe the factorization of the densities of a regular PC sequence generated by its innovation. As a byproduct we obtain a certain factorization of vector analytic functions which may be of interest in the theory of Hardy spaces.

2000 AMS Mathematics Subject Classification: Primary: 60G25; Secondary: 30H10.

Key words and phrases: Periodically correlated sequence, prediction, innovation process, factorization of analytic vector functions.

1. INTRODUCTION

In this paper a (*stochastic*) *sequence* $\big(x(n)\big)$ is a not identically zero sequence in some complex, separable Hilbert space $\mathcal{H}$, indexed by the set of integers Z. If $\big(x(n)\big)$ is a stochastic sequence, then the *covariance function* R_x of $\big(x(n)\big)$ is the function on $Z \times Z$ defined by $R_x(n,m) = \big(x(n), x(m)\big)$, where $(\cdot,\cdot)$ is the inner product in $\mathcal{H}$. Two sequences $\big(x(n)\big)$ and $\big(y(n)\big)$, in possibly different Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, are said to be *unitary equivalent* if $R_x(m,n) = R_y(m,n)$ for every $m, n \in Z$. Unitary equivalent sequences will be identified. This identification makes the space $\mathcal{H}$ where the values of $\big(x(n)\big)$ are physically located completely irrelevant. The symbol $(x|M)$ will denote the orthogonal projection of a vector $x \in \mathcal{H}$ onto a closed subspace M of $\mathcal{H}$.

Given a sequence $\big(x(n)\big)$ in $\mathcal{H}$, the following subspaces of $\mathcal{H}$ will be of interest:

$$M_x(n) = \overline{\mathrm{sp}}\,\{x(m) : m \leqslant n\} \quad \text{and} \quad M_x = M_x(\infty) = \overline{\mathrm{sp}}\,\{x(m) : m \in Z\},$$

where $\overline{\mathrm{sp}}\,\{N\}$ stands for the closed linear subspace of $\mathcal{H}$ spanned by linear combinations of vectors from N. A sequence $\big(x(n)\big)$ is called *regular* if $\bigcap_n M_x(n) = \{0\}$.

* This research was supported by ARO Contract No. W911NF-11-1-0051.

The main objective of the theoretical prediction theory is to describe the projections $\big(x(n+k)|M_x(n)\big)$ and the error of prediction $\big\|x(n+k) - \big(x(n+k)|M_x(n)\big)\big\|$. This can be achieved by computing innovation coefficients of the sequence $\big(x(n)\big)$. So far the prediction problem has been completely solved only for univariate stationary sequences (see, e.g., [5], [1], [10]). In particular, it is known that innovation coefficients of a regular stationary sequence $\big(x(n)\big)$ are the coefficients of an outer square-factor of the spectral density $\gamma_0'(t)$ of $\big(x(n)\big)$. The last observation connects the prediction problem with the theory of Hardy spaces and eventually leads to the solution.

The main purpose of this paper is to examine the factorization of the spectral densities of a periodically correlated sequence generated by its innovation representation, and this is done in Section 3. In Section 2 we give preliminary results. Section 4 contains remarks on the relation between an innovation of a PC sequence and an innovation of the corresponding multivariate stationary sequence, and an interpretation of our main theorem in terms of factorization of C^T-valued analytic functions.

We adopt the following notation. T will always be a positive integer, C will stand for the field of complex numbers, C^T will denote the Cartesian product of T copies of C. Elements of the Cartesian product C^T will be represented as row vectors $a = [a^0, a^1, \ldots, a^{T-1}]$ with coordinates labeled from 0 to $T-1$, and e_p, $p = 0, \ldots, T-1$, will denote the standard orthonormal basis in C^T. If $a \in C^T$ then a^t is the transpose of a, and a^* is the column vector whose k-th coordinate is the complex conjugate $\overline{a^k}$ of a^k. With this notation the standard scalar product of $a, b \in C^T$ can be expressed as ab^*. The interval $[0, 2\pi)$ will be treated as a group with addition modulo 2π. The symbol $L^2(C^T)$ will denote the Hilbert space of all C^T-valued functions on $[0, 2\pi)$ which are square integrable with respect to the Lebesgue measure du on $[0, 2\pi)$, and $L^2_T(C^T)$ will be the subspace of $L^2(C^T)$ consisting of all $2\pi/T$-periodic functions. $H^2(C^T)$ will stand for the subspace of $L^2(C^T)$ consisting of functions whose negative Fourier coefficients are zero, that is, functions from $L^2(C^T)$ of the form $h(u) = \sum_{k=0}^{\infty} e^{iku} h_k$, $h_k \in C^T$. Elements of $H^2(C^T)$ will be called *analytic functions*. $H^2_T(C^T) = H^2(C^T) \cap L^2_T(C^T)$ will stand for the space of $2\pi/T$-periodic analytic functions with values in C^T. Finally, $M_T(C)$ will denote the set of all $T \times T$ matrices with complex entries. The rows and columns of a matrix $A \in M_T(C)$ will be numbered from 0 to $T-1$.

If $m \in Z$ then $\mathrm{q}(m)$ and $\langle m \rangle$ will denote the quotient and the nonnegative remainder in division of m by T, so that $m = \mathrm{q}(m)T + \langle m \rangle$, $\mathrm{q}(m) \in Z$, $\langle m \rangle \in \{0, \ldots, T-1\}$. Other notation will be introduced later when needs occur.

2. MA REPRESENTATION

A sequence $\big(x(n)\big)$ in $\mathcal{H}$ is called *periodically correlated* (PC) with period T, or simply T-PC, if $\big(x(n)\big)$ is not identically zero and $R_x(m+T, n+T) = R_x(m, n)$ for every $m, n \in Z$. A PC sequence with $T = 1$ is called *stationary*. If

$\big(x(n)\big)$ is T-PC, then the mapping $W : x(n) \to x(n+T)$, $n \in Z$, extends linearly to the unitary operator in M_x, which is customarily called the T-*shift* of $\big(x(n)\big)$. For every $j, n \in Z$ define

$$b_j(n) = \frac{1}{T} \sum_{r=0}^{T-1} e^{-2\pi ijr/T} R_x(n+r, r), \quad j = 0, \ldots, T-1. \tag{2.1}$$

The correlation function R_x of a T-PC sequence $\big(x(n)\big)$ is completely determined by the sequences $\big(b_j(n)\big)$, namely

$$R_x(n+r, r) = \sum_{j=0}^{T-1} e^{2\pi ijr/T} b_j(n), \quad n, r \in Z. \tag{2.2}$$

It is well known (see, e.g., [2] or [4]) that for every j there is a complex finite measure γ_j on $[0, 2\pi)$ such that

$$b_j(n) = \int_0^{2\pi} e^{-int} \gamma_j(dt), \quad j = 0, \ldots, T-1, \ n \in Z. \tag{2.3}$$

Measures (γ_j) above are called the *spectral measures* of the T-PC sequence $\big(x(n)\big)$. We say that a PC sequence $\big(x(n)\big)$ has an *absolutely continuous spectrum* if each γ_j is absolutely continuous with respect to the Lebesgue measure du on $[0, 2\pi)$. If this is the case, then the Radon–Nikodym derivatives of γ_j with respect to du will be denoted by $\gamma_j'(u)$. By the *spectral density* γ_x' of the sequence $\big(x(n)\big)$ we will understand the family of T functions $\gamma_x' = (\gamma_j')$, $j = 0, \ldots, T-1$. It has been shown in [7] that if $\gamma_x' = (\gamma_j')$ is the spectral density of a T-PC sequence $\big(x(n)\big)$, then there is a function $g \in L^2(C^T)$ such that for every $j = 0, \ldots, T-1$

$$\gamma_j'(u) = \frac{1}{T} g(u) g(u + 2\pi j/T)^* \ du\text{-a.e.} \tag{2.4}$$

Any function $g \in L^2(C^T)$ satisfying (2.4) will be referred to as a *factor* of the spectral density γ_x' of $\big(x(n)\big)$. It is known that a T-PC sequence $\big(x(n)\big)$ is regular iff its spectrum is absolutely continuous and its spectral density γ_x' admits an analytic factor $g \in H^2(C^T)$ (see [7], Theorem 4.2).

DEFINITION 2.1. A T-PC sequence $\big(x(n)\big)$ is called a *moving average* (MA) if there exist an orthonormal system (ξ_n) in some Hilbert space $\mathcal{H} \supseteq M_x(n)$, and a set of scalars $\big(a_k(n)\big)$, $n, k \in Z$, such that the following conditions are satisfied:

(A1) for every $n \in Z$, $x(n) = \sum_{k=-\infty}^{\infty} a_k(n) \xi_{n-k}$,

(A2) the *shift operator* U *of* (ξ_n), defined as a unitary operator in $\mathcal{H}$ such that $U\xi_n = \xi_{n+1}$, $n \in Z$, has the property that for every $n \in Z$ it follows that $U^T x(n) = x(n+T)$.

If (A1) and (A2) are satisfied then a representation in (A1) will be referred to as a *moving average* or *MA representation* of $\big(x(n)\big)$. The condition (A2) says that U^T restricted to M_x is equal to the T-shift W of $\big(x(n)\big)$ and it implies that the MA coefficients $a_k(n)$ are T-periodic in n. Indeed, from (A2) it follows that

$$\sum_{k=-\infty}^{\infty} a_k(n+T)\xi_{n+T-k} = x(n+T) = U^T x(n) = \sum_{k=-\infty}^{\infty} a_k(n)\xi_{n+T-k},$$

and hence $a_k(n+T) = a_k(n)$.

Each MA representation of a T-PC sequence $\big(x(n)\big)$ generates a certain factor of γ'_x and *vice versa* – each factor of γ'_x produces an MA representation of $\big(x(n)\big)$. Before we prove it we will need the following lemma that was proved in [7] (Lemma 3.3) in a much more general setting. In the case of sequences with absolutely continuous spectra the proof is easy and we include it below.

LEMMA 2.1. *Let $g \in L^2(C^T)$. If*

$$y(n)(u) = e^{-inu}\left(\frac{1}{T}\sum_{j=0}^{T-1} e^{-2\pi ijn/T} g(u+2\pi j/T)\right), \tag{2.5}$$

then $\big(y(n)\big)$ is a T-PC sequence in $L^2(C^T)$ with absolutely continuous spectrum and for every $p = 0,\dots,T-1$

$$\gamma'_p(u) = \frac{1}{T} g(u) g(u+2\pi p/T)^*.$$

P r o o f. The fact that $\big(y(n)\big)$ is T-PC follows from the T-periodicity in n of the expression in parentheses. By definition (2.1),

$$b_p(n) = \frac{1}{T}\sum_{r=0}^{T-1} e^{-2\pi ipr/T}\int_0^{2\pi} y(n+r)(u)y(r)(u)^* dy.$$

Substituting (2.5) into the above we obtain

$$b_p(n) =$$

$$\frac{1}{T^3}\sum_{r=0}^{T-1}\sum_{j=0}^{T-1}\sum_{k=0}^{T-1} e^{-2\pi i(p-k+j)r/T}\int_0^{2\pi} e^{-in(u+2\pi j/T)} g(u+2\pi j/T) g(u+2\pi k/T)^* du.$$

The sum over r is zero except when $k = p + j$ modulo T, and in the latter case it is equal to T. Hence

$$b_p(n) = \frac{1}{T^2}\sum_{j=0}^{T-1}\int_0^{2\pi} e^{-in(u+2\pi j/T)} g(u+2\pi j/T) g\big(u+2\pi(p+j)/T\big)^* du.$$

Putting $t = u + 2\pi j/T$ in the integral we see that the value of the integral does not depend on j. Hence

$$b_p(n) = \frac{1}{T}\int_0^{2\pi} e^{-int} g(t) g(t + 2\pi p/T)^* dt.$$

Consequently, $\gamma_p(dt) = (1/T) g(t) g(t + 2\pi p/T)^* dt$. ∎

We will also use repeatedly the following simple observation.

LEMMA 2.2. *Let $\phi \in L^2(C^T)$, $\phi(u) = \sum_{k=-\infty}^{\infty} d_k e^{iku}$, $d_k \in C^T$, and let P be defined as*

$$P\phi(u) = \frac{1}{T}\sum_{j=0}^{T-1} \phi(u + 2\pi j/T).$$

Then $P\phi(u) = \sum_{q=-\infty}^{\infty} d_{qT} e^{iqTu}$.

P r o o f. Indeed, we have

$$P\phi(u) = \sum_{k=-\infty}^{\infty} d_k e^{iku} \left(\frac{1}{T} \sum_{j=0}^{T-1} e^{2\pi ijk/T} \right) = \sum_{q=-\infty}^{\infty} d_{qT} e^{iqTu}. \quad ∎$$

Lemma 2.2 shows that P is the orthogonal projection in $L^2(C^T)$ onto $L_T(C^T)$.

PROPOSITION 2.1. *Let $\big(x(n)\big)$ be a T-PC sequence with absolutely continuous spectrum, and let $\gamma'_x = (\gamma'_j)$ be its spectral density. Suppose that $g(u) = \sum_{k=-\infty}^{\infty} g_k e^{iku}$, $g_k = \sum_{p=0}^{T-1} g_k^p e_p \in C^T$, is a factor of γ'_x. Then there is an orthonormal system in some Hilbert space $\mathcal{H} \supseteq M_x$ such that*

$$x(n) = \sum_{k=-\infty}^{\infty} g_{k+\langle n-k\rangle}^{\langle n-k\rangle} \xi_{n-k}.$$

P r o o f. Define the sequence $\big(G(n)\big)$ of functions in $L^2(C^T)$ as follows:

$$G(n)(u) = \frac{1}{T}\sum_{j=0}^{T-1} e^{-in(u+2\pi j/T)} g(u + 2\pi j/T). \tag{2.6}$$

Using Lemma 2.2 for $\phi(u) = e^{-inu} g(u) = \sum_{m=-\infty}^{\infty} g_{n+m} e^{imu}$, we obtain

$$G(n)(u) = \sum_{q=-\infty}^{\infty} g_{n+qT} e^{iqTu} = \sum_{q=-\infty}^{\infty} \sum_{p=0}^{T-1} e^{iqTu} g_{n+qT}^p e_p. \tag{2.7}$$

Substituting first $m = -qT + p$, so that $q = -\mathrm{q}(m)$ and $p = \langle m \rangle$, and then $m = n - k$, we get

$$\begin{aligned} G(n)(u) &= \sum_{m=-\infty}^{\infty} e^{-i\mathrm{q}(m)Tu} g^{\langle m \rangle}_{n-\mathrm{q}(m)T} e_{\langle m \rangle} \\ &= \sum_{k=-\infty}^{\infty} g^{\langle n-k \rangle}_{k+\langle n-k \rangle} (e^{-i\mathrm{q}(n-k)Tu} e_{\langle n-k \rangle}), \end{aligned}$$

because $n - \mathrm{q}(n-k)T = n - (n-k) + \langle n-k \rangle = k + \langle n-k \rangle$. Note that each $G(n)$ is $2\pi/T$-periodic, and consequently $M_G \subseteq L^2_T(C^T)$. If we define $\zeta_n(u) = \big(1/\sqrt{2\pi}\big) e^{-i\mathrm{q}(n)Tu} e_{\langle n \rangle}$, then the last formula says that

$$G(n)(u) = \sqrt{2\pi} \sum_{k=-\infty}^{\infty} g^{\langle n-k \rangle}_{k+\langle n-k \rangle} \zeta_{n-k}(u). \tag{2.8}$$

The sequence (ζ_n) is an orthonormal system in $L^2_T(C^T)$ and the shift U of (ζ_n) satisfies $U^T G(n) = G(n+T)$, so (2.8) is an MA representation of $\big(G(n)\big)$ with MA coefficients $a_k(n) = g^{\langle n-k \rangle}_{k+\langle n-k \rangle}$, $k, n \in Z$. Since, by Lemma 2.1, $G(n)$ and $\big(x(n)\big)$ have the same density, the proposition is proved. ∎

PROPOSITION 2.2. *Let $\big(x(n)\big)$ be a T-PC MA sequence and let*

$$x(n) = \sum_{k=-\infty}^{\infty} a_k(n) \xi_{n-k}(u) = \sum_{m=-\infty}^{\infty} a_{n-m}(n) \xi_m(u)$$

be its MA representation. For any $k \in Z$ define $g_k = \big(1/\sqrt{2\pi}\big) \sum_{p=0}^{T-1} a_{k-p}(k) e_p$, and let

$$g(u) = \sum_{k=0}^{\infty} e^{iku} g_k.$$

Then $\big(x(n)\big)$ is absolutely continuous, and g is a factor of γ'_x.

P r o o f. The proof is just the reverse of an argument used in Proposition 2.1. Define

$$\zeta_n(u) = \big(1/\sqrt{2\pi}\big) e^{-i\mathrm{q}(n)Tu} e_{\langle n \rangle}. \tag{2.9}$$

Then (ζ_n) form an orthonormal system in the space $L^2_T(C^T)$. Let

$$G(n)(u) = \sum_{m=-\infty}^{\infty} a_{n-m}(n) \zeta_m(u).$$

Obviously, $\big(x(n)\big)$ and $\big(G(n)\big)$ have the same correlation, and hence the same spectrum. Substituting (2.9) into the above and writing $m = -qT + p$ we obtain

$$\begin{aligned} G(n)(u) &= \big(1/\sqrt{2\pi}\big) \sum_{m=-\infty}^{\infty} a_{n-m}(n) e^{-iq(m)Tu} e_{\langle m \rangle} \\ &= \sum_{q=-\infty}^{\infty} e^{iqTu} \Big[\big(1/\sqrt{2\pi}\big) \sum_{p=0}^{T-1} a_{n-p+qT}(n) e_p\Big] = \sum_{q=-\infty}^{\infty} e^{iqTu} g_{n+qT}. \end{aligned}$$

From Lemma 2.2 we conclude that

$$G(n)(u) = e^{-inu} \frac{1}{T} \sum_{j=0}^{T-1} e^{-2\pi ijn/T} g(u + 2\pi j/T).$$

By Lemma 2.1, g is a factor of the spectral density of $\big(G(n)\big)$, and hence also of γ_x'. ∎

Matrix visualization. The relationship between MA coefficients $\big(a_k(n)\big)$ of $\big(x(n)\big)$ and the coefficients $g_k = [g_k^p]$ of the corresponding factor $g(u) = \sum g_k e^{iku}$ of γ_x',

$$\text{(2.10)} \qquad a_k(n) = \big(\sqrt{2\pi}\big) g_{k+\langle n-k \rangle}^{\langle n-k \rangle}, \qquad g_k^p = \big(1/\sqrt{2\pi}\big) a_{k-p}(k),$$

can be easily visualized in terms of matrices. First arrange MA coefficients $\big(a_k(n)\big)$ of $\big(x(n)\big)$ into a $\{0, \dots, T-1\} \times Z$ matrix A with rows labeled from $i = 0$ (top) to $T-1$ (bottom), and columns j labeled by Z (from left to right) in such a way that the i,j-th entry of A is $A^{i,j} = a_{i-j}(i)$. Recall that the sequences $a_k(n)$ are T-periodic in n, so the matrix A contains all MA coefficients of $\big(x(n)\big)$. If now $x = [x(n)]_{T-1}^{0}$ and $\xi = [\xi_n]_{\infty}^{-\infty}$ are column vectors with coordinates labeled as shown, then the representation (A1) can be written as $x = A\xi$. Let further $A(k)$ be the $T \times T$ block of the matrix A with the top left corner at the position $(0, -Tk)$, that is, $A(k)^{i,j} = A^{i,j-kT} = a_{i-j+kT}(i)$:

$$A =$$

$$\left[\begin{array}{c|ccc|ccc|c} \dots & a_T(0) & \dots & a_1(0) & a_0(0) & \dots & a_{-T+1}(0) & \dots \\ \dots & \dots & \mathbf{A(1)} & \dots & \dots & \mathbf{A(0)} & \dots & \dots \\ \dots & a_{2T-1}(T-1) & \dots & a_T(T-1) & a_{T-1}(T-1) & \dots & a_0(T-1) & \dots \end{array}\right].$$

The formula (2.10) says that if $g(u) = \sum g_k e^{iku}$ is the factor of γ_x' corresponding to the MA representation $\big(a_k(n)\big)$, then the vector g_{kT+r}, $r = 0, \dots, T-1, k \in Z$, is equal to the r-th row of the matrix $A(k)$.

3. INNOVATION

Since the word *innovation* is used in different ways in the prediction theory, we start with the definition.

DEFINITION 3.1. Let $\big(x(n)\big)$ be a regular stochastic sequence. An *innovation of a sequence* $\big(x(n)\big)$ is a pair, an orthonormal system $\xi = (\xi_n)$ in some Hilbert space $\mathcal{H} \supseteq M_x(n)$, and a set of coefficients $c = \big(c_k(n)\big)$, $n, k \in Z, k \geqslant 0$, called *innovation coefficients* of $\big(x(n)\big)$, such that the following conditions hold:

(I1) for every $n \in Z$, $x(n) = \sum_{k=0}^{\infty} c_k(n)\xi_{n-k}$;

(I2) for every $n \in Z$, $x(n) - \big(x(n)|M_x(n-1)\big) = c_0(n)\xi_n$.

Our definition is slightly different from the one used in [3] and [4], but the change is purely cosmetic and is dictated by convenience. In [3], for example, if $M_x(n-1) = M_x(n)$ then ξ_n is defined to be zero, while in this paper we instead define ξ_n to be a vector of norm one from outside of M_x and the corresponding coefficient $c_0(n) = 0$. Each regular sequence has an innovation. For the future use we put $\overline{\text{sp}}\,\{c_0(m)\xi_m : m \leqslant n\} = M_{c\xi}(n)$. With this notation the condition (I2) can be replaced by the condition

(I2)$'$ for every $n \in Z$, $M_x(n) = M_{c\xi}(n)$.

For convenience, we also set $c_k(n) = 0$ for each $k < 0$ and $n \in Z$, and then the representation (I1) can be written as

$$x(n) = \sum_{m=-\infty}^{\infty} c_{n-m}(n)\xi_m. \tag{3.1}$$

An innovation is not unique; however, from (I2) it follows that if (c, ξ) and (d, ζ) are two innovations of $\big(x(n)\big)$, then for every $n \in Z$ and $k \geqslant 0$, $|c_0(n)| = |d_0(n)|$. We can change the vectors (ξ_n) slightly and require that all $c_0(n) \geqslant 0$, $n \in Z$. Under this assumption the set $\big(c_k(n)\big)$ is uniquely determined by conditions (I1) and (I2). The series representation in (I1) will be called an *innovation representation* of $\big(x(n)\big)$. The set of all $n \in Z$ such that $c_0(n) \neq 0$ will be denoted by S_x and will be called the *support of the innovation* of $\big(x(n)\big)$. Because of (I2), the set S_x does not depend on a choice of innovation. If (c, ξ) is an innovation and $c_0(n) = 0$, then $c_k(k+n) = 0$ for all $k \geqslant 0$. Indeed, projecting orthogonally $x(n)$ onto $M_x = M_{c\xi}(\infty)$ we obtain

$$0 = x(n) - \big(x(n)|M_x\big) = \sum_{m \notin S_x} c_{n-m}(n)\xi_m,$$

which implies that $\sum_{m \notin S_x} |c_{n-m}(n)|^2 = 0$.

LEMMA 3.1. *Let $\big(x(n)\big)$ be a regular T-PC sequence, W be its T-shift operator, and $\big(\big(c_k(n)\big), (\xi_n)\big)$ be an innovation of $\big(x(n)\big)$. Then:*

(i) $S_x = \{n \in Z : c_0(n) \neq 0\}$ *is* T*-periodic, i.e.*, $n \in S_x \Leftrightarrow n + T \in S_x$;

(ii) *one can choose the sequence* (ξ_n) *in such a way that the shift* U *of* (ξ_n) *satisfies* $U^{Tp}x(n) = x(n + pT) = W^p x(n)$, $p, n \in Z$.

P r o o f. If $\big(x(n)\big)$ is regular and $\big(\big(c_k(n)\big), (\xi_n)\big)$ is its innovation, then from the facts that $M_{c\xi}(n) = M_x(n)$ and $W\big(M_x(n)\big) = M_x(n + T)$, $n \in Z$, it follows that for every $n, p \in Z$

$$\begin{aligned} W^p\big(c_0(n)\xi_n\big) &= W^p\Big(x(n) - \big(x(n)|M_x(n-1)\big)\Big) \\ &= x(n + pT) - \big(x(n + pT)|M_x(n + pT - 1)\big) \\ &= c_0(n + pT)\xi_{n+pT}. \end{aligned}$$

This shows that for every $n, p \in Z$ we have $|c_0(n)| = |c_0(n + pT)|$. In particular, $c_0(n) = 0$ iff $c_0(n + pT) = 0$, which proves (i). If $n \notin S_x$, then $c_0(n) \neq 0$ and $W^p(\xi_n) = \big(c_0(n + pT)/c_0(n)\big)\xi_{n+pT} = \alpha_{p,n}\xi_{n+pT}$, where $|\alpha_{p,n}| = 1$. Starting from ξ_r, $r = 0, \ldots, \xi_{T-1}$, $r \notin S_x$, we can therefore change the vectors ξ_{r+pT} so that $\xi_{r+pT} = W^p(\xi_r)$, $r \notin S_x$. ∎

An innovation satisfying condition (ii) of Lemma 3.1 will be called a *periodic innovation*. The number of nonzero vectors in the set $D_x = S_x \cap \{0, \ldots, T - 1\}$ will be called the *rank* of $\big(x(n)\big)$. If $\big(\big(c_k(n)\big), (\xi_n)\big)$ is a periodic innovation of $\big(x(n)\big)$, then (3.1) is an MA representation of $\big(x(n)\big)$. Hence everything that we said in the previous section applies to the coefficients $\big(c_k(n)\big)$, in particular, that the sequence $\big(x(n)\big)$ is absolutely continuous, the sequences $c_k(n)$ are T-periodic in n, and that $\big(c_k(n)\big)$ generate a certain factor of the spectral density γ'_x of $\big(x(n)\big)$. The last statement is singled out below.

THEOREM 3.1. *Let us suppose that* $\big(x(n)\big)$ *is a* T*-PC regular sequence and* $\big(\big(c_k(n)\big), (\xi_n)\big)$ *is a periodic innovation of* $\big(x(n)\big)$. *Let* $c_k = \sum_{p=0}^{T-1} c_{k-p}(\langle k\rangle)e_p$ *and*

$$f(u) = \sum_{k=0}^{\infty} e^{iku}c_k. \tag{3.2}$$

Then $\big(x(n)\big)$ *is absolutely continuous and its density* $\gamma'_x = (\gamma'_j)$ *admits a factorization*

$$\gamma'_j(u) = (1/T)f(u)f(u + 2\pi j/T)^* \ \text{a.e.}, \qquad j = 0, \ldots, T - 1.$$

A factor of γ'_x that corresponds to some periodic innovation of $\big(x(n)\big)$ will be called an *innovation generated factor* or, shortly, an *i-factor* of γ'_x.

In the case of regular stationary sequences $\big(x(n)\big)$ (i.e., if $T = 1$) it is well known that $f(u) = \sum_{k=0}^{\infty} e^{iku}f_k$, $f_k \in C$, is an i-factor of the spectral density γ'_0 of $\big(x(n)\big)$ iff $\gamma'_0(u) = |f(u)|^2$ a.e., and $\overline{\text{sp}}\,\{\phi(u)f(u) : \phi \in \mathcal{P}\} = H^2(C)$, where

$\mathcal{P}$ denotes the set of all scalar analytic polynomials. Below we try to characterize i-factors of a regular T-PC sequence in a similar spirit. First we introduce needed notation.

The symbols $L^2(C^T)$, $H^2(C^T)$, $L^2_T(C^T)$, $H^2_T(C^T)$, $\mathrm{q}(m)$ and $\langle m\rangle$ were defined in Introduction. If M is a subspace of $L^2(C)$ then $e^{imu}M = \{e^{imu}h(u); h \in M\}$. By $\mathcal{P}$ we will denote the set of all scalar analytic polynomials, that is, elements of $H^2(C)$ with only finitely many nonzero Fourier coefficients, and $\mathcal{P}_m = e^{imu}\mathcal{P}$. If M_p, $p = 0, \ldots, T-1$, are subspaces of $L^2(C)$, then $M = \sum_{p=0}^{T-1} M_p e_p = [M_0, \ldots, M_{T-1}]$ will stand for the subspace $L^2(C^T)$ consisting of all the functions $f = \sum_{p=0}^{T-1} f^p e_p$ such that $f^p \in M_p$. Especially important for us will be the spaces

$$(3.3)\qquad H^2_T(C^T, r) = \sum_{p=0}^{r} H^2_T e_p + \sum_{p=r+1}^{T-1} e^{iTu} H^2_T e_p, \quad r = 0, \ldots, T-1,$$

where $H^2_T = H^2_T(C) = L^2_T(C) \cap H^2(C)$, and if $r = T-1$ then $H^2_T(C^T, T-1) = \sum_{p=0}^{T-1} H^2_T e_p$. If $a = \sum_{p=0}^{T-1} a^p e_p \in C^T$ and $D \subseteq \{0, \ldots, T-1\}$, then $a_{|D} = \sum_{p=0}^{T-1} a^p 1_D(p) e_p$ will denote the vector a in which the coordinates a^p for $p \notin D$ are replaced by zero. The same convention is used for the direct sum of subspaces; in particular,

$$(3.4)\qquad H^2_T(C^T, r)_{|D} = \sum_{p=0}^{r} H^2_T 1_D(p) e_p + \sum_{p=r+1}^{T-1} e^{iTu} H^2_T 1_D(p) e_p$$

is a subspace of $H^2_T(C^T, r)$ in which all p-th components for $p \notin D$ are replaced by $\{0\}$.

THEOREM 3.2. *Let $\big(x(n)\big)$ be a regular T-PC sequence and γ'_x be its spectral density. A function $f \in H^2(C^T)$, $f(u) = \sum_{k=0}^{\infty} f_k e^{iku}$, is an i-factor of γ'_x iff*

(i) *f is a factor of γ'_x,*

(ii) *there is a nonempty $D \subseteq \{0, \ldots, T-1\}$ such that for every $r = 0, \ldots, T-1$ we have*

$$\overline{\mathrm{sp}}\Big\{\sum_{j=0}^{T-1} p(u + 2\pi j/T) f(u + 2\pi j/T) : p \in \mathcal{P}_{-r}\Big\} = H^2_T(C^T, r)_{|D}.$$

P r o o f. ($\Rightarrow$) Let $c = \big(c_k(n)\big)$, $\xi = (\xi_n)$ be an innovation of $\big(x(n)\big)$, D be its support, and f be a factor of γ'_x constructed in Theorem 3.1. Consider the sequences

$$F(n)(u) = e^{-inu} \frac{1}{T} \sum_{j=0}^{T-1} e^{-2\pi ijn/T} f(u + 2\pi j/T)$$

and $\zeta_n(u) = \big(1/\sqrt{2\pi}\big) e^{-i\mathrm{q}(n)Tu} e_{\langle n\rangle}$, $n \in Z$, as constructed in Proposition 2.1. From the proof of Proposition 2.1 it follows that (c, ζ) is an innovation of $\big(F(n)\big)$.

For $r = 0, \dots, T-1$, we have

$$M_\zeta(r) = \overline{\mathrm{sp}}\{e^{-iq(n)Tu} e_{\langle n\rangle} : n \leqslant r\} = \bigoplus_{p=0}^{T-1} \overline{\mathrm{sp}}\{e^{iqTu} e_p : -qT + p \leqslant r\}.$$

Since the span

$$\overline{\mathrm{sp}}\{e^{iqTu} : -qT + p \leqslant r\} = \begin{cases} H_T^2(C) & \text{if } 0 \leqslant p \leqslant r, \\ e^{iTu} H_T^2(C) & \text{if } r < p \leqslant T-1, \end{cases}$$

$M_\zeta(m) = H_T^2(C^T, r)$, and hence $M_{c\zeta(m)} = H_T^2(C^T, r)_{|D}$. The space $M_F(m)$ is the closed span of

$$\begin{aligned} \sum_{n=-\infty}^{m} a_n F(n) &= \frac{1}{T} \sum_{j=0}^{T-1} \sum_{n=-\infty}^{m} a_n e^{-in(u+2\pi j/T)} f(u + 2\pi j/T) \\ &= \frac{1}{T} \sum_{j=0}^{T-1} p(u + 2\pi j/T) f(u + 2\pi j/T), \end{aligned}$$

where $p(u) = \sum_{n=-\infty}^{m} a_n e^{-inu} \in P_{-m}$. Since (c, ζ) is an innovation of $\big(F(n)\big)$, $M_{c\zeta}(m) = M_F(m)$ for every $m \in Z$, and in particular for every $r = 0, \dots, T-1$.

($\Leftarrow$) Conversely, suppose that $f(u) = \sum_{k=0}^{\infty} f_k e^{iku}$, $f_k = \sum_{p=0}^{T-1} f_k^p e_p \in C^T$, satisfies the conditions (i) and (ii) of the theorem. Define

$$F(n)(u) = \frac{1}{T} \sum_{j=0}^{T-1} e^{-in(u+2\pi j/T)} f(u + 2\pi j/T).$$

From the assumption (i) and Lemma 2.1 it follows that γ_x' is the spectral density of $\big(F(n)\big)$. The T-shift of $\big(F(n)\big)$ is the operator of multiplication by e^{-iTu}. As usual, we define $f_k = 0$ if $k < 0$. From Lemma 2.2 it follows that

$$F(n)(u) = \sum_{q=-\infty}^{\infty} f_{n+qT} e^{iqTu} = \sum_{q=-\infty}^{\infty} \sum_{p=0}^{T-1} e^{iqTu} f_{n+qT}^p e_p.$$

Let us first look at $F(r)$, where $r = 0, \dots, T-1$. By assumption (ii), $M_F(r) = H_T^2(C^T, r)_{|D}$. In particular, $F(r) \in H_T^2(C^T, r)_{|D}$, which implies that $f_r^p = 0$ for $p = r+1, \dots, T-1$. Therefore,

$$F(r)(u) = \sum_{q=1}^{\infty} \sum_{p=0}^{T-1} e^{iqTu} f_{r+qT}^p e_p + \sum_{p=0}^{r} f_r^p e_p,$$

$r = 0, \dots, T-1$. Since $M_F(r) = H_T^2(C^T, r)_{|D}$, we conclude that

$$F(r) - \big(F(r) | M_F(r-1)\big) = F(r) - \big(F(r) | H_T^2(C^T, r-1)_{|D}\big) = f_r^r e_r, \tag{3.5}$$

$r = 1, \ldots, T-1$. Since $F(r+qT)(u) = e^{-iqTu}F(r)(u)$, we have

$$M_F(-1) = e^{iTu}M_T(T-1) = e^{iTu}H_T^2(C^T, T-1)_{|D} = e^{iTu}\sum_{p=0}^{T-1} H_T^2 e_p,$$

so the formula (3.5) holds true also for $r = 0$. If now $n = r + qT$, then

$$\begin{aligned} F(n)(u) - \big(F(n)|M_F(n-1)\big)(u) &= e^{-iqTu}\big[F(r)(u) - \big(F(r)|M_F(r-1)\big)\big] \\ &= e^{-iqTu} f_r^r e_r = f_{\langle n\rangle}^{\langle n\rangle}\big(e^{-\mathrm{iq}(n)Tu}e_{\langle n\rangle}\big). \end{aligned}$$

Defining $\zeta_n(u) = \big(1/\sqrt{2\pi}\big)e^{-\mathrm{iq}(n)Tu}e_{\langle n\rangle}$ we conclude from (3) that the pair $\big(\big(c_k(n)\big), (\zeta_n)\big)$, where $c_k(n) = \big(\sqrt{2\pi}\big)f_{k+\langle n-k\rangle}^{\langle n-k\rangle}$, is a periodic innovation of $\big(F(n)\big)$. Since $\big(x(n)\big)$ and $\big(F(n)\big)$ have the same correlation, one can find an orthonormal system in some space $\mathcal{H} \supseteq M_x$ such that $\big(\big(c_k(n)\big), (\xi_n)\big)$ is a periodic innovation of $\big(x(n)\big)$. To complete the proof note that $D = D_F = D_x$. ■

4. TWO REMARKS

We finish this paper with two remarks. In the first we discuss a relationship between an innovation of a PC sequence and an innovation of the corresponding multivariate stationary sequence. In the second we restate our main theorem in terms of factorization of vector-valued analytic functions, which may be of interest in the theory of vector Hardy spaces.

A family $\big(x^p(n)\big)$, $p = 0, \ldots, T-1$, $n \in Z$, of elements of a complex Hilbert space $\mathcal{H}$ is called a *T-variate stationary sequence* if for every r, s, n, m

$$\big(x^r(n), x^s(m)\big) = \big(x^r(n-m), x^s(0)\big) = K^{r,s}(n-m).$$

It is convenient to look at $\big(x^p(n)\big)$ as a column vector $\boldsymbol{x}(n) = [x^p(n)]_{T-1}^0$ with rows indexed from 0 (top) to $T-1$ and the r-th row being equal to $x^r(n)$. The matrix function $K(n) = [K^{r,s}(n)]$, $n \in Z$, $r, s = 0, \ldots, T-1$, is called the *correlation function* of $\big(\boldsymbol{x}(n)\big)$. The past of $\big(\boldsymbol{x}(n)\big)$ at a moment $n \in Z$ is defined as $M_{\boldsymbol{x}}(n) = \overline{\mathrm{sp}}\{d_m\boldsymbol{x}(m) : m \leqslant n, d_m \in C^T\}$. A sequence $\big(\boldsymbol{x}(n)\big)$ is called *regular* if $\bigcap_n M_{\boldsymbol{x}}(n) = \{0\}$. A T-variate stationary sequence $\boldsymbol{\xi}(n) = [\xi_n^p]$, $n \in Z$, is said to be *T-variate orthonormal* if its correlation function $K_{\boldsymbol{\xi}}(n) = 0$ for $n \neq 0$, and $K_{\boldsymbol{\xi}}(0) = I$, where here and in the sequel I stands for the identity $T \times T$ matrix. An *MA representation of a T-variate stationary sequence* $\big(\boldsymbol{x}(n)\big)$ is a pair $\big(\big(A(n)\big), (\boldsymbol{\xi}_n)\big)$, where $A(n) \in M_T(C)$, $n \in Z$, and $(\boldsymbol{\xi}_n)$ is a T-variate orthonormal sequence in some space $\mathcal{H} \supseteq M_{\boldsymbol{x}}$, such that

1. for every $n \in Z$, $\boldsymbol{x}(n) = \sum_{k=-\infty}^{\infty} A(k)\boldsymbol{\xi}_{n-k}$;

2. the unitary operator U in $M_{\boldsymbol{\xi}}$, defined by the requirement $U(\xi_n^p) = \xi_{n+1}^p$, has the property that $U\big(x^p(n)\big) = x^p(n+1)$, $n \in Z$, $p = 0, \ldots, T-1$.

If $\big(\boldsymbol{x}(n)\big)$ is a T-variate stationary sequence then there exists a $T \times T$ matrix valued measure Γ of $[0, 2\pi)$ such that $K(n) = \int_0^{2\pi} e^{-int}\Gamma(dt)$. If Γ is absolutely continuous (with respect to the Lebesgue measure on $[0, 2\pi)$), then its Randon–Nikodym derivative will be denoted by $\Gamma'(t)$ and will be referred to as the *density* of $\big(\boldsymbol{x}(n)\big)$. A T-variate stationary sequence $\big(\boldsymbol{x}(n)\big)$ has an MA representation iff its spectral measure Γ is absolutely continuous. If $\big(\big(A(n)\big), (\boldsymbol{\xi}_n)\big)$ is an MA representation of $\big(\boldsymbol{x}(n)\big)$, then $F(t) = \big(1/\sqrt{2\pi}\big)\sum_{k=-\infty}^{\infty} A(k)e^{ikt}$ is a factor of $\Gamma'(t)$ in the sense that $\Gamma'(t) = F(t)F(t)^*$ dt-a.e. A factor F of Γ' is called *outer* (cf. [9], p. 190) if F is analytic (i.e., each $F^{ij} \in H^2(C)$) and

$$\overline{\mathrm{sp}}\{\phi(t)F(t) : \phi \in \mathcal{P}(C^T)\} = H^2(\mathcal{M}) \quad \text{for some subspace } \mathcal{M} \subseteq C^T, \tag{4.1}$$

where $H^2(\mathcal{M})$ denotes the subspace of $H^2(C^T)$ consisting of functions f such that $f(t) \in \mathcal{M}$ a.e., and $\mathcal{P}(C^T)$ is the set of all C^T-valued analytic polynomials. An MA representation $\big(\big(A(n)\big), (\boldsymbol{\xi}_n)\big)$ of $\big(\boldsymbol{x}(n)\big)$ is called an *innovation of* $\big(\boldsymbol{x}(n)\big)$ if $A(k) = 0$ for every $k < 0$ and, for every $n \in Z$,

$$\boldsymbol{x}(n) - \big(\boldsymbol{x}(n)|M_{\boldsymbol{x}}(n-1)\big) = A(0)\boldsymbol{\xi}_n.$$

Here $\big(\boldsymbol{x}(n)|M_{\boldsymbol{x}}(n-1)\big) = \big[\big(x^p(n)|M_{\boldsymbol{x}}(n-1)\big)\big]$ is a column vector whose p-th coordinate is equal to $\big(x^p(n)|M_{\boldsymbol{x}}(n-1)\big)$. Every regular T-variate stationary sequence has an innovation. It is well known that $\big(\big(A(n)\big), (\boldsymbol{\xi}_n)\big)$ is an innovation of a regular T-variate stationary sequence $\big(\boldsymbol{x}(n)\big)$ iff the corresponding factor $F(t)$ of the density of $\big(\boldsymbol{x}(n)\big)$ is outer. All the above facts are standard and in one or another form can be found in any publication on prediction theory of multivariate stationary processes, the best being Masani's summary paper [8].

There is an obvious one-to-one correspondence between T-PC sequences $\big(x(n)\big)$ and T-variate stationary sequences $\big(\boldsymbol{x}(n)\big)$ given by

$$\boldsymbol{x}(n) = [x^r(n)] \leftrightarrow x(n) = x^{\langle n\rangle}\big(\mathrm{q}(n)\big), \tag{4.2}$$

where, as before, $\mathrm{q}(m)$ and $\langle m\rangle$ are the quotient and the remainder in division of m by T. We will refer to $\big(\boldsymbol{x}(n)\big)$ and $\big(x(n)\big)$ above as *corresponding to each other*. Note that the shift V of $\boldsymbol{x}(n)$ (defined as $Vx^p(n) = x^p(n+1)$) is equal to the T-shift W of $\big(x(n)\big)$. Clearly, a T-PC sequence $\big(x(n)\big)$ is regular iff the corresponding sequence $\big(\boldsymbol{x}(n)\big)$ is regular. Every MA representation $\big(\big(a_k(n)\big), (\xi_n)\big)$ of a T-PC sequence $\big(x(n)\big)$ generates an MA representation of the corresponding stationary sequence $\big(\boldsymbol{x}(n)\big)$ and *vice versa*. To see this arrange $a_k(n)$ into a matrix $A = [\ldots\ A(2)\ A(1)\ A(0)\ A(-1)\ \ldots]$ as described at the end of Section 2. With the notation introduced in that section we have

$$\boldsymbol{x}(0) = [x(r)]_{T-1}^{0} = A[\xi_n]_{\infty}^{-\infty} = \sum_{k=-\infty}^{\infty} A(k)[\xi_n]_{-kT+T-1}^{-kT} = \sum_{k=-\infty}^{\infty} A(k)\boldsymbol{\xi}_{-k},$$

where $\boldsymbol{\xi}_m = [\xi_{mT}, \xi_{mT+1}, \ldots, \xi_{mT+T-1}]^t$ and a^t denotes the transpose of the vector a. Applying V^T to both sides we conclude that $\boldsymbol{x}(n) = \sum_{k=-\infty}^{\infty} A(k)\boldsymbol{\xi}_{n-k}$. The relation between sequences $a_k(n)$ and $\big(A(n)\big)$ is as follows:

$$A(k)^{i,j} = a_{i-j+kT}(i), \quad i,j = 0,\ldots,T-1,\ k \in Z. \tag{4.3}$$

Every factor $f(u) = \sum_{k=-\infty}^{\infty} f_k e^{iku}$ of γ_x' produces a factor

$$F(u) = \sum_{k=-\infty}^{\infty} F(k) e^{iku}$$

of the density $\Gamma'(u)$ of the corresponding stationary sequence $\boldsymbol{x}(n)$ and *vice versa*. The relation is the following:

$$F(k)^{i,j} = f^j_{kT+i}, \quad i,j = 0,\ldots,T-1,\ k \in Z. \tag{4.4}$$

Every periodic innovation $\big(\big(a_k(n)\big), (\xi_n)\big)$ of a T-PC sequence $\big(x(n)\big)$ generates, via formula (4.3), an innovation representation $\boldsymbol{x}(n) = \sum_{k=0}^{\infty} A(k)\boldsymbol{\xi}_{n-k}$ of the corresponding stationary sequence $\big(\boldsymbol{x}(n)\big)$. But the converse is not true. The reason is that in construction of an innovation of a T-variate stationary sequence we have a freedom in choosing an order of components of the innovation vector $\boldsymbol{\xi}_n$. Requiring that $A(0)$ in innovation representation of $\big(\boldsymbol{x}(n)\big)$ is a lower triangular matrix still is not sufficient in general to assure that $\big(\big(a_k(n)\big), (\xi_k)\big)$, where $a_{i-j+kT}(i) = A(k)^{i,j}$ and $\xi_{kT+p} = \xi_k^p$, is an innovation of the corresponding T-PC sequence. An example is below.

EXAMPLE 4.1. Let (ξ_n) be any orthonormal system in $\mathcal{H}$ and for every $q \in Z$ define $x(3q) = \xi_{3q}$, $x(3q+1) = 0$, and $x(3q+2) = \xi_{3q+2}$. The sequence $\big(x(n)\big)$ is PC with period $T = 3$. Since $\big(x(n)\big)$ is orthogonal, it follows that each projection $\big(x(n)|M_x(n-1)\big) = 0$, and hence the above is an innovation representation of $\big(x(n)\big)$. The only nonzero innovation coefficients are $c_0(0) = 1$ and $c_0(2) = 1$. Putting $C(0)$ to be a diagonal matrix with diagonal entries $1, 0, 1$, we see that the corresponding innovation representation of the block sequence $\big(\boldsymbol{x}(n)\big)$ is $\boldsymbol{x}(n) = C(0)\boldsymbol{\xi}_n$, $\boldsymbol{\xi}_n = [\xi_{3n}, \xi_{3n+1}, \xi_{3n+2}]^t$. We change the order of elements in $\boldsymbol{\xi}_n$ and define $\boldsymbol{\zeta}_n = [\xi_{3n}, \xi_{3n+2}, \xi_{3n+1}]'$, that is,

$$\boldsymbol{\zeta}_n = B\boldsymbol{\xi}_n, \quad \text{where } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then (ζ_n) is a 3-variate orthonormal system and $\boldsymbol{x}(n) = C(0)\boldsymbol{\xi}_n = B(0)\boldsymbol{\zeta}_n$, where

$$B(0) = C(0)B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that $B(0)$ is lower triangular. Because $\big(\boldsymbol{x}(n)\big)$ are orthogonal, $\boldsymbol{x}(n) = B(0)\boldsymbol{\zeta}_n$ is an innovation representation of $\big(\boldsymbol{x}(n)\big)$. In terms of the corresponding PC sequence the equation $\boldsymbol{x}(n) = B(0)\boldsymbol{\zeta}_n$ reads

$$x(3q) = \zeta_{3q}, \;\; x(3q+1) = 0, \;\; x(3q+2) = \zeta_{3q+1}, \;\; q \in Z.$$

For the system (ζ_n) above we have $b_0(0) = 1$, $b_1(2) = 1$, and the remaining $b_k(n)$ are zero. But the pair $\big(\big(b_k(n)\big), (\zeta_n)\big)$ is not an innovation of $\big(x(n)\big)$ in our sense because $x(2) - \big(x(2)|M_x(1)\big) = \zeta_1 \neq b_0(2)\zeta_2 = 0$.

Another reason that $\big(\big(b_k(n)\big), (\zeta_n)\big)$ above is not an innovation of $\big(x(n)\big)$ is that $b_0(1) = 0$ but $b_1(2) \neq 0$, which contradicts the fact that if (c, ξ) is an innovation and $c_0(n) = 0$, then $c_k(k+n) = 0$ for all $k \geqslant 0$. One can easily see that given an innovation $\big(\big(C(n)\big), (\boldsymbol{\xi}_n)\big)$ of $\big(\boldsymbol{x}(n)\big)$, the pair $\big(\big(c_k(n)\big), (\zeta_n)\big)$, where $c_{i-j+kT}(i) = C(k)^{i,j}$ and $\zeta_{nT+r} = \xi^r(n)$, is an innovation of the corresponding T-PC sequence $x(nT+r) = x^r(n)$, $n \in Z$, $r = 0, \ldots, T-1$, if and only if the matrix $C(0)$ is lower triangular and $C(0)^{i,i} = 0$ implies that the i-th column of each $C(k)$ is zero. In the case of sequences of full rank the second condition is obviously always satisfied. Example 4.1 also shows that the outerness (4.1) of $F(t)$ is not sufficient for f given by (4.4) to be an i-factor of the density of the corresponding T-PC sequence.

Since every $g \in H^2(C^T)$ is a factor of the density γ' of a certain T-PC sequence (Lemma 2.1), and any two factors differ by $2\pi/T$-periodic partial isometric multiple ([7], Corollary 4.2), Theorem 3.2 translates into the following (possibly known) theorem about factorization of vector analytic functions.

THEOREM 4.1. *Suppose that $g \in H^2(C^T)$, $g \neq 0$. Then there exists an $f \in H^2(C^T)$, a unique nonempty $D \subseteq \{0, \ldots, T-1\}$, and a $2\pi/T$-periodic analytic $T \times T$-matrix function $V(u)$ such that*

(i) *$V(u)$ is a partial isometry (du-a.e.),*

(ii) *$g(u) = f(u)V(u)$ a.e.,*

(iii) *for every $r = 0, \ldots, T-1$,*

$$\overline{\mathrm{sp}}\Big\{\sum_{j=0}^{T-1} p(u + 2\pi j/T) f(u + 2\pi j/T) : p \in \mathcal{P}_{-r}\Big\} = H_T^2(C^T, r)_{|D}.$$

The function f can be chosen in such a way that $f_r^r \geqslant 0$ for each $r = 0, \ldots, T-1$, and such f is then unique.

REFERENCES

[1] J. L. Doob, *Stochastic Processes*, Wiley, New York 1953.

[2] E. G. Gladyshev, *Periodically correlated random sequences*, Soviet Math. 2 (1961), pp. 385–388.

[3] H. L. Hurd, *Periodically correlated sequences of less than full rank*, J. Statist. Plann. Inference 129 (2005), pp. 279–303.

[4] H. L. Hurd and A. Miamee, *Periodically Correlated Random Sequences: Spectral Theory and Practice*, Wiley, 2007.

[5] A. N. Kolmogorov, *Stationary Sequences in Hilbert Space*, Moscow Univ. Math. Bull. 2 (1941), pp. 1–40.

[6] A. Makagon, *Stationary sequences associated with a periodically correlated sequence*, Probab. Math. Statist. 31 (2) (2011), pp. 263–283.

[7] A. Makagon and A. Miamee, *Spectral representation of periodically correlated sequences*, Probab. Math. Statist. 33 (1) (2013), pp. 175–188.

[8] P. Masani, *Recent trends in multivariate prediction theory*, in: *Multivariate Analysis: Proceedings. International Symposium on Multivariate Analysis. Dayton, Ohio, 1965*, P. R. Krishnaiah (Ed.), Academic Press, 1966, pp. 351–382.

[9] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, 1970.

[10] K. Urbanik, *Lectures on Prediction Theory*, Lecture Notes in Math., Vol. 44, Springer, 1967.

Andrzej Makagon
Department of Mathematics
Hampton University
100 E. Queen St.
Hampton, VA 23668, USA
E-mail: andrzej.makagon@hamptonu.edu

Received on 20.12.2013;
revised version on 4.4.2014