PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 35, Fasc. 1 (2015), pp. 143–159

CAPTIVITY OF MEAN-FIELD PARTICLE SYSTEMS AND THE RELATED EXIT PROBLEMS*

BY

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I dedicate this article to Marina Sertić

Abstract. A mean-field system is a weakly interacting system of N particles in $\mathbb{R}^d$ confined by an external potential. The aim of this work is to establish a simple result about the exit problem of mean-field systems from some domains when the number of particles goes to infinity. More precisely, we prove the existence of some subsets of $\mathbb{R}^{dN}$ such that the probability of leaving these sets before any $T > 0$ is arbitrarily small by taking N large enough. On the one hand, we show that the number of steady states in the small-noise limit is arbitrarily large with a sufficiently large number of particles. On the other hand, using the long-time convergence of the hydrodynamical limit, we identify the steady states as N goes to infinity with the invariant probabilities of the McKean–Vlasov diffusion so that some steady states in the small-noise limit are not steady states in the large N limit.

2000 AMS Mathematics Subject Classification: Primary: 82C22, 60F10; Secondary: 60J60, 60G10.

Key words and phrases: Interacting particle system, propagation of chaos, exit time, nonconvexity, free energy, invariant probabilities.

1. INTRODUCTION

We are interested in some exit problems for a class of weakly interacting, mean-field particle systems. We know the asymptotics of the exit time of the mean-field system in the small-noise limit, see [11], [10]. Indeed, if the number of particles, N, is finite, the system corresponds to a classical diffusion in $\mathbb{R}^{dN}$. The exit problem has also been studied as the number of particles goes to infinity and the diffusion coefficient goes to zero, see [24]. However, here, we take the large

* Supported by the DFG-funded CRC 701, Spectral Structures and Topological Methods in Mathematics, at the University of Bielefeld.

** Vrlo velika hvala Marini za sve. Finalement, un très grand merci à Manue, à Sandra et à Virginie pour tout.

N limit, N being the number of particles and the diffusion coefficient is fixed. In other words, we consider N diffusions in $\mathbb{R}^d$ with independent d-dimensional Wiener processes and independent initial random values. We add a friction term, that is, the gradient of an external potential V. Moreover, we assume that each particle is under the influence of the global behaviour of the particle system, which justifies the expression "mean-field system". One way to understand this model is to think about several individuals maximizing their utility function according to the global data of the market. Here, we assume that each particle is attracted by any other one and that the interaction depends only on the distance between the particles. Thus, the equation satisfied by each diffusion Z^i is

$$Z_t^i = Z_0^i + \sigma B_t^i - \int_0^t \nabla V(Z_s^i)ds - \frac{1}{N}\sum_{j=1}^N \int_0^t \nabla F(Z_s^i - Z_s^j)ds$$

for any $1 \leqslant i \leqslant N$. Here, $B^1, \ldots, B^N$ are N independent d-dimensional Wiener processes. By taking $\mu_t^N := N^{-1}(\delta_{Z_t^1} + \ldots + \delta_{Z_t^N})$, we can write

$$Z_t^i = Z_0^i + \sigma B_t^i - \int_0^t \nabla V(Z_s^i)ds - \int_0^t \nabla F * \mu_s^N(Z_s^i)ds.$$

The random variables $Z_0^1, \ldots, Z_0^N$ are i.i.d. with common law μ_0. Moreover, the initial positions $Z_0^1, \ldots, Z_0^N$ are independent of the Brownian motions $B^1, \ldots, B^N$. Here, the function V is called the *confining potential*. Indeed, it attracts each diffusion to its minimizers. The potential F is the so-called *interacting potential*. Due to the assumptions on the interaction, the function $x \mapsto \nabla F(x)$ is radial. The specific hypotheses are given after the introduction. Let us notice that we write Z_t^i instead of $Z_t^{i,N,\sigma}$ in order to simplify the notation.

Such models intervene in many applications. Let us mention [8] in which the McKean–Vlasov diffusion (which corresponds to the hydrodynamical limit of $(Z^1, \ldots, Z^N)$) is used to obtain a representation of a solution to a particular stochastic partial differential equation. Also, mean-field systems are relevant to study the social interaction, see [7].

We introduce the notation: $\mathcal{Z}^N := (Z^1, \ldots, Z^N)$ and $\mathcal{B}^N := (B^1, \ldots, B^N)$. Thereby, $\mathcal{Z}^N$ ties in a classical diffusion in $\mathbb{R}^{dN}$,

$$\mathcal{Z}_t^N = \mathcal{Z}_0^N + \sigma\mathcal{B}_t^N - N\int_0^t \nabla\Upsilon_0^N(\mathcal{Z}_s^N)ds, \tag{1.1}$$

the potential Υ_0^N being defined by

$$\Upsilon_0^N(\mathcal{Z}) := \frac{1}{N}\sum_{i=1}^N V(Z_i) + \frac{1}{2N^2}\sum_{i=1}^N\sum_{j=1}^N F(Z_i - Z_j)$$

for any $\mathcal{Z} = (Z_1, \ldots, Z_N) \in \mathbb{R}^{dN}$. Three concurrent forces generate the motion of the process $\mathcal{Z}^N$. The first one is the gradient of the diagonal potential $(Z_1, \ldots, Z_N) \mapsto V(Z_1) + \ldots + V(Z_N)$. The second term represents the average tension of the interacting potential F between the coordinates. The first two forces generate $N\Upsilon_0^N$. We are interested in the trajectorial behaviour as N, the number of particles, is large. Thus, we need to approximate each particle in $\mathbb{R}^{dN}$ by taking the hydrodynamical limit. We remark that the potential Υ_0^N has a sense as N goes to infinity. Indeed, if $\{Z^i\,;\, i \in \mathbb{N}^*\}$ is a family of i.i.d. random variables with common law μ, we have

$$\lim_{N\to+\infty} \Upsilon_0^N(Z^1, \ldots, Z^N) = \Upsilon_0(\mu) := \int_{\mathbb{R}^d} \left\{ V(x) + \frac{1}{2} F * \mu(x) \right\} \mu(dx),$$

the convergence being almost sure. Here, $*$ denotes the convolution, that is, $F * \mu(x) := \int_{\mathbb{R}^d} F(x-y)\mu(dy)$. The third influence is a dN-dimensional Wiener process $\mathcal{B}^N$ which allows the diffusion $\mathcal{Z}^N$ to escape from the stable domains of the potential $N\Upsilon_0^N$.

By $\mu_t^N := N^{-1}(\delta_{Z_t^1} + \ldots + \delta_{Z_t^N})$ we denote the empirical measure of the particle system. Using the Itô formula, we obtain

$$\frac{d}{dt}\mathbb{E}\{ \int_{\mathbb{R}^d} f\mu_t^N \} = \mathbb{E}\left\{ \frac{\sigma^2}{2} \int_{\mathbb{R}^d} \Delta f \mu_t^N - \int_{\mathbb{R}^d} \langle \nabla f\,;\, \nabla V + \nabla F * \mu_t^N \rangle \mu_t^N \right\}$$

for any smooth function with compact support f from $\mathbb{R}^d$ to $\mathbb{R}$. Let us notice that if a family of deterministic measures $\{\nu_t\,;\, t \geqslant 0\}$ were satisfying the previous equation, it would be a solution of the so-called *granular media equation*

$$\frac{\partial}{\partial t}\mu_t = \operatorname{div}\left\{ \frac{\sigma^2}{2}\nabla\mu_t + (\nabla V + \nabla F * \mu_t)\,\mu_t \right\}. \tag{1.2}$$

Heuristically, if the family of random measures $\{\mu_t^N\,;\, 0 \leqslant t \leqslant T\}$ converges to a family of deterministic measures $\{\mu_t\,;\, 0 \leqslant t \leqslant T\}$, this deterministic family satisfies the non-linear partial differential equation (1.2). Since we take N arbitrarily large, it motivates to focus on this family of measures. Indeed, we can prove this convergence.

The idea of the propagation of chaos is the following. Let us assume that $\{Z_0^i\,;\, i \in \mathbb{N}\}$ is a family of i.i.d. random variables with common law μ_0. The law of large numbers implies the convergence of the empirical measure at time zero, that is, $\mu_0^N := N^{-1}(\delta_{Z_0^1} + \ldots + \delta_{Z_0^N})$, to μ_0 as N goes to infinity. We say that propagation of chaos *holds on the interval* $[0, T]$ with $T < \infty$ if the family of random measures $\{\mu_t^N\,;\, 0 \leqslant t \leqslant T\}$ converges to the family of deterministic measures $\{\mu_t\,;\, 0 \leqslant t \leqslant T\}$ satisfying the equation (1.2). About details on the propagation of chaos, we refer the reader to [22], [21], [4], [18], [19]. Another way to understand propagation of chaos is the following. By hypotheses, the particles are independent (chaotic) at time zero. The propagation of chaos on the interval $[0,T]$ means that the larger is N, the more independent the two particles are. Besides, we have a coupling result on $[0,T]$, that is,

$$\lim_{N\to\infty}\mathbb{E}\{\sup_{0\leqslant t\leqslant T}\|Z_t^1-X_t^1\|^2\}=0,$$

X^1 being the so-called McKean–Vlasov diffusion,

$$X_t^1=Z_0^1+\sigma B_t^1-\int_0^t\nabla V(X_s^1)ds-\int_0^t\nabla F*\mu_s(X_s^1)ds,\qquad(1.3)$$
$$\mu_s=\mathcal{L}(X_s^1).$$

Cattiaux et al. [6] provide a uniform propagation of chaos, that is,

$$\lim_{N\to\infty}\sup\{\mathbb{E}[\|Z_t^1-X_t^1\|^2]\,;\,t\geqslant 0\}=0,$$

under simple assumptions.

About propagation of chaos phenomena, let us also mention that it has recently been studied in the context well beyond that of the Brownian motion, namely, in the situation where the driving Brownian motions have been replaced by Lévy processes and anomalous diffusions, see [15]–[17].

The own law of X_t^1 intervenes in the drift. We say that it is non-linear in the sense of McKean, see [20]. About the existence of a solution, we refer the reader to [1], [12]. Furthermore, it is well known (see [20]) that the probability measure μ_t is absolutely continuous with respect to the Lebesgue measure provided that $t>0$. From now on, let us denote by u_t its density. We notice that the family of functions $\{u_t\,;\,t\geqslant 0\}$ satisfies the granular media equation (1.2). This equation allows us to characterize the invariant probabilities of diffusion (1.3) and its long-time behaviour. See [19], [2], [5], [6] with a convex confining potential, and [13], [14], [25]–[27], [23] with a multi-wells confining potential.

In [9], the authors go further than the propagation of chaos by studying large deviations. The family of empirical measures $\{\mu_t^N\,;\,0\leqslant t\leqslant T\}$ is a small perturbation, with respect to N, of the family $\{\mu_t\,;\,0\leqslant t\leqslant T\}$.

The exit time of the diffusions in the small-noise limit can be estimated by a Kramers type law theorem (see [10], [11]). However, in this work, we do not want σ to be small but N to be large and we cannot apply this method. Consequently, the functional Υ_0 is not appropriate to understand the long-time behaviour of $\mathcal{Z}^N$. Thus, we need to introduce the entropy $S(\mu)$. If μ is absolutely continuous with respect to the Lebesgue measure with density u, the entropy is equal to $-\int_{\mathbb{R}^d}u(x)\log\big(u(x)\big)dx$ if this quantity is well defined. Alternatively, $S(\mu):=-\infty$. In order to obtain the free energy, we subtract the dissipated energy, that is, the product of the temperature $\sigma^2/2$ with the entropy $S(\mu)$ to $\Upsilon_0(\mu)$,

$$\Upsilon_\sigma(\mu) := \int_{\mathbb{R}^d} \left\{ \frac{\sigma^2}{2} \log\big(u(x)\big) + V(x) + \frac{1}{2} F * u(x) \right\} u(x) dx \tag{1.4}$$

for the measures μ absolutely continuous with respect to the Lebesgue measure with density equal to u. Otherwise, $\Upsilon_\sigma(\mu) = +\infty$.

In this work, we deal with general settings and we simply use the propagation of chaos and the study made in [25], [27] about the adherence values of the family $\{\mu_t\,;\, t \geqslant 0\}$. The first two sections deal with the assumptions, the notation and the potential geometry. Then the propagation of chaos and the main results on the exit problems are provided in the last section.

2. PRELIMINARIES

First, let us denote by $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^d$, $\|x\|^2 := x_1^2 + \ldots + x_d^2$ for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Now, we give the assumptions used in the current work.

ASSUMPTIONS (M). *The triple* (V, F, μ_0) *satisfies the set of assumptions* (M) *if the following conditions hold true*:

(M1) *V is a smooth function on $\mathbb{R}^d$. Moreover, there exist $m \in \mathbb{N}^*$ and $C > 0$ such that* $\lim_{\|x\|\to+\infty} V(x)\|x\|^{-2m} = C$.

(M2) *There exists $\mathcal{K}$, a compact subset of $\mathbb{R}^d$, such that $\nabla^2 V(x) > 0$ for all $x \notin \mathcal{K}$. Besides,* $\lim_{\|x\|\to\infty} \nabla^2 V(x) = +\infty$.

(M3) *The gradient ∇V is slowly increasing: there exist $C' > 0$ and a function $\mathcal{R}$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ such that $\nabla V(x) = C'\|x\|^{2m-2}x + \mathcal{R}(x)$ for all $x \in \mathbb{R}^d$. Moreover, the function $\mathcal{R}$ satisfies* $\lim_{\|x\|\to+\infty} \mathcal{R}(x)\|x\|^{-(2m-1)} = 0$.

(M4) *There exists an even polynomial function G on $\mathbb{R}$ such that $F(x) = G(\|x\|)$. Moreover,* $\deg(G) := 2n \geqslant 2$.

(M5) *The function G is convex.*

(M6) *The $8q^2$-th moment of the measure μ_0 is finite with* $q := \max\{m, n\}$.

(M7) *The measure μ_0 admits a $\mathcal{C}^\infty$-continuous density u_0 with respect to the Lebesgue measure. Moreover, the entropy $-\int_{\mathbb{R}^d} u_0(x)\log\big(u_0(x)\big)dx$ is finite.*

By Theorem 2.13 in [12], we know that equation (1.3) admits a unique strong solution on $\mathbb{R}_+$. Besides, there exists $M_0 > 0$ satisfying

$$\max_{1\leqslant j\leqslant 8q^2} \sup_{t\geqslant 0} \mathbb{E}[\|X_t\|^j] \leqslant M_0\,. \tag{2.1}$$

We deduce immediately from this inequality that the family $\{\mu_t\,;\, t \geqslant 0\}$ is tight.

DEFINITION 2.1. By $\mathcal{A}_\sigma$ (respectively, $\mathcal{S}_\sigma$) we denote the set of the limiting values of the family $\{\mu_t\,;\, t \geqslant 0\}$ (respectively, the set of the invariant probabilities of diffusion (1.3)).

In the following, $\mathcal{E}(x)$ is the unique integer such that $x-1 < \mathcal{E}(x) \leqslant x$. Let us present some notation about $\mathbb{R}^{dN}$.

DEFINITION 2.2. (1) The space $\mathbb{R}^{dN}$ is equipped with the Euclidean norm $\|\cdot\|_N$ defined by

$$\|\mathcal{Z}\|_N^2 := N^{-1}(\|Z_1\|^2 + \ldots + \|Z_N\|^2) \quad \text{for all } \mathcal{Z} = (Z_1, \ldots, Z_N) \in \mathbb{R}^{dN}.$$

(2) For all $r > 0$ and for all $\mathcal{Z}_0 \in \mathbb{R}^{dN}$, we introduce the ball

$$\mathbb{B}_r^N(\mathcal{Z}_0) := \{\mathcal{Z} \in \mathbb{R}^{dN} : \|\mathcal{Z} - \mathcal{Z}_0\|_N \leqslant r\}.$$

(3) For any $\mathcal{Z}_0^N \in \mathbb{R}^{dN}$, we introduce the diffusion $\mathcal{Z}^N$ defined by

$$\mathcal{Z}_t^N = \mathcal{Z}_0^N + \sigma \mathcal{B}_t^N - N \int_0^t \nabla \Upsilon_0^N(\mathcal{Z}_s^N) ds.$$

By $Z_t^i \in \mathbb{R}^d$ we denote the i-th coordinate of $\mathcal{Z}_t^N$.

We recall Theorem A in [27].

PROPOSITION 2.1. *Let us assume that the set of assumptions* (M) *is satisfied. Then $\mathcal{A}_\sigma$ is either a single element $\mu^\sigma \in \mathcal{S}_\sigma$ or a path-connected subset of $\mathcal{S}_\sigma$ such that $\Upsilon_\sigma(\mathcal{A}_\sigma) = \{\lim_{t\to\infty} \Upsilon_\sigma(\mu_t)\}$.*

We finish this section by introducing

$$\mathcal{V} := \{x \in \mathbb{R}^d : \nabla V(x) = 0\},$$

the set of all the critical points of the potential V.

3. POTENTIAL GEOMETRY

Now, we study the geometry of the potential Υ_0^N. Each point of the form $(a_0, \ldots, a_0)$ with $a_0 \in \mathcal{V}$ is a critical point of Υ_0^N. Let us prove that they are the only ones under the set of assumptions (M) and an additional assumption.

PROPOSITION 3.1. *Let us assume that the set of assumptions* (M) *is satisfied and that synchronization occurs, that is, $G''(0) + \inf\{\nabla^2 V(x)\,;\, x \in \mathbb{R}^d\} > 0$. Then $\mathcal{Z} \in \mathbb{R}^{dN}$ is a critical point of Υ_0^N if and only if there exists $a \in \mathcal{V}$ such that $\mathcal{Z} = (a, \ldots, a)$. Moreover, if the signature of the Hessian matrix $\nabla^2 V(a)$ is (p, q) then the signature of the Hessian matrix $\nabla^2 \Upsilon^N(a, \ldots, a)$ is $\big((N-1)d + p, q\big)$.*

P r o o f. S t e p 1. For all $1 \leqslant i \leqslant N$, the differential of the potential Υ_0^N with respect to z_i is

$$\frac{\partial}{\partial z_i}\Upsilon_0^N(z_1,\dots,z_N) = \frac{1}{N}\left\{\nabla V(z_i) + \frac{1}{N}\sum_{j=1}^N \nabla F(z_i - z_j)\right\}. \tag{3.1}$$

Let $\mathcal{Z} = (Z_1,\dots,Z_N)$ be a critical point of Υ_0^N. We introduce the function from $\mathbb{R}^d$ to $\mathbb{R}$,

$$\rho_{\mathcal{Z}}(z) := V(z) + \frac{1}{N}\sum_{j=1}^N F(z - Z_j).$$

Due to the synchronization, $\rho_{\mathcal{Z}}$ is convex. Thus, it admits only one critical point. However, according to equality (3.1), Z_i is a critical point of $\rho_{\mathcal{Z}}$ for all $1 \leqslant i \leqslant N$. It implies the existence of $a \in \mathbb{R}^d$ such that $Z_i = a$ for all $1 \leqslant i \leqslant N$. Besides, the equality $\nabla\rho_{\mathcal{Z}}(a) = 0$ leads to $\nabla V(a) = 0$, which means $a \in \mathcal{V}$.

S t e p 2. By I_d we denote the identity matrix of dimension d. Let us look at $\nabla^2\,\Upsilon_0^N\,(a,\dots,a)$. We compute the second derivatives:

$$\frac{\partial^2}{\partial z_i^2}\Upsilon_0^N(a,\dots,a) = \frac{1}{N}\left\{\nabla^2\,V(a) + G''(0)\left(1 - \frac{1}{N}\right)\,I_d\right\}$$

for all $1 \leqslant i \leqslant N$. Moreover, for all $1 \leqslant i \neq j \leqslant N$, we have

$$\frac{\partial^2}{\partial z_i \partial z_j}\Upsilon_0^N(a,\dots,a) = -\frac{G''(0)}{N^2}\,I_d\,.$$

Simple results on linear algebra imply that the eigenvalues of $\nabla^2\,\Upsilon_0^N(a,\dots,a)$ are those of $N^{-1}\nabla^2\,V(a)$ (each one being associated with a vector space of dimension one) and $N^{-1}\big(\nabla^2\,V(a) + G''(0)I_d\big)$ (each one being associated with a vector space of dimension $N-1$). ■

Under simple assumptions, Theorem 4.5 in [13] and Proposition 3.7 in [14] establish that diffusion (1.3) admits an invariant probability arbitrarily close to the Dirac measure of a local maximizer a of the confining potential V. However, the vector $(a,\dots,a)\in\mathbb{R}^{dN}$ is *never* a local minimizer of Υ_0^N if $\inf\{\nabla^2 V(x);\ x\in\mathbb{R}^d\} + G''(0) > 0$. *This points out the importance of the entropy and σ since there is no correspondence between the local minimizers of Υ_0^N and the invariant probabilities of diffusion* (1.3).

Now, we show that under easily verified assumptions the number of local minimizers of the potential Υ_0^N goes to infinity as N goes to infinity.

PROPOSITION 3.2. *We assume the existence of $a \in \mathcal{V}$ such that $\nabla^2\,V(a) > 0$ and $b \neq a$ such that $\nabla V(b) + \nabla F(b-a) = 0$ and $\nabla^2\,V(b) + \nabla^2\,F(b-a)$ is positive definite. Then the number of local minimizers of the potential Υ_0^N converges to infinity as N goes to infinity.*

P r o o f. First, we notice that Υ_0^N and $N\Upsilon_0^N$ have the same minimizers. Thereby, we study $N\Upsilon_0^N$. In the following, we say that $\mathcal{Z} \in \mathbb{R}^{dN}$ is an (a_1, a_2, p)-*vector* if there exists $\tau \in \mathcal{S}_N$ such that $Z_{\tau(i)} = a_1$ for all $1 \leqslant i \leqslant \mathcal{E}\,(pN)$ and $Z_{\tau(i)} = a_2$ for all $\mathcal{E}\,(pN) + 1 \leqslant i \leqslant N$.

S t e p 1. According to (3.1), an (a_1, a_2, p)-vector is a critical point of $N\Upsilon_0^N$ if and only if the triple (a_1, a_2, p) solves the following two equations:

$$\Psi_1(a_1, a_2) := \nabla V(a_1) - \nabla V(a_2) - \nabla F(a_2 - a_1) = 0 \tag{3.2}$$

and

$$p\nabla V(a_1) + (1-p)\nabla V(a_2) = 0. \tag{3.3}$$

By the definition of a and b, we have $\Psi_1(a, b) = 0$ and $\nabla V(a) = 0$. Consequently, the triple $(a, b, 1)$ satisfies equations (3.2)–(3.3). Since $\nabla^2 V(a) > 0$, we have $\nabla^2 V(a) + \nabla^2 F(b-a) > 0$. This implies that the matrix $(\partial/\partial a_1)\Psi_1(a, b)$ is positive definite. We apply the implicit function theorem and we infer the existence of two connected open sets $I \ni a$ and $J \ni b$ and a bijection ξ from I to J such that $\Psi_1\big(a_1, \xi(a_1)\big) = 0$ for all $a_1 \in I$. Moreover, $\xi(a) = b$.

S t e p 2. Now, we look at equation (3.3). Let us define

$$\Psi_2(p, a_1) := p\nabla V(a_1) + (1-p)\nabla V\big(\xi(a_1)\big)$$

for any $a_1 \in I$ and $p \in [0, 1]$. We already know that $\Psi_2(1, a) = 0$. Furthermore, by equation (3.2) with a and b, we have $\nabla V(a) - \nabla V(b) = \nabla F(b-a)$. However, $b \neq a$, so the differential $(\partial/\partial p)\Psi_2(p, a) = \nabla V(a) - \nabla V(b) \neq 0$. Applying the implicit function theorem, we deduce the existence of $\rho > 0$, an open set $L \subset I$ which contains a, and a bijection φ_1 from $]1-\rho, 1]$ to L such that $\Psi_2\big(p, \varphi_1(p)\big) = 0$ for all $p \in]1-\rho, 1]$. Let us define $\varphi_2 := \xi \circ \varphi_1$. This function is a bijection from $]1-\rho, 1]$ to an open set $K \subset J$. Thus, for all $p \in]1-\rho, 1]$, any $\big(\varphi_1(p), \varphi_2(p), p\big)$-vector is a critical point of $N\Upsilon_0^N$.

S t e p 3. Now, we study $N\nabla^2 \Upsilon_0^N\big(\varphi_1(p), \varphi_2(p), p\big)$. By making linear algebra computations, we can show that the eigenvalues are those of four matrices $\lambda_1(N, \rho)$, $\lambda_2(N, \rho)$, $\lambda_3(N, \rho)$ and $\lambda_4(N, \rho)$ satisfying

$$\begin{aligned}
&\lim_{\rho\to 0}\lim_{N\to\infty} \lambda_1(N, \rho) = \nabla^2 V(a) + \nabla^2 F(0),\\
&\lim_{\rho\to 0}\lim_{N\to\infty} \lambda_2(N, \rho) = \lim_{\rho\to 0}\lim_{N\to\infty} \lambda_3(N, \rho) = \nabla^2 V(b) + \nabla^2 F(b-a),\\
&\lim_{\rho\to 0}\lim_{N\to\infty} \lambda_4(N, \rho) = \nabla^2 V(a).
\end{aligned}$$

Since $\nabla^2 V(b) + \nabla^2 F(b-a)$ and $\nabla^2 V(a)$ are positive definite, λ_1, λ_2 and λ_3 are positive and definite. We proceed as in the previous steps by applying the implicit function theorem. Hence, there exists $\rho_0 > 0$ such that for all $1 - \rho_0 < \rho \leqslant 1$ and for N large enough, any $\big(\varphi_1(\rho), \varphi_2(\rho), \rho\big)$-vector is a local minimizer of $N\Upsilon_0^N$, and so is a local minimizer of Υ_0^N.

Step 4. Now, we remark that the number of $\big(\varphi_1\,(k/N)\,, \varphi_2\,(k/N)\,, k/N\big)$-vectors is equal to $N!/\big(k!(N-k)!\big)$. Consequently, the number of local minimizers constructed in Step 3 is

$$\sum_{k=\mathcal{E}((1-\rho_0)N)}^{N} \frac{N!}{k!(N-k)!}\,,$$

which converges to infinity as N goes to infinity. ∎

Proposition 3.2 gives us a result which has previously been established in [3] for a near-neighbour system, that is, the convergence to infinity of the number of wells as N goes to infinity.

Proposition 3.2 also points out that the number of steady states, as σ goes to zero, is arbitrarily large for N large enough.

REMARK 3.1. *In the proof of Proposition* 3.2, *we recover the family of equations* (3.11) *in* [14]. *Due to the restriction to the* (a_1, a_2, p)*-vectors with* a_1 *close to* $a \in \mathbb{R}^d$ *such that* $\nabla^2\, V(a) > 0$, a_2 *close to* b *such that* $\nabla V(b) + \nabla F(b-a) = 0$ *and* $\nabla^2\, V(b) + \nabla^2\, F(b-a) > 0$ *and* p *close to* 1, *the minimizers that we constructed satisfy the following two inequalities*:

$$\nabla^2\, V(a_1) + p\nabla^2\, F(0) + (1-p)\nabla^2\, F(a_2 - a_1) > 0$$

and

$$\nabla^2\, V(a_2) + p\nabla^2\, F(a_2 - a_1) + (1-p)\nabla^2\, F(0) > 0,$$

that is, *the family of inequalities* (3.13) *in* [14]. *However*, *there is no correspondence between the local minimizers of* Υ_0^N *and the invariant probabilities of the non-linear diffusion since the family of equations* (3.12) *in* [14], *that is*,

$$\frac{\nabla F(a_2 - a_1)}{F(a_2 - a_1)} = \frac{\nabla V(a_2) + \nabla V(a_1)}{V(a_2) - V(a_1)},$$

is not a priori *satisfied. However*, *a discrete measure is the small-noise limit of a sequence of invariant probabilities only if it satisfies* (3.11)–(3.13).

Nevertheless, even if the number of wells of Υ_0^N goes to infinity, we establish in the following that there is no correspondence between the steady states of the mean-field system (1.1) and these wells as N goes to infinity.

4. MAIN RESULTS

Now, we remind the reader of the classical results about the propagation of chaos. Particularly, we give a coupling result on the interval $[0, T]$ between the mean-field system and the self-stabilizing process as N converges to infinity. Each particle of the mean-field system satisfies the equation

$$Z_t^i = Z_0^i + \sigma B_t^i - \int_0^t \nabla V(Z_s^i) ds - \int_0^t \frac{1}{N} \sum_{j=1}^N \nabla F(Z_s^i - Z_s^j) ds,$$

and the associated McKean–Vlasov diffusions are

$$X_t^i = Z_0^i + \sigma B_t^i - \int_0^t \nabla V(X_s^i) ds - \int_0^t \nabla F * \mu_s(X_s^i) ds$$

with $\mu_s = \mu_s^i := \mathcal{L}(X_s^i)$. Here, $B^1, \ldots, B^N$ are independent d-dimensional Wiener processes. The proof is similar to that of Theorem 5.3 in [1]. Consequently, it is left to the reader.

PROPOSITION 4.1. *Let us assume that the triple (V, F, μ_0) satisfies the set of assumptions* (M). *Let $\{Z_0^i\,;\, i \in \mathbb{N}^*\}$ be a family of i.i.d. random variables with common law μ_0. Then there exist two positive constants C and K such that*

$$\max_{1\leqslant i\leqslant N} \mathbb{E}\{\sup_{0\leqslant t\leqslant T} \|X_t^i - Z_t^i\|^2\} \leqslant \frac{CT}{N} \exp[K\,T].$$

This result is not uniform with respect to the time. Now, we give the following essential lemma.

LEMMA 4.1. *We assume that the triple (V, F, μ_0) satisfies the set of assumptions* (M). *Let T be a positive real and $\{Z_0^i\,;\, i \in \mathbb{N}^*\}$ be a family of i.i.d. random variables with law μ_0. Let $\mu^N := \{N^{-1}(\delta_{Z_t^1} + \ldots + \delta_{Z_t^N});\, 0 \leqslant t \leqslant T\}$ be the family of random empirical measures on the Skorokhod path space $E_T := \mathbb{D}([0,T]\,;\, \mathbb{R}^d)$. Then μ^N converges, as N goes to infinity, to the family of deterministic measures $\mu^\infty := \{\mu_t\,;\, 0 \leqslant t \leqslant T\}$ in law and in probability.*

P r o o f. The proof is similar to that of Theorem 4.4 in [21]. By π^N we denote the law of the family of random measures μ^N. In other words, for any $f \in E_T$, we have

$$\int_{\mathcal{P}(E_T)} \int_{[0,T]\times\mathbb{R}^d} f(t,x) m(dt,dx) \pi^N(dm) = \mathbb{E}\Big\{ \int_{[0,T]\times\mathbb{R}^d} f(t,x) \mu_t^N(dx) dt \Big\}.$$

We need to verify the following three arguments:

- The family $\{\pi^N\,;\, N \in \mathbb{N}^*\}$ is tight.

- Each adherence value of the family $\{\mu^N\,;\,N\in\mathbb{N}^*\}$ satisfies the martingale problem associated with equation (1.3).
- There is a unique solution to this martingale problem.

The tightness of the family $\{\pi^N\,;\,N\in\mathbb{N}^*\}$ is a consequence of Proposition 4.6 in [21]. Indeed, if E is a Polish space, this proposition points out that a family of probability measures $\{\eta^N\,;\,N\in\mathbb{N}^*\}$ on $\mathcal{P}(E)$ is tight provided that the family of intensity measures $\{I(\eta^N)\,;\,N\in\mathbb{N}^*\}$ is tight, the measure $I(\eta^N)$ being defined as follows:

$$\int_E f(x)I(\eta^N)(dx) := \int_{\mathcal{P}(E)}\Big(\int_E f(x)m(dx)\Big)\eta^N(dm).$$

Here, the intensity measure is equal to $\{\mathcal{L}(Z_t^1)\,;\,0\leqslant t\leqslant T\}$. However, by Proposition 4.1, the family of intensity measures converges to $\{\mu_t\,;\,0\leqslant t\leqslant T\}$ in Wasserstein distance. Consequently, the family of intensity measures is tight. Hence, the family $\{\pi^N\,;\,N\in\mathbb{N}^*\}$ is tight. The identification between the limiting values and the solutions of the martingale problem is the same as in the proof of Theorem 4.4 in [21]. For the last point, we refer the reader to Theorem 2.13 in [12]. ∎

Let us recall that $\mathcal{A}_\sigma$ is the set of the adherence values of the family of probability measures $\{\mu_t\,;\,t\geqslant 0\}$, $\mathcal{S}_\sigma$ is the set of the invariant probabilities of diffusion (1.3), and u_t is the density of μ_t with respect to the Lebesgue measure. Now, we are able to provide the main result of the paper.

THEOREM 4.1. *We assume that the triple (V, F, μ_0) satisfies the set of assumptions* (M). *Let $\{Z_0^i\,;\,i\geqslant 1\}$ be a family of i.i.d. random variables with common law μ_0. We take a smooth function f from $\mathbb{R}^d$ to itself satisfying the inequality*

$$\|f(x)-f(y)\|\leqslant C\|x-y\|(1+\|x\|+\|y\|),$$

C being a positive constant. Then, for all $\delta>0$ and for all $T\geqslant 0$, we have

$$\lim_{N\to+\infty}\mathbb{P}\left\{\sup_{0\leqslant t\leqslant T}\left\|\frac{1}{N}\sum_{i=1}^N f(Z_t^i)-\int_{\mathbb{R}^d}f(x)u_t(x)dx\right\|<\delta\right\}=1. \tag{4.1}$$

Furthermore, there exists $T_\delta\geqslant 0$ deterministic such that

$$\lim_{N\to+\infty}\mathbb{P}\left\{\sup_{T_\delta\leqslant t\leqslant T_\delta+T}\inf_{\mu\in\mathcal{A}_\sigma}\left\|\frac{1}{N}\sum_{i=1}^N f(Z_t^i)-\int_{\mathbb{R}^d}f(x)\mu(dx)\right\|<\delta\right\}=1 \tag{4.2}$$

for any $T>0$.

Proof. Step 1. We aim to prove (4.1). The triangular inequality provides

$$\left\|\frac{1}{N}\sum_{i=1}^{N} f(Z_t^i) - \int_{\mathbb{R}^d} f(x)u_t(x)dx\right\| \leqslant \frac{1}{N}\sum_{i=1}^{N}\sup_{0\leqslant t\leqslant T}\|f(Z_t^i) - f(X_t^i)\| + \sup_{0\leqslant t\leqslant T}\left\|\frac{1}{N}\sum_{i=1}^{N} f(X_t^i) - \int_{\mathbb{R}^d} f(x)u_t(x)dx\right\|$$

for any $t \in [0,T]$. By A_1^N (respectively, A_2^N) we denote the first term (respectively, the second one). Therefore, we obtain the inequality

$$\mathbb{P}\left\{\sup_{0\leqslant t\leqslant T}\left\|\frac{1}{N}\sum_{i=1}^{N} f(Z_t^i) - \int_{\mathbb{R}^d} f(x)u_t(x)dx\right\| \geqslant \delta\right\} \leqslant \mathbb{P}\left\{A_1^N \geqslant \frac{\delta}{2}\right\} + \mathbb{P}\left\{A_2^N \geqslant \frac{\delta}{2}\right\}.$$

According to Lemma 4.1, the second term, $\mathbb{P}\left\{A_2^N \geqslant \delta/2\right\}$, goes to zero as N goes to infinity. Let us prove that $\lim_{N\to+\infty}\mathbb{P}\{A_1^N \geqslant \delta/2\} = 0$.

Step 2. For the moment we take $f_0(x) := x$. Applying Markov's inequality we get

$$\mathbb{P}\left\{\frac{1}{N}\sum_{i=1}^{N}\sup_{0\leqslant t\leqslant T}\|Z_t^i - X_t^i\| \geqslant \frac{\delta}{2}\right\} \leqslant \frac{2}{\delta}\mathbb{E}\left\{\sup_{0\leqslant t\leqslant T}\frac{1}{N}\sum_{i=1}^{N}\|Z_t^i - X_t^i\|\right\}.$$

The particles Z^i are exchangeable and the McKean–Vlasov diffusions are independent. So, we obtain

$$\mathbb{P}\left\{A_1^N \geqslant \frac{\delta}{2}\right\} \leqslant \frac{2}{\delta}\mathbb{E}\{\sup_{0\leqslant t\leqslant T}\|Z_t^1 - X_t^1\|\} \leqslant \frac{2}{\delta}\sqrt{\frac{CT}{N}}\exp\left[\frac{KT}{2}\right]$$

after applying the Cauchy–Schwarz inequality and Proposition 4.1. Using inequality (2.1) and the propagation of chaos established in Proposition 4.1, we see that the following holds for all $T > 0$:

$$\mathbb{E}[\sup_{0\leqslant t\leqslant T}\|X_t^1\|^2] + \mathbb{E}[\sup_{0\leqslant t\leqslant T}\|Z_t^1\|^2] \leqslant \lambda(T) < \infty.$$

Step 3. In the general case, again the Markov inequality, the Jensen inequality and Proposition 4.1 imply

$$\mathbb{P}\left\{A_1^N \geqslant \frac{\delta}{2}\right\} \leqslant \frac{2}{\delta}\mathbb{E}\{\sup_{0\leqslant t\leqslant T}\|f(Z_t^1) - f(X_t^1)\|\} \leqslant \frac{2}{\delta}\sqrt{\frac{CT}{N}}\sqrt{1+\lambda(T)}\exp\left[\frac{KT}{2}\right],$$

which goes to zero as N goes to infinity.

S t e p 4. In order to prove the second statement it is sufficient to notice that the tightness of the family $\{\mu_t\,;\,t\geqslant 0\}$ and the definition of $\mathcal{A}_\sigma$ yield

$$\lim_{T_0\to+\infty}\sup_{t\geqslant T_0}\inf_{\mu\in\mathcal{A}_\sigma}\Big\|\int_{\mathbb{R}^d}f(x)u_t(x)dx-\int_{\mathbb{R}^d}f(x)\mu(dx)\Big\|=0.$$

Consequently, for all $\delta>0$, there exists $T_\delta\geqslant 0$ such that

$$\inf\Big\{\Big\|\int_{\mathbb{R}^d}f(x)u_t(x)dx-\int_{\mathbb{R}^d}f(x)\mu(dx)\Big\|;\,\mu\in\mathcal{A}_\sigma\Big\}\leqslant\delta/2$$

for any $t\geqslant T_\delta$. Then we apply the first statement with $\delta/2$ on the interval $[0,T_\delta+T]$, which completes the proof. ∎

The time T_δ is deterministic and is linked to the rate of convergence to the invariant probabilities of $\mathcal{A}_\sigma$, so it does depend on σ.

REMARK 4.1. *Under the set of assumptions* (M), *the limits* (4.1) *and* (4.2) *in Theorem* 4.1 *hold with a smooth function f from $\mathbb{R}^d$ to $\mathbb{R}$ satisfying*, *for some* $C>0$,

$$|f(x)-f(y)|\leqslant C\|x-y\|(1+\|x\|+\|y\|).$$

From Theorem 4.1 and the remark above we derive two corollaries. The first one establishes that the mean-field system is a prisoner one, as N goes to infinity, of some union of balls.

COROLLARY 4.1. *Let us assume that the triple (V,F,μ_0) satisfies the set of assumptions* (M). *Let $\{Z_0^i\,;\,i\geqslant 1\}$ be a family of i.i.d. random variables with common law μ_0. Then*, *for all $r>0$, there exists $T_r\geqslant 0$ such that*

$$\lim_{N\to+\infty}\mathbb{P}\Big\{\mathcal{Z}_t^N\in\bigcup_{\mu\in\mathcal{A}_\sigma}\mathbb{S}_r^N(\mu);\forall\,T_r\leqslant t\leqslant T_r+T\Big\}=1,\tag{4.3}$$

the set $\mathbb{S}_r^N(\mu)$ being defined by

$$\mathbb{S}_r^N(\mu):=\left\{\mathcal{Z}\in\mathbb{R}^{dN}\,:\,\mathrm{Var}(\mu)-r\leqslant\frac{1}{N}\sum_{i=1}^N\|Z_i-\mathbb{E}\{\mu\}\|^2\leqslant\mathrm{Var}(\mu)+r\right\}.$$

Here, *the limit* (4.3) *holds for any* $T\geqslant 0$.

P r o o f. S t e p 1. First of all, for all $t\geqslant 0$ and for all $\mu\in\mathcal{A}_\sigma$, we define the quantity

$$\Lambda_t^N(\mu):=\frac{1}{N}\sum_{i=1}^N\|Z_t^i-\mathbb{E}\{\mu\}\|^2-\mathrm{Var}(\mu).$$

In the same way, we define

$$\Lambda_t^\infty(\mu):=\int_{\mathbb{R}^d}\|x-\mathbb{E}\{\mu\}\|^2\mu_t(dx)-\mathrm{Var}(\mu).$$

S t e p 2. Let us prove that $\inf\{\Lambda_t^\infty(\mu)\,;\,\mu\in\mathcal{A}_\sigma\}$ converges to zero as t goes to infinity. We proceed by *reductio ad absurdum*, assuming the existence of $r_0>0$ and a family $\{T_n\,;\,n\in\mathbb{N}^*\}$ such that $T_n\geqslant n$ and $\inf\{\Lambda_{T_n}^\infty(\mu)\,;\,\mu\in\mathcal{A}_\sigma\}\geqslant r_0$ for all $n\geqslant 0$. The family of measures $\{\mu_t\,;\,t\geqslant 0\}$ is tight, so the family $\{\mu_{T_n}\,;\,n\in\mathbb{N}^*\}$ has an adherence value $\mu^\sigma\in\mathcal{A}_\sigma$ (by the definition of the set $\mathcal{A}_\sigma$). Moreover, this element μ^σ satisfies

$$\inf_{\mu\in\mathcal{A}_\sigma}\Big\{\int_{\mathbb{R}^d}\|x-\mathbb{E}\{\mu\}\|^2\mu^\sigma(dx)-\mathrm{Var}(\mu)\Big\}\geqslant r_0.$$

This is absurd since it is equal to zero, μ^σ belonging to $\mathcal{A}_\sigma$.

S t e p 3. Let r be a positive real. There exists $T_r>0$ such that, for all $t\geqslant T_r$, we have $|\inf\{\Lambda_t^\infty(\mu)\,;\,\mu\in\mathcal{A}_\sigma\}|<r/2$. Let T be a positive real. We apply Theorem 4.1 on the interval $[T_r,T_r+T]$; more precisely, the limit (4.1) with $f_1(x):=x$ and the limit (4.1) (plus Remark 4.1) with $f_2(x):=\|x\|^2$. We obtain the following two limits:

$$\lim_{N\to+\infty}\mathbb{P}\bigg\{\sup_{T_r\leqslant t\leqslant T_r+T}\bigg\|\frac{1}{N}\sum_{i=1}^N Z_t^i-\int_{\mathbb{R}^d}xu_t(x)dx\bigg\|<\delta\bigg\}=1$$

and

$$\lim_{N\to+\infty}\mathbb{P}\bigg\{\sup_{T_r\leqslant t\leqslant T_r+T}\bigg|\frac{1}{N}\sum_{i=1}^N \|Z_t^i\|^2-\int_{\mathbb{R}^d}\|x\|^2u_t(x)dx\bigg|<\delta\bigg\}=1.$$

However, for all $(Z_1,\dots,Z_N)\in\mathbb{R}^{dN}$ such that

$$\bigg\|\frac{1}{N}\sum_{j=1}^N Z_j-\int_{\mathbb{R}^d}xu_t(x)dx\bigg\|<\delta\quad\text{and}\quad\bigg|\frac{1}{N}\sum_{j=1}^N\|Z_j\|^2-\int_{\mathbb{R}^d}\|x\|^2u_t(x)dx\bigg|<\delta,$$

the following inequality holds for all $\mu\in\mathcal{A}_\sigma$:

$$\bigg|\frac{1}{N}\sum_{i=1}^N\|Z_i-\mathbb{E}\{\mu\}\|^2-\mathrm{Var}(\mu)-\Lambda_t^\infty(\mu)\bigg|\leqslant(1+2\|\mathbb{E}\{\mu\}\|)\,\delta.$$

However, due to the inequality (2.1), we have $\sup\{\|\mathbb{E}\{\mu\}\|\,;\,\mu\in\mathcal{A}_\sigma\}\leqslant M_0$. Taking $\delta:=r/\big(2(1+2M_0)\big)$, we obtain the limit (4.3). ■

This result means that, for all $T\geqslant 0$, $\lim_{N\to+\infty}\mathbb{P}\{\tau_N^r\leqslant T\}=0$, τ_N^r being the first exit time of the domain

$$\bigcup_{\mu\in\mathcal{A}_\sigma}\mathbb{B}^N_{\sqrt{\mathrm{Var}(\mu)+r}}\left(\mathbb{E}\left[\mu\right],\dots,\mathbb{E}\left[\mu\right]\right)\cap\left(\mathbb{B}^N_{\sqrt{\mathrm{Var}(\mu)-r}}\left(\mathbb{E}\left[\mu\right],\dots,\mathbb{E}\left[\mu\right]\right)\right)^c.$$

Let us present two examples of application of Corollary 4.1, when the set $\mathcal{A}_\sigma$ is a single element and when it is not.

EXAMPLE 4.1. Take $d := 1$, $V(x) := x^4/4 - x^2/2$ and $F(x) := (\alpha/2)x^2$ with $\alpha > 1$. Let μ_0 be a measure which satisfies the assumptions (M6) and (M7). We consider a family $\{Z_0^i\,;\, i \geqslant 1\}$ of i.i.d. random variables with common law μ_0. Thus, for all $r > 0$, by taking σ sufficiently small there exists $T_r > 0$ and $m \in \{-1, 0, 1\}$ such that

$$\lim_{N\to+\infty} \mathbb{P}\{\mathcal{Z}_t^N \in \mathbb{B}_r^N(m, \dots, m); \forall\, T_r \leqslant t \leqslant T_r + T\} = 1$$

for any $T \geqslant 0$.

Proof. By Theorems 3.2 and 4.5 in [13] and by Proposition 3.7 in [14], we know that diffusion (1.3) admits exactly three invariant probabilities if σ is small enough and these measures converge in the small-noise limit to δ_{-1}, δ_0 or δ_1. This implies that $\sup\{\mathrm{Var}(\mu)\,;\, \mu \in \mathcal{S}_\sigma\}$ goes to zero as σ goes to zero. Then, we apply Corollary 4.1 since the set of assumptions (M) is satisfied. ■

However, the set $\mathcal{A}_\sigma$ is not necessarily a single element. Then, we give a similar result in the case in which the set $\mathcal{S}_\sigma$ is not discrete.

EXAMPLE 4.2. Take $d>1$, $V(x):=\frac{1}{4}\|x\|^4-\frac{1}{2}\|x\|^2$ and $F(x):=(\alpha/2)\|x\|^2$ with $\alpha > 1$. Let μ be a measure satisfying (M6), (M7) and

$$\int_{\mathbb{R}^d} V(x)\mu(dx) + \frac{1}{2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} F(x-y)\mu(dx)\mu(dy) < 0. \tag{4.4}$$

We consider $\{Z_0^i\,;\, i \geqslant 1\}$, a family of i.i.d. random variables with common law μ_0. Thus, for all $r > 0$, by taking σ sufficiently small, there exists $T_r > 0$ such that, for all $T \geqslant 0$,

$$\lim_{N\to+\infty} \mathbb{P}\{1 - r < \|\mathcal{Z}_t^N\|_N < 1 + r; \forall\, T_r \leqslant t \leqslant T_r + T\} = 1.$$

Proof. Let us denote by μ_0^σ the unique radial invariant probability of diffusion (1.3). We can easily prove that $\lim_{\sigma\to0} \Upsilon_\sigma(\mu_0^\sigma) = 0$. By inequality (4.4), we see that $\Upsilon_\sigma(\mu) < \Upsilon_\sigma(\mu_0^\sigma)$ for σ sufficiently small. Thus, by Theorem A in [27] and Proposition 2.1 in [5], we know that $\mu_0^\sigma \notin \mathcal{A}_\sigma$. By Proposition 3.10 in [23], we know that each family $\{\mu^\sigma\,;\, \sigma > 0\}$ with $\mu^\sigma \in \mathcal{S}_\sigma$ admits an adherence value. And, by Proposition 3.8 in [23], since $\alpha > 1$, we know that the only possible small-noise limits of these sequences are Dirac measures δ_a, a being in $\mathcal{V}$, the set of the critical points of V. Here, we have $\mathcal{V} = \{0\} \cup \{x \in \mathbb{R}^d : \|x\| = 1\}$. Since $\Upsilon_\sigma(\mu_0) < \Upsilon_\sigma(\mu_0^\sigma)$, we infer that δ_0 is not the small-noise limit of any family of adherence values of $\{\mu_t\,;\, t \geqslant 0\}$. Besides, since the small-noise limits are Dirac measures, it implies $\lim_{\sigma\to0}\sup\{\mathrm{Var}(\mu)\,;\, \mu \in \mathcal{S}_\sigma\} = 0$. Then, we apply Corollary 4.1. ■

The second corollary provides a sufficient condition to forbid the crossing of any hyperplane $\{\mathcal{Z} \in \mathbb{R}^{dN} : N^{-1}(Z_1 + \ldots + Z_N) = m\}$, $m \in \mathbb{R}^d$.

COROLLARY 4.2. *We assume the triple* (V, F, μ_0) *satisfies the set of assumptions* (M) *and* $m_0 \in \mathbb{R}^d$ *is such that* $\Upsilon_\sigma(\mu_0) < \inf\left\{\Upsilon_\sigma(\mu) : \int_{\mathbb{R}^d} x\mu(dx) = m_0\right\}$. *Then, for all* $T \geqslant 0$, *the following limit holds*:

$$\lim_{N\to+\infty} \mathbb{P}\left\{\frac{1}{N}\sum_{i=1}^{N} Z_t^i \neq m_0; \forall\, 0 \leqslant t \leqslant T\right\} = 1.$$

Proof. By Proposition 2.1 in [5], the free energy is non-increasing along the orbit $\{\mu_t\,;\, t \geqslant 0\}$. Consequently, $\Upsilon_\sigma(\mu_t) < \inf\left\{\Upsilon_\sigma(\mu) : \int_{\mathbb{R}^d} x\mu(dx) = m_0\right\}$ for all $t \in [0, T]$, and so $\int_{\mathbb{R}^d} x\mu_t(dx) \neq m_0$ for all $t \geqslant 0$. We conclude by applying (4.1) with $f(x) := x$ and $\delta := \inf\left\{\left\|\int_{\mathbb{R}^d} x\mu_t(dx) - m_0\right\|; 0 \leqslant t \leqslant T\right\} > 0$. ∎

This result means that $\lim_{N\to+\infty} \mathbb{P}\{T_N^{m_0} \leqslant T\} = 0$ for any $T \geqslant 0$, where $T_N^{m_0}$ is the first hitting time of $\{\mathcal{Z} \in \mathbb{R}^{dN} : N^{-1}(Z_1 + \ldots + Z_N) = m_0\}$.

As a conclusion, let us make the following remark. When N is fixed, the Freidlin–Wentzell theory takes into account the microscopic wells for the computations of the exit time in the small-noise limit. However, when σ is fixed, they do not intervene in the dynamics as N goes to infinity.

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Received on 10.8.2012;
revised version on 10.9.2014