PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 35, Fasc. 2 (2015), pp. 343–365

STOCHASTIC DIFFERENTIAL EQUATIONS WITH CONSTRAINTS DRIVEN BY PROCESSES WITH BOUNDED p-VARIATION*

BY

ADRIAN FALKOWSKI (TORUŃ) AND LESZEK SŁOMIŃSKI (TORUŃ)

Abstract. We study the existence, uniqueness and approximation of solutions of stochastic differential equations with constraints driven by processes with bounded p-variation. Our main tool are new estimates showing Lipschitz continuity of the deterministic Skorokhod problem in p-variation norm. Applications to fractional SDEs with constraints are given.

2010 AMS Mathematics Subject Classification: Primary: 60H20; Secondary: 60G22.

Key words and phrases: Skorokhod problem, p-variation, integral equations, stochastic differential equations with constraints, reflecting boundary condition.

1. INTRODUCTION

In the present paper we study the problems of existence, uniqueness and approximation of solutions of finite-dimensional stochastic differential equations (SDEs) with constraints driven by general processes with bounded p-variation, $p \geqslant 1$. More precisely, let $f:\mathbb{R}^d \to \mathbb{R}^d$, $g:\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ be measurable functions, A be a one-dimensional process with locally bounded variation, and Z be a d-dimensional process with locally bounded p-variation. We consider SDEs with reflecting boundary condition of the form

$$X_t = X_0 + \int_0^t f(X_{s-})\,dA_s + \int_0^t g(X_{s-})\,dZ_s + K_t, \quad t \in \mathbb{R}^+. \tag{1.1}$$

By a *solution* to (1.1) we mean a pair (X, K) consisting of a process X living over a given d-dimensional barrier process L and a d-dimensional process K, called a *regulator term*, whose each component K^i is nondecreasing and increases only when X^i is living on L^i (for details see Section 3). Equation (1.1) is called the *Skorokhod SDE* by analogy with the case $L = 0$ first discussed by Skorokhod [26]

* Research supported by Polish NCN grant No. 2012/07/B/ST1/03508.

for a standard Brownian motion in place of Z and $A_t = t$, $t \in \mathbb{R}^+$. Next, many attempts have been made to extend Skorokhod's results to a larger class of domains or a larger class of driving processes (see, e.g., [3], [9], [19], [28], [33]). The equations of this kind have many applications, for instance in queueing systems, seismic reliability analysis and finance (see, e.g., [1], [10], [16], [25]). In recent papers by Besalu and Rovira [2] and Ferrante and Rovira [13] the SDE with non-negativity constraints driven by fractional Brownian motion B^H with Hurst index $H > 1/2$ and $A_t = t$, $t \in \mathbb{R}^+$, is studied. This equation is a particular case of (1.1) because B^H has locally bounded p-variation for $p > 1/H$. In the main theorem of [13] the existence of a solution is proved under the assumption that the coefficients f, g are Lipschitz continuous. The proof is based on a quite natural, in the context of SDEs driven by B^H, technics based on λ-Hölder norms. Unfortunately, in [13] it is only shown that the solution is unique for some small time interval. To our knowledge, global uniqueness for fractional SDEs with constraints is still an open problem. In contrast to [13], in our paper we use p-variation norm (for the theory of functions of p-variation and its various applications see, e.g., [7], [8]).

In our paper we consider two conditions (see Section 3): continuity and linear growth of f and Hölder continuity of g (condition (H1)) and local Lipschitz continuity of f and local Hölder continuity of the derivative of each component $g_{i,j}$, $i, j = 1, \ldots, d$ (condition (H2)). We show that under (H1) and (H2) there exists a unique (globally in time) solution to (1.1), which can be approximated by some natural approximation schemes.

The paper is organized as follows.

In Section 2 we consider the deterministic Skorokhod problem $x = y + k$ associated with $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and time-dependent lower barrier $l \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ with $l_0 \leqslant y_0$. We show that the mapping $(y, l) \mapsto (x, k)$ is Lipschitz continuous in p-variation norm. In fact, we show that if (x, k) is a solution associated with $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and barrier $l \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, and (x', k') is a solution associated with $y' \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and barrier $l' \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, then, for any $T \in \mathbb{R}^+$,

$$\bar{V}_p(x - x')_T \leqslant (d+1)\bar{V}_p(y - y')_T + d\bar{V}_p(l - l')_T \tag{1.2}$$

and

$$\bar{V}_p(k - k')_T \leqslant d\bar{V}_p(y - y')_T + d\bar{V}_p(l - l')_T. \tag{1.3}$$

It is worth noting here that in [13], Remark 3.6, it is observed that $(y, l) \mapsto (x, k)$ is not Lipschitz continuous in the λ-Hölder norm and for that reason in [13] the authors were not able to obtain global uniqueness.

In Section 3 we consider a deterministic counterpart to (1.1). We prove that under (H1) the deterministic equation has a solution. If, moreover, (H2) is satisfied, then it is unique. Then we show convergence of some natural approximation schemes for a deterministic equation of the form (1.1). In the proofs of convergence we use the Skorokhod topology J_1 and general methods of approximations of stochastic integrals and solutions of SDEs developed in [15], [21], [27], [29]. For the convenience of the reader we prove in Appendix a general tightness criterion and a functional limit theorem for sequences of integrals with respect to càdlàg functions with bounded p-variation.

In Section 4 we apply our deterministic results to obtain the existence, uniqueness and approximation of solutions to SDEs of the form (1.1). In particular, we show that if f, g satisfy (H1) and (H2), then (1.1) has a unique strong solution (X, K). Moreover, we show convergence to (X, K) of some easily implementable approximations (X^n, K^n) constructed by analogy with the classical Euler scheme. To illustrate how our results work in practice, at the end of the paper we consider fractional SDEs with constraints of the form

$$X_t = X_0 + \int_0^t f(X_{s-})\, da_s + \int_0^t g(X_{s-})\, dZ_s^H + K_t, \quad t \in \mathbb{R}^+. \tag{1.4}$$

Here $a : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function with locally bounded variation, and $Z^{H,i} = \int_0^{\cdot} \sigma_s^i\, dB_s^{H,i}$, $t \in \mathbb{R}^+$, where $B^{H,1}, \ldots, B^{H,d}$ are independent fractional Brownian motions and $\sigma^i : \mathbb{R}^+ \to \mathbb{R}$ are such that $\|\sigma^i\|_{\mathbb{L}^{1/H}_{[0,T]}} := \big(\int_0^T |\sigma_s^i|^{1/H} ds\big)^H < \infty$, $T > 0$, $i = 1, \ldots, d$. Under the last assumption Z^H is a centered Gaussian process with continuous trajectories such that $P\big(V_p(Z^H)_T < \infty\big) = 1$, $p > 1/H$, $T \in \mathbb{R}^+$ (see Section 4), so (1.4) is a particular case of (1.1). Of course, (1.4) generalizes classical fractional SDEs driven by B^H.

In the sequel we will use the following notation. $\mathbb{M}^d$ is the space of $d \times d$ real matrices A with the matrix norm $\|A\| = \sup\{|Au|; u \in \mathbb{R}^d, |u| = 1\}$, where $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^d$, $\mathbb{R}^+ = [0, \infty)$. $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ is the space of càdlàg mappings $x : \mathbb{R}^+ \to \mathbb{R}^d$, i.e., mappings which are right continuous and admit left-hand limits equipped with the Skorokhod topology J_1. For $x \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $t > 0$, we put $x_{t-} = \lim_{s \uparrow t} x_s$, $v_p(x)_{[a,b]} = \sup_\pi \sum_{i=1}^n |x_{t_i} - x_{t_{i-1}}|^p < \infty$, where the supremum is taken over all subdivisions $\pi = \{a = t_0 < \ldots < t_n = b\}$ of $[a, b]$. $V_p(x)_{[a,b]} = \big(v_p(x)_{[a,b]}\big)^{1/p}$ and $\bar{V}_p(x)_{[a,b]} = V_p(x)_{[a,b]} + |x_a|$ is the usual variation norm. For simplicity of the notation we write $v_p(x)_T = v_p(x)_{[0,T]}$, $V_p(x)_T = V_p(x)_{[0,T]}$ and $\bar{V}_p(x)_T = \bar{V}_p(x)_{[0,T]}$. If $x \in \mathbb{D}(\mathbb{R}^+, \mathbb{M}^d)$, then in the definition of p-variation v_p we use the matrix norm $\|\cdot\|$ in place of the Euclidean norm. We write $x \leqslant x'$, $x, x' \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, if $x_t^i \leqslant x_t'^i$, $t \in \mathbb{R}^+$, $i = 1, \ldots, d$. Every process Y appearing in the sequel is assumed to have càdlàg trajectories.

2. LIPSCHITZ CONTINUITY OF THE SOLUTION OF THE SKOROKHOD PROBLEM IN p-VARIATION NORM

Let $y, l \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ be such that $l_0 \leqslant y_0$. We recall that a pair $(x, k) \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ is called a *solution of the Skorokhod problem* associated with y and lower barrier l ($(x, k) = SP_l(y)$ for short) if

(i) $x_t = y_t + k_t \geqslant l_t$, $t \in \mathbb{R}^+$;

(ii) $k_0 = 0$, $k = (k^1, \ldots, k^d)$, where k^i are nondecreasing functions such that, for every $t \in \mathbb{R}^+$,

$$\int_0^t (x_s^i - l_s^i)\, dk_s^i = 0, \qquad i = 1, \ldots, d.$$

The Lipschitz continuity of the mapping $(y, l) \mapsto (x, k)$ in the supremum norm is well known. More precisely, let $(x, k) = SP_l(y)$, $(x', k') = SP_{l'}(y')$. Since $k_t = \sup_{s \leqslant t}(y_s - l_s)^-$ and $k'_t = \sup_{s \leqslant t}(y'_s - l'_s)^-$, for any $T \in \mathbb{R}^+$ we have

$$\sup_{t \leqslant T} |x_t - x'_t| \leqslant 2 \sup_{t \leqslant T} |y_t - y'_t| + \sup_{t \leqslant T} |l_t - l'_t| \tag{2.1}$$

and

$$\sup_{t \leqslant T} |k_t - k'_t| \leqslant \sup_{t \leqslant T} |y_t - y'_t| + \sup_{t \leqslant T} |l_t - l'_t|. \tag{2.2}$$

On the other hand, it was observed in Ferrante and Rovira [13] that the above property does not hold in the λ-Hölder norm. We will show that the Lipschitz continuity of the mapping $(y, l) \mapsto (x, k)$ holds in the variation norm. A key step in proving it is the following estimate.

THEOREM 2.1. *For any* $y^1, y^2 \in \mathbb{D}(\mathbb{R}^+, \mathbb{R})$ *and* $T \in \mathbb{R}^+$,

$$v_p(\sup_{s \leqslant \cdot} y_s^1 - \sup_{s \leqslant \cdot} y_s^2)_T \leqslant v_p(y^1 - y^2)_T.$$

Proof. It is clear that without loss of generality we may and will assume that

$$v_p(\sup_{s \leqslant \cdot} y_s^1 - \sup_{s \leqslant \cdot} y_s^2)_T > 0. \tag{2.3}$$

Step 1. We assume additionally that y^1, y^2 are step functions of the form

$$y_t^j = y_{j,i}, \quad t \in [t_{i-1}, t_i), \quad i = 1, \ldots, n-1,$$

and $y_t^j = y_{j,n}$, $t \in [t_{n-1}, t_n = T]$, $j = 1, 2$, for some partition $0 = t_0 < t_1 < \ldots < t_n = T$ of the interval $[0, T]$.

Set $Y_k^j = \max_{1 \leqslant i \leqslant k} y_{j,i}$, $k = 1, \ldots, n$, $j = 1, 2$. By (2.3) it is clear that there exists k such that $Y_k^1 > Y_{k-1}^1$ or $Y_k^2 > Y_{k-1}^2$. Without loss of generality we will assume that, for any $k = 2, \ldots, n$,

$$Y_k^1 > Y_{k-1}^1 \quad \text{or} \quad Y_k^2 > Y_{k-1}^2. \tag{2.4}$$

Indeed, if (2.4) does not hold, then we set

$$u_0 = 0, \quad u_k = \inf\{i > u_{k-1}; Y^1_i > Y^1_{i-1} \,\text{or}\, Y^2_i > Y^2_{i-1}\} \wedge n,\ k = 1, \dots, n.$$

and $\tilde n = \inf\{k; u_k = n\}$, $\tilde y^j_t = y_{j,u_k}$, $t \in [t_{u_{k-1}}, t_{u_k})$ for $k = 1, \dots, \tilde n - 1$, $\tilde y^j_t = y_{j,\tilde n}$ for $t \in [t_{u_{\tilde n - 1}}, t_{u_{\tilde n}} = T]$, $j = 1, 2$. Then (2.4) holds for the functions $\tilde y^1, \tilde y^2$ and $v_p(\sup_{s\leqslant \cdot} y^1_s - \sup_{s\leqslant \cdot} y^2_s)_T = v_p(\sup_{s\leqslant \cdot} \tilde y^1_s - \sup_{s\leqslant \cdot} \tilde y^2_s)_T$, $v_p(\tilde y^1 - \tilde y^2)_T \leqslant v_p(y^1 - y^2)_T$.

It is clear that there exist numbers $0 = i_0 < i_1 < \ldots < i_m = n$ such that

$$v_p(\sup_{s\leqslant \cdot} y^1_s - \sup_{s\leqslant \cdot} y^2_s)_T = \sum_{k=1}^m |(Y^1_{i_k} - Y^1_{i_{k-1}}) - (Y^2_{i_k} - Y^2_{i_{k-1}})|^p \tag{2.5}$$

and

$$(Y^1_{i_k} - Y^1_{i_{k-1}}) - (Y^2_{i_k} - Y^2_{i_{k-1}}) \neq 0 \tag{2.6}$$

for $k = 1, \dots, m$. In particular, this implies that if $m \geqslant 2$, then for $k = 2, \dots, m$ we have

$$\big((Y^1_{i_{k-1}} - Y^1_{i_{k-2}}) - (Y^2_{i_{k-1}} - Y^2_{i_{k-2}})\big)\big((Y^1_{i_k} - Y^1_{i_{k-1}}) - (Y^2_{i_k} - Y^2_{i_{k-1}})\big) < 0. \tag{2.7}$$

Indeed, if (2.7) is not satisfied, then, by (2.6),

$$\begin{aligned} &|(Y^1_{i_{k-1}} - Y^1_{i_{k-2}}) - (Y^2_{i_{k-1}} - Y^2_{i_{k-2}})|^p + |(Y^1_{i_k} - Y^1_{i_{k-1}}) - (Y^2_{i_k} - Y^2_{i_{k-1}})|^p \\ &\quad < |(Y^1_{i_{k-1}} - Y^1_{i_{k-2}}) - (Y^2_{i_{k-1}} - Y^2_{i_{k-2}}) + (Y^1_{i_k} - Y^1_{i_{k-1}}) - (Y^2_{i_k} - Y^2_{i_{k-1}})|^p \\ &\quad = |(Y^1_{i_k} - Y^1_{i_{k-2}}) - (Y^2_{i_k} - Y^2_{i_{k-2}})|^p, \end{aligned}$$

which contradicts (2.5). Set $l^j_k = \max\{i \leqslant i_k : y^j_i = Y^j_i\}$, $j = 1, 2$, and $l^\wedge_k = \min\{l^1_k, l^2_k\}$, $l^\vee_k = \max\{l^1_k, l^2_k\}$, $k = 1, \dots, m$. Then

$$y_{1,l^1_k} = Y^1_{l^1_k} = Y^1_{l^1_k+1} = \ldots = Y^1_{i_k} \quad \text{and} \quad y_{2,l^2_k} = Y^2_{l^2_k} = Y^2_{l^2_k+1} = \ldots = Y^2_{i_k}. \tag{2.8}$$

We claim that, for any $k = 1, \dots, m$,

$$i_{k-1} \leqslant l^\wedge_k \leqslant l^\vee_k \leqslant i_k. \tag{2.9}$$

The last two inequalities are obvious. Moreover, $0 = i_0 \leqslant l^\wedge_1$. Assume that there exists $2 \leqslant k \leqslant m$ such that $i_{k-1} > l^\wedge_k$. In what follows we will consider only the case $l^\wedge_k = l^1_k$ (the case $l^\wedge_k = l^2_k$ can be handled in much the same way). We have $i_{k-2} < l^\wedge_k$ because if $i_{k-2} \geqslant l^\wedge_k$, then, by (2.8),

$$\begin{aligned} |(Y^1_{i_{k-1}} - Y^1_{i_{k-2}}) - (Y^2_{i_{k-1}} - Y^2_{i_{k-2}})|^p &+ |(Y^1_{i_k} - Y^1_{i_{k-1}}) - (Y^2_{i_k} - Y^2_{i_{k-1}})|^p \\ &= |(Y^2_{i_{k-1}} - Y^2_{i_{k-2}})|^p + |(Y^2_{i_k} - Y^2_{i_{k-1}})|^p \\ &< |(Y^2_{i_k} - Y^2_{i_{k-2}})|^p \\ &= |(Y^1_{i_k} - Y^1_{i_{k-2}}) - (Y^2_{i_k} - Y^2_{i_{k-2}})|^p, \end{aligned}$$

which contradicts (2.5). From the inequality $i_{k-2} < l_k^\wedge$ and (2.4) it follows that

$$Y^1_{i_{k-2}} \leqslant Y^1_{l^1_k} = Y^1_{l^1_k+1} = \ldots = Y^1_{i_{k-1}} = \ldots = Y^1_{i_k} \tag{2.10}$$

and

$$Y^2_{i_{k-2}} \leqslant Y^2_{l^1_k} < Y^2_{l^1_k+1} < \ldots < Y^2_{i_{k-1}} < \ldots < Y^2_{i_k}. \tag{2.11}$$

Since

$$(Y^1_{i_k} - Y^1_{i_{k-1}}) - (Y^2_{i_k} - Y^2_{i_{k-1}}) = -(Y^2_{i_k} - Y^2_{i_{k-1}}) < 0,$$

(2.7), (2.10) and (2.11) imply

$$\begin{aligned} 0 < (Y^1_{i_{k-1}} - Y^1_{i_{k-2}}) - (Y^2_{i_{k-1}} - Y^2_{i_{k-2}}) &= (Y^1_{l^1_k} - Y^1_{i_{k-2}}) - (Y^2_{i_{k-1}} - Y^2_{i_{k-2}}) \\ &< (Y^1_{l^1_k} - Y^1_{i_{k-2}}) - (Y^2_{l^1_k} - Y^2_{i_{k-2}}), \end{aligned}$$

and hence

$$|(Y^1_{i_{k-1}} - Y^1_{i_{k-2}}) - (Y^2_{i_{k-1}} - Y^2_{i_{k-2}})|^p < |(Y^1_{l^1_k} - Y^1_{i_{k-2}}) - (Y^2_{l^1_k} - Y^2_{i_{k-2}})|^p. \tag{2.12}$$

Similarly,

$$\begin{aligned} 0 < -\big((Y^1_{i_k} - Y^1_{i_{k-1}}) - (Y^2_{i_k} - Y^2_{i_{k-1}})\big) &= Y^2_{i_k} - Y^2_{i_{k-1}} \\ &< Y^2_{i_k} - Y^2_{l^1_k} \\ &= (Y^1_{i_k} - Y^1_{l^1_k}) - (Y^2_{i_k} - Y^2_{l^1_k}), \end{aligned}$$

which implies

$$|(Y^1_{i_k} - Y^1_{i_{k-1}}) - (Y^2_{i_k} - Y^2_{i_{k-1}})|^p < |(Y^1_{i_k} - Y^1_{l^1_k}) - (Y^2_{i_k} - Y^2_{l^1_k})|^p. \tag{2.13}$$

Combining (2.12) with (2.13), we obtain

$$\begin{aligned} &|(Y^1_{i_{k-1}} - Y^1_{i_{k-2}}) - (Y^2_{i_{k-1}} - Y^2_{i_{k-2}})|^p + |(Y^1_{i_k} - Y^1_{i_{k-1}}) - (Y^2_{i_k} - Y^2_{i_{k-1}})|^p \\ &\quad < |(Y^1_{l^1_k} - Y^1_{i_{k-2}}) - (Y^2_{l^1_k} - Y^2_{i_{k-2}})|^p + |(Y^1_{i_k} - Y^1_{l^1_k}) - (Y^2_{i_k} - Y^2_{l^1_k})|^p, \end{aligned}$$

which contradicts (2.5) and completes the proof of (2.9). It is clear that for any k we have $y_{1,l^1_k} \geqslant y_{1,l^2_k}$ and $y_{2,l^2_k} \geqslant y_{2,l^1_k}$. Consequently, in the case $Y^1_{i_k} - Y^1_{i_{k-1}} > Y^2_{i_k} - Y^2_{i_{k-1}}$ we have

$$\begin{aligned} 0 < (Y^1_{i_k} - Y^1_{i_{k-1}}) - (Y^2_{i_k} - Y^2_{i_{k-1}}) &= (y_{1,l^1_k} - y_{1,l^1_{k-1}}) - (y_{2,l^2_k} - y_{2,l^2_{k-1}}) \\ &\leqslant (y_{1,l^1_k} - y_{1,l^2_{k-1}}) - (y_{2,l^1_k} - y_{2,l^2_{k-1}}). \end{aligned}$$

Hence

(2.14) $|(Y^1_{i_k} - Y^1_{i_{k-1}}) - (Y^2_{i_k} - Y^2_{i_{k-1}})|^p \leqslant |(y_{1,l^1_k} - y_{1,l^2_{k-1}}) - (y_{2,l^1_k} - y_{2,l^2_{k-1}})|^p.$

Similarly one can check that if $Y^1_{i_k} - Y^1_{i_{k-1}} < Y^2_{i_k} - Y^2_{i_{k-1}}$, then

(2.15) $|(Y^1_{i_k} - Y^1_{i_{k-1}}) - (Y^2_{i_k} - Y^2_{i_{k-1}})|^p \leqslant |(y_{1,l^2_k} - y_{1,l^1_{k-1}}) - (y_{2,l^2_k} - y_{2,l^1_{k-1}})|^p.$

By (2.14) and (2.15),

$$\sum_{k=1}^m |(Y^1_{i_k} - Y^1_{i_{k-1}}) - (Y^2_{i_k} - Y^2_{i_{k-1}})|^p \leqslant \sum_{k=1}^m |(y_{1,l_k} - y_{1,l_{k-1}}) - (y_{2,l_k} - y_{2,l_{k-1}})|^p,$$

where $l_k = l^1_k$ or $l_k = l^2_k$. Moreover, by (2.9), $i_{k-1} \leqslant l_k \leqslant i_k$ for $k = 1, \ldots, m$. Hence

$$v_p(\sup_{s\leqslant\cdot} y^1_s - \sup_{s\leqslant\cdot} y^2_s)_T \leqslant \sum_{k=1}^m |(y^1_{t_{l_k}} - y^1_{t_{l_{k-1}}}) - (y^2_{t_{l_k}} - y^2_{t_{l_{k-1}}})|^p$$

for some partition $0 = t_{l_0} < t_{l_1} < \ldots < t_{l_m} \leqslant T$, which proves the theorem under our additional assumption.

Step 2. The general case. Let us assume that $\{y^{1,n}\}$ and $\{y^{2,n}\}$ are sequences of discretizations of y^1 and y^2, respectively, i.e., $y^{1,n}_t = y^1_{k/n}$, $y^{2,n}_t = y^2_{k/n}$, $t \in \big[k/n, (k+1)/n\big)$, $k \in \mathbb{N} \cup \{0\}$. By Step 1, for any $n \in \mathbb{N}$ and $T \in \mathbb{R}^+$ we have

$$v_p(\sup_{s\leqslant\cdot} y^{1,n}_s - \sup_{s\leqslant\cdot} y^{2,n}_s)_T \leqslant v_p(y^{1,n} - y^{2,n})_T.$$

Clearly, $v_p(y^{1,n} - y^{2,n})_T \leqslant v_p(y^1 - y^2)_T$, $n \in \mathbb{N}$, $T \in \mathbb{R}^+$. By using, e.g., [11], Chapter 3, Proposition 6.5, one can check that

$$y^{1,n} \to y^1 \quad \text{and} \quad y^{2,n} \to y^2 \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}).$$

Consequently, by Proposition 2.2 in [14], Chapter VI,

$$(y^{1,n}, y^{2,n}) \to (y^1, y^2) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^2),$$

which together with Proposition 2.4 in [14], Chapter VI, implies that

$$\sup_{s\leqslant\cdot} y^{1,n}_s - \sup_{s\leqslant\cdot} y^{2,n}_s \to \sup_{s\leqslant\cdot} y^1_s - \sup_{s\leqslant\cdot} y^2_s \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}).$$

Therefore, for any T such that $\Delta y^1_T = \Delta y^2_T = 0$,

$$\begin{aligned} v_p(\sup_{s\leqslant\cdot} y^1_s - \sup_{s\leqslant\cdot} y^2_s)_T &\leqslant \liminf_{n\to\infty} v_p(\sup_{s\leqslant\cdot} y^{1,n}_s - \sup_{s\leqslant\cdot} y^{2,n}_s)_T \\ &\leqslant \sup_n v_p(y^{1,n} - y^{2,n})_T \leqslant v_p(y^1 - y^2)_T. \end{aligned}$$

If $\Delta y^1_T \neq 0$ or $\Delta y^2_T \neq 0$, then there exists a sequence $\{T_k\}$ such that $T_k \downarrow T$ and $\Delta y^1_{T_k} = \Delta y^2_{T_k} = 0$, $k \in \mathbb{N}$. Then $v_p(\sup_{s\leqslant\cdot} y^1_s - \sup_{s\leqslant\cdot} y^2_s)_{T_k} \leqslant v_p(y^1 - y^2)_{T_k}$, $k \in \mathbb{N}$, so letting $k \to \infty$ we obtain the desired result. ∎

THEOREM 2.2. *Assume* $l, y, l', y' \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ *are such that* $l_0 \leqslant y_0$ *and* $l'_0 \leqslant y'_0$. *Let* $(x, k) = SP_l(y)$ *and* $(x', k') = SP_{l'}(y')$. *Then, for any* $T \in \mathbb{R}^+$,

$$V_p(x - x')_T \leqslant (d+1)V_p(y - y')_T + d|y_0 - y'_0| + dV_p(l - l')_T + d|l_0 - l'_0|$$

and

$$V_p(k - k')_T \leqslant dV_p(y - y')_T + d|y_0 - y'_0| + dV_p(l - l')_T + d|l_0 - l'_0|.$$

Proof. Observe that $k_t = \sup_{s \leqslant t}(y_s - l_s)^- = \sup_{s \leqslant 1+t} \bar{y}_s$, where $\bar{y}_s = 0$ for $s \in [0, 1)$ and $\bar{y}_s = l_{s-1} - y_{s-1}$ for $s \geqslant 1$. Similarly, $k'_t = \sup_{s \leqslant 1+t} \bar{y}'_s$, where $\bar{y}'_s = 0$ for $s \in [0, 1)$ and $\bar{y}'_s = l'_{s-1} - y'_{s-1}$ for $s \geqslant 1$. By Theorem 2.1,

$$\begin{aligned} V_p(k - k')_T &= V_p(\sup_{s \leqslant \cdot} \bar{y}_s - \sup_{s \leqslant \cdot} \bar{y}'_s)_{T+1} \\ &\leqslant d^{(p-1)/p}\Big(\sum_{i=1}^{d} v_p(\sup_{s \leqslant \cdot} \bar{y}^i_s - \sup_{s \leqslant \cdot} \bar{y}'^i_s)_{T+1}\Big)^{1/p} \\ &\leqslant d^{(p-1)/p}\Big(\sum_{i=1}^{d} v_p(\bar{y}^i - \bar{y}'^i)_{T+1}\Big)^{1/p} \\ &\leqslant d \max_i V_p(\bar{y}^i - \bar{y}'^i)_{T+1} \leqslant dV_p(\bar{y} - \bar{y}')_{[0,T+1]}. \end{aligned}$$

Since

$$\begin{aligned} V_p(\bar{y} - \bar{y}')_{T+1} &\leqslant V_p(\bar{y} - \bar{y}')_{[0,1]} + V_p(\bar{y} - \bar{y}')_{[1,T+1]} \\ &= |(y_0 - y'_0) - (l_0 - l'_0)| + V_p\big((y - y') - (l - l')\big)_T \\ &\leqslant V_p(y - y')_T + |y_0 - y'_0| + V_p(l - l')_T + |l_0 - l'_0| \end{aligned}$$

and

$$V_p(x - x')_T \leqslant V_p(y - y')_T + V_p(k - k')_T,$$

the proof is complete. ■

COROLLARY 2.1. *Under the assumptions of Theorem* 2.2, *for every* $T \in \mathbb{R}^+$ *the estimates* (1.2) *and* (1.3) *hold true.*

Proof. It suffices to observe that $x_0 = y_0$, $x'_0 = y'_0$ and $k_0 = k'_0 = 0$. ■

REMARK 2.1. (a) *The case* $p = d = 1$ *was studied earlier in* [31] (see also [24]).

(b) *Let* $(x^n, k^n) = SP_{l^n}(y^n)$, $(x, k) = SP_l(y)$. *By* (2.1) *and* (2.2) *it is clear that if* (y^n, l^n) *tends to* (y, l) *in the uniform norm, then* (x^n, k^n) *tends to* (x, k) *in the uniform norm. From this one can deduce that*

$$(y^n, l^n) \to (y, l) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}) \Rightarrow (x^n, k^n, y^n, l^n) \to (x, k, y, l) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}),$$

and if $\{(y^n, l^n)\}$ *is relatively compact in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$, *then* $\{(x^n, k^n, y^n, l^n)\}$ *is relatively compact in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d})$ (see, e.g., [31]). *From Corollary* 2.1 *it also follows that if* (y^n, l^n) *tends to* (y, l) *in the variation norm*, *then* (x^n, k^n) *tends to* (x, k) *in the variation norm.*

REMARK 2.2. *Let* $(x, k) = SP_l(y)$. *Since* $k_t = \sup_{s\leqslant t}(y_s - l_s)^-$, *for any* $T \in \mathbb{R}^+$ *we have*

$$\bar{V}_p(k)_T \leqslant d\sup_{t\leqslant T}|y_t| + d\sup_{t\leqslant T}|l_t|$$

and

$$\bar{V}_p(x)_T \leqslant (d+1)\bar{V}_p(y)_T + d\sup_{t\leqslant T}|l_t|.$$

3. DETERMINISTIC INTEGRAL EQUATIONS

Let $x \in \mathbb{D}(\mathbb{R}^+, \mathbb{M}^d)$, $z \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ be such that $V_q(x)_T < \infty$, $V_p(z)_T < \infty$, $T \in \mathbb{R}^+$, where $1/p + 1/q > 1$, $p, q \geqslant 1$. It is well known (see, e.g., [6]–[8], [34]) that the Riemann–Stieltjes integral $\int_0^{\cdot} x_{s-} dz_s$ is a well-defined càdlàg function such that, for any $a < b$,

$$V_p\Big(\int_a^{\cdot} x_{s-} dz_s\Big)_{[a,b]} \leqslant C_{p,q}\bar{V}_q(x)_{[a,b)}V_p(z)_{[a,b]}, \tag{3.1}$$

where $C_{p,q} = 2\zeta(p^{-1} + q^{-1})$ and ζ denotes the Riemann zeta function, i.e., $\zeta(x) = \sum_{n=1}^{\infty} 1/n^x$.

Let $a \in \mathbb{D}(\mathbb{R}^+, \mathbb{R})$, $z, l \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ be such that $V_1(a)_T < \infty$, $V_p(z)_T < \infty$, $T \in \mathbb{R}^+$, and $x_0 \geqslant l_0$. We consider equations with constraints of the form

$$x_t = x_0 + \int_0^t f(x_{s-})\, da_s + \int_0^t g(x_{s-})\, dz_s + k_t, \quad t \in \mathbb{R}^+, \tag{3.2}$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{M}^d$ are given functions and the integral with respect to z is a Riemann–Stieltjes integral.

DEFINITION 3.1. We say that a pair $(x, k) \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ is a *solution* of (3.2) if $V_p(x)_T < \infty$, $T \in \mathbb{R}^+$, and $(x, k) = SP_l(y)$, where

$$y_t = x_0 + \int_0^t f(x_{s-})\, da_s + \int_0^t g(x_{s-})\, dz_s, \quad t \in \mathbb{R}^+.$$

We will need the following conditions.

(H1) (a) $f : \mathbb{R}^d \to \mathbb{R}^d$ is continuous and satisfies the linear growth condition, i.e., there is $L > 0$ such that

$$|f(x)| \leqslant L(1 + |x|), \quad x \in \mathbb{R}^d.$$

(b) $g : \mathbb{R}^d \to \mathbb{M}^d$ is a Hölder continuous function of order $\alpha \in (p-1, 1]$, i.e., there is $C_\alpha > 0$ such that

$$\|g(x) - g(y)\| \leqslant C_\alpha |x-y|^\alpha, \quad x, y \in \mathbb{R}^d.$$

(H2) (a) $f : \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz continuous, i.e., for any $k \in \mathbb{N}$ there is $L_k > 0$ such that

$$|f(x) - f(y)| \leqslant L_k |x-y|, \quad |x|, |y| \leqslant k.$$

(b) $g : \mathbb{R}^d \to \mathbb{M}^d$, each of its components $g_{i,j}$ is differentiable, and there are $\gamma \in (p-1, 1]$ and $C_{k,\gamma} > 0$ such that, for every $k \in \mathbb{N}$,

$$|\nabla_x g_{i,j}(x) - \nabla_x g_{i,j}(y)| \leqslant M_{k,\gamma} |x-y|^\gamma, \quad |x|, |y| \leqslant k, \ i, j = 1, \ldots, d.$$

Similar sets of conditions were considered in papers on equations without constraints driven by functions (processes) with bounded p-variation (see, e.g., [6], [12], [17], [18], [20], [22], [23]).

The outline of the rest of Section 3 is as follows. First we study convergence of solutions of equations of the type (3.2) in the Skorokhod topology J_1. In our proofs we use a general tightness criterion and a functional limit theorem for sequences of integrals with respect to càdlàg functions (see Appendix). As a simple corollary to our convergence result we show that under (H1) there exists a solution of (3.2). Next, assuming additionally (H2), we prove that (3.2) has a unique solution (x, k). Under (H1) and (H2), we show at the end of Section 3 that (x, k) can be approximated by simple and easily implementable approximation schemes.

Let us assume that $z^n, l^n \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and $a^n \in \mathbb{D}(\mathbb{R}^+, \mathbb{R})$ be such that $x_0^n \geqslant l_0^n$ and $V_1(a^n)_T$, $V_p(z^n)_T < \infty$ for $T \in \mathbb{R}^+$. We will consider solutions $(x^n, k^n) \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ of equations with constraints of the form

$$x_t^n = x_0^n + \int_0^t f(x_{s-}^n)\, da_s^n + \int_0^t g(x_{s-}^n)\, dz_s^n + k_t^n, \quad t \in \mathbb{R}^+, \tag{3.3}$$

i.e., $(x^n, k^n) = SP_{l^n}\big(x_0^n + \int_0^\cdot f(x_{s-}^n)\, da_s^n + \int_0^\cdot g(x_{s-}^n)\, dz_s^n\big)$ and $V_p(x^n)_T < \infty$, $T \in \mathbb{R}^+$.

THEOREM 3.1. *Suppose that functions f and g satisfy* (H1). *Let* $\{a^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R})$, $\{z^n\}$, $\{l^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ *be sequences such that* $\sup_n V_1(a^n)_T < \infty$, $\sup_n V_p(z^n)_T < \infty$, $T \in \mathbb{R}^+$, *and*

$$(x_0^n, a^n, z^n, l^n) \to (x_0, a, z, l) \quad \textit{in } \mathbb{R}^d \times \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d+1}).$$

If $\{(x^n, k^n)\}$ *is a sequence of solutions of* (3.3), *then*

$$\{(x^n, k^n)\} \textit{ is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}),$$

and its every limit point is a solution of (3.2).

The lemma below will be our main tool in the proof.

LEMMA 3.1. *Assume f and g satisfy* (H1). *Let (x,k) be a solution of* (3.2), *and let $b,T>0$. If*

$$\max\big(V_1(a)_T, V_p(z)_T, \sup_{t\leqslant T}|l_t|\big)\leqslant b,$$

then there is $\bar{C}=C\big(d,p,\alpha,L,g(0),x_0,b\big)>0$ such that $\bar{V}_p(x)_T\leqslant\bar{C}$.

Proof of Lemma 3.1. By Remark 2.2, for any $t\leqslant T$,

$$\begin{aligned}\bar{V}_p(x)_t &\leqslant (d+1)\bar{V}_p(y)_t + d\sup_{s\leqslant t}|l_s|\\ &\leqslant (d+1)\Big[|x_0| + V_p\Big(\int_0^{\cdot} f(x_{s-})\,da_s\Big)_t + V_p\Big(\int_0^{\cdot} g(x_{s-})\,dz_s\Big)_t\Big] + d\sup_{s\leqslant t}|l_s|.\end{aligned}$$

We have

$$V_p\Big(\int_0^{\cdot} f(x_{s-})\,da_s\Big)_t \leqslant V_1(a)_t\sup_{s\leqslant t}|f(x_{s-})| \leqslant LV_1(a)_t\big(1+\bar{V}_p(x)_t\big)$$

and, by (3.1),

$$\begin{aligned}V_p\Big(\int_0^{\cdot} g(x_{s-})\,dz_s\Big)_t &\leqslant C_{p,p/\alpha}\bar{V}_{p/\alpha}\big(g(x)\big)_t V_p(z)_t\\ &\leqslant C_{p,p/\alpha}\big(C_\alpha V_p^\alpha(x)_t + C_\alpha|x_0|^\alpha + |g(0)|\big)V_p(z)_t\\ &\leqslant C_{p,p/\alpha}\big[C_\alpha\big(\alpha\bar{V}_p(x)_t + 2(1-\alpha)\big) + |g(0)|\big]V_p(z)_t\\ &\leqslant DV_p(z)_t\big(1+\bar{V}_p(x)_t\big),\end{aligned}$$

where $D=C_{p,p/\alpha}\big(C_\alpha(2-\alpha)+|g(0)|\big)$.

Set $t_1=\inf\big\{t; LV_1(a)_t>\frac{1}{4(d+1)} \text{ or } DV_p(z)_t>\frac{1}{4(d+1)}\big\}\wedge T$. By the above,

$$\bar{V}_p(x)_{[0,t_1)}\leqslant (d+1)|x_0| + \frac{1}{2}\big(1+\bar{V}_p(x)_{[0,t_1)}\big) + d\sup_{s\leqslant t_1}|l_s|,$$

which implies that $\bar{V}_p(x)_{[0,t_1)}\leqslant 2(d+1)|x_0|+1+2d\sup_{s\leqslant t_1}|l_s|$. Since

$$\begin{aligned}|\Delta x_{t_1}| &\leqslant |f(x_{t_1-})\Delta a_{t_1}| + |g(x_{t_1-})\Delta z_{t_1}| + |\Delta l_{t_1}|\\ &\leqslant \big(L(1+|x_{t_1-}|) + C_\alpha|x_{t_1-}|^\alpha + |g(0)| + 2\big)b,\end{aligned}$$

there exist $C_1, C_2>0$ depending only on $d,p,\alpha,L,g(0),b$ such that

$$\bar{V}_p(x)_{[0,t_1]}\leqslant C_1 + C_2|x_0|.$$

Set $t_k = \inf\big\{t > t_{k-1}; LV_1(a)_{[t_{k-1},t]} > \frac{1}{4(d+1)}$ or $DV_p(z)_{[t_{k-1},t]} > \frac{1}{4(d+1)}\big\} \wedge T$, $k = 2, 3, \dots$, and observe that, for the same constants C_1, C_2,

$$V_p(x)_{[t_{k-1},t_k]} \leqslant C_1 + C_2|x_{t_{k-1}}| \leqslant C_1 + C_2\bar{V}_p(x)_{[0,t_{k-1}]}.$$

What is left is to show that $m = \sup\{k; t_k < T\}$ is finite and depends only on $p, \alpha, L, g(0), b$. To see this, let us observe that

$$\begin{aligned} m\left(\frac{1}{4(d+1)}\right)^p &\leqslant \sum_{k=1}^{m} LV_1(a)_{[t_{k-1},t_k]} + D^p v_p(z)_{[t_{k-1},t_k]} \\ &\leqslant Lb + D^p b^p, \end{aligned}$$

which yields $m \leqslant \big(4(d+1)\big)^p[Lb + D^p b^p]$. This completes the proof. ∎

Proof of Theorem 3.1. By Lemma 3.1, $\sup_n \bar{V}_p(x^n)_T < \infty$. Since $\bar{V}_{p/\alpha}\big(g(x^n)\big)_T \leqslant C_\alpha V_p^\alpha(x^n)_T + |g(x_0^n)|$, we also have

$$\sup_n \bar{V}_{p/\alpha}\big(g(x^n)\big)_T < \infty, \quad T \in \mathbb{R}^+. \tag{3.4}$$

Since f is continuous, there exists a sequence of Lipschitz continuous functions $\{f^k\}$ such that, for any compact $K \subset \mathbb{R}^d$, $\sup_{u\in K} |f^k(u) - f(u)| \to 0$. Hence and from the fact that $\sup_n \bar{V}_p(x^n)_T < \infty$, $T \in \mathbb{R}^+$, it follows that

$$\sup_n \bar{V}_p\big(f^k(x^n)\big)_T < \infty, \quad T \in \mathbb{R}^+,\ k \in \mathbb{N}, \tag{3.5}$$

and

$$\lim_{k\to\infty} \limsup_{n\to\infty} \sup_{t\leqslant T} |f^k(x_t^n) - f(x_t)| = 0, \quad T \in \mathbb{R}^+. \tag{3.6}$$

Putting $q = p/\alpha$ in Corollary 5.1 (see the Appendix) and using (3.4), (3.5) we show that, for any $k \in \mathbb{N}$,

$$\Big\{\Big(\int_0^{\cdot} f^k(x_{s-}^n)\, da_s^n, a^n, \int_0^{\cdot} g(x_{s-}^n)\, dz_s^n, z^n\Big)\Big\} \text{ is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1}).$$

By the above and (3.6),

$$\Big\{\Big(\int_0^{\cdot} f(x_{s-}^n)\, da_s^n, a^n, \int_0^{\cdot} g(x_{s-}^n)\, dz_s^n, z^n\Big)\Big\} \text{ is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1}).$$

Therefore, $\{(a^n, y^n, z^n, l^n)\}$, with $y^n = x_0^n + \int_0^{\cdot} f(x_{s-}^n)\, da_s^n + \int_0^{\cdot} g(x_{s-}^n)\, dz_s^n$, is relatively compact in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1})$. Now, put $(x^n, k^n) = SP_{l^n}(y^n)$, $n \in \mathbb{N}$, and observe that, by Remark 2.1 (b),

$$\{(x^n, a^n, z^n, l^n)\} \text{ is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1}).$$

Assume that $(x^n,\, a^n,\, z^n,\, l^n) \to (x,\, a,\, z,\, l)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1})$. By Corollary 5.2, $(y^n, l^n) \to (y, l)$, where $y = x_0 + \int_0^{\cdot} f(x_{s-})\, da_s + \int_0^{\cdot} g(x_{s-})\, dz_s$. Consequently, by Remark 2.1 (a),

$$(x^n, k^n) = SP_{l^n}(y^n) \to SP_l(y) = (x, k) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}),$$

which completes the proof of the theorem. ∎

COROLLARY 3.1. *Assume f, g satisfy* (H1) *and let $a \in \mathbb{D}(\mathbb{R}^+, \mathbb{R})$, $z, l \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ be such that $V_1(a)_T < \infty$, $V_p(z)_T < \infty$ with $x_0 \geqslant l_0$. Set $x_0^n = x_0$, $k_0^n = 0$, and*

$$\begin{aligned}
\Delta y^n_{(k+1)/n} &= f(x^n_{k/n})(a_{(k+1)/n} - a_{k/n}) + g(x^n_{k/n})(z_{(k+1)/n} - z_{k/n}),\\
x^n_{(k+1)/n} &= \max(x^n_{k/n} + \Delta y^n_{(k+1)/n}, l_{(k+1)/n}),\\
k^n_{(k+1)/n} &= k^n_{k/n} + (x^n_{(k+1)/n} - x^n_{k/n}) - \Delta y^n_{(k+1)/n},
\end{aligned}$$

and $x^n_t = x^n_{k/n}$, $k^n_t = k^n_{k/n}$, $l^n_t = l^n_{k/n}$, $t \in \big[k/n, (k+1)/n\big)$, $k \in \mathbb{N} \cup \{0\}$. Then $\{(x^n, k^n)\}$ is relatively compact in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$, and its every limit point is a solution of (3.2)*. Consequently, equation* (3.2) *has a solution (possibly nonunique).*

Proof. It suffices to observe that (x^n, k^n) is a solution of (3.3) with $a^n_t = a_{k/n}$, $z^n_t = z_{k/n}$, $l^n_t = l_{k/n}$, $t \in \big[k/n, (k+1)/n\big)$, $k \in \mathbb{N} \cup \{0\}$. Also observe that $\sup_n V_1(a^n)_T \leqslant V_1(a)_T < \infty$, $\sup_n V_p(z^n)_T \leqslant V_p(z)_T < \infty$ for $T \in \mathbb{R}^+$ and

$$(a^n, z^n, l^n) \to (a, z, l) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d+1}).$$

Therefore, the result follows from Theorem 3.1. ∎

THEOREM 3.2. *Assume f, g satisfy* (H1) *and* (H2)*. Then there exists a unique solution (x, k) of* (3.2)*.*

Proof. Assume that there exist two solutions (x^j, k^j), $j = 1, 2$.

Step 1. We first replace (H2) by the following stronger condition:

(H2*) (a) $f : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous, i.e., there is $L > 0$ such that

$$|f(x) - f(y)| \leqslant L|x - y|, \quad x, y \in \mathbb{R}^d.$$

(b) $g : \mathbb{R}^d \to \mathbb{M}^d$, each of its components $g_{i,j}$ is differentiable,

$$C = \max_{i,j} \sup_x |\nabla_x g_{i,j}(x)| < \infty,$$

and there are $\gamma \in (p-1, 1]$ and $M_\gamma > 0$ such that

$$|\nabla_x g_{i,j}(x) - \nabla_x g_{i,j}(y)| \leqslant M_\gamma |x - y|^\gamma, \quad x, y \in \mathbb{R}^d,\ i, j = 1, \ldots, d.$$

Fix $T \in \mathbb{R}^+$. By Corollary 2.1, for any $t \leqslant T$ we have

$$\bar{V}_p(x^1 - x^2)_t \leqslant (d+1)\bar{V}_p\Big(\int_0^{\cdot} f(x^1_{s-}) - f(x^2_{s-})\, da_s\Big)_t + (d+1)\bar{V}_p\Big(\int_0^{\cdot} g(x^1_{s-}) - g(x^2_{s-})\, dz_s\Big)_t.$$

Moreover,

$$\bar{V}_p\Big(\int_0^{\cdot} f(x^1_{s-}) - f(x^2_{s-})\, da_s\Big)_t \leqslant LV_1(a)_T \sup_{s\leqslant t} |x^1_s - x^2_s| \leqslant LV_1(a)_t \bar{V}_p(x^1 - x^2)_t$$

and, by (3.1),

$$\bar{V}_p\Big(\int_0^{\cdot} g(x^1_{s-}) - g(x^2_{s-})\, dy_s\Big)_t \leqslant C_{p,p/\gamma}\bar{V}_{p/\gamma}\big(g(x^1) - g(x^2)\big)_t V_p(z)_t.$$

By [6], Theorem 2, for $i, j = 1, \ldots, d$ we have

$$V_{p/\gamma}\big(g_{i,j}(x^1) - g_{i,j}(x^2)\big)_t \leqslant CV_{p/\gamma}(x^1 - x^2)_t + M_\gamma \sup_{s\leqslant t} |x^1_s - x^2_s| \big(V_p(x^1)_T\big)^\gamma.$$

Therefore,

$$V_{p/\gamma}\big(g(x^1) - g(x^2)\big)_t \leqslant \sum_{i,j=1}^{d} V_{p/\gamma}\big(g_{i,j}(x^1) - g_{i,j}(x^2)\big)_t \leqslant \tilde{C}_1 V_{p/\gamma}(x^1 - x^2)_t + \tilde{C}_2 \sup_{s\leqslant t} |x^1_s - x^2_s| \big(V_p(x^1)_T\big)^\gamma,$$

where $\tilde{C}_1 = C^{d^2}$, $\tilde{C}_2 = (M_\gamma)^{d^2}$. Set

$$t_1 = \inf\left\{t; \max\big[LV_1(a)_t, C_{p,p/\gamma}\big(\tilde{C}_1 + \tilde{C}_2 V_p(x^1)^\gamma_t\big)V_p(z)_t\big] > \frac{1}{4(d+1)}\right\} \wedge T.$$

Then $\bar{V}_p(x^1 - x^2)_{[0,t_1)} \leqslant \frac{1}{2}\bar{V}_p(x^1 - x^2)_{[0,t_1)}$, thus $x^1 = x^2$ on $[0, t_1)$. Since for $j = 1, 2$ we have

$$x^j_{t_1} = \max\big(x^j_{t_1-} + f(x^j_{t_1-})\Delta a_{t_1} + g(x^j_{t_1-})\Delta y_{t_1}, l_{t_1}\big),$$

$x^1_{t_1} = x^2_{t_1}$, too. For $k \geqslant 2$ set

$$t_k = \inf\Big\{t > t_{k-1}; \max\big[LV_1(a)_{[t_{k-1},t]}, C_{p,p/\gamma}\big(\tilde{C}_1 + \tilde{C}_2 V_p(x^1)^\gamma_T\big)V_p(z)_{[t_{k-1},t]}\big] > \frac{1}{4(d+1)}\Big\} \wedge T.$$

Arguing as above we show recurrently that $x^1 = x^2$ on each interval $[t_{k-1}, t_k]$. Since, by the same arguments as in the proof of Theorem 3.1, $m = \sup\{k; t_k < T\}$ is finite, $x^1 = x^2$ on the interval $[0, T]$, which completes the proof under (H2*).

S t e p 2. T h e g e n e r a l c a s e. Set $s_k = \inf\{t; \max(|x_t^1|, |x_t^2|) > k\}$, $k \in \mathbb{N}$. By the first part of the proof,

$$x_t^1 = x_t^2, \quad t < s_k,\ k \in \mathbb{N}.$$

Since $s_k \to \infty$, this proves the corollary. ∎

COROLLARY 3.2. *Assume* (H1) *and* (H2) *are satisfied. Let* $a \in \mathbb{D}(\mathbb{R}^+, \mathbb{R})$ *be such that* $V_1(a)_T < \infty$, $z, l \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, *with* $V_p(z)_T < \infty$, $T \in \mathbb{R}^+$, *and* $x_0 \geqslant l_0$. *Let* $\{(x^n, k^n)\}$ *be a sequence of approximations defined in Corollary* 3.1. *Then*

$$(x^n, k^n) \to (x, k) \quad \textit{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}) \tag{3.7}$$

and, for any $T \in \mathbb{R}^+$,

$$\max_{k/n \leqslant T} |x^n_{k/n} - x_{k/n}| \to 0 \quad \textit{and} \quad \max_{k/n \leqslant T} |k^n_{k/n} - k_{k/n}| \to 0, \tag{3.8}$$

where (x, k) *is a unique solution of* (3.2).

P r o o f. The convergence (3.7) easily follows from Corollary 3.1. If we set $x_t^{(n)} = x_{k/n}$, $k_t^{(n)} = k_{k/n}$, $t \in \big[k/n, (k+1)/n\big)$, $k \in \mathbb{N} \cup \{0\}$, then by [14], Proposition 2.2,

$$(x^n, x^{(n)}, k^n, k^{(n)}) \to (x, x, k, k) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}).$$

Consequently, $x^n - x^{(n)} \to 0$ and $k^n - k^{(n)} \to 0$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, which is equivalent to (3.8). ∎

COROLLARY 3.3. *Assume* (H1) *and* (H2) *are satisfied. Let* a, z, l *satisfy the assumptions of Corollary* 3.2 *and let* $\{a^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R})$, $\{z^n\}$, $\{l^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ *be such that* $\sup_n V_1(a^n)_T < \infty$, $\sup_n V_p(z^n)_T < \infty$, $T \in \mathbb{R}^+$, *and*

$$\sup_{t \leqslant T} (|a_t^n - a_t| + |z_t^n - z_t| + |l_t^n - l_t|) \to 0, \quad T \in \mathbb{R}^+. \tag{3.9}$$

If $\{(x^n, k^n)\}$ *is a sequence of solutions of* (3.3), *then*

$$\sup_{t \leqslant T} (|x_t^n - x_t| + |k_t^n - k_t|) \to 0, \quad T \in \mathbb{R}^+,$$

where (x, k) *is a unique solution of* (3.2).

P r o o f. By Theorem 3.1,

$$(x^n, k^n, a^n, z^n, l^n) \to (x, k, a, z, l) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d+1}).$$

Since, for any $t \in \mathbb{R}^+$,

$$\Delta x_t \neq 0 \Rightarrow \Delta a_t \neq 0 \text{ or } \Delta z_t \neq 0 \text{ or } \Delta l_t \neq 0,$$

Lemma C in [27] implies that $\sup_{t \leqslant T} |x_t^n - x_t| \to 0, T \in \mathbb{R}^+$. Similarly we show the uniform convergence of k^n to k. ∎

COROLLARY 3.4. *Assume* (H1) *and* (H2) *are satisfied. Let* a, z, l *satisfy the assumptions of Corollary* 3.2. *Set* $x_0^n = x_0$, $k_0^n = 0$, $t_0^n = 0$,

$$t_k^n = \inf\left\{t > t_{k-1}^n; \max(|\Delta a_t|, |\Delta z_t|, |\Delta l_t|) > \frac{1}{n}\right\} \wedge \left(t_{k-1}^n + \frac{1}{n}\right), \quad k \in \mathbb{N},$$

and

$$\begin{aligned} \Delta y^n_{t^n_{k+1}} &= f(x^n_{t^n_k})(a_{t^n_{k+1}} - a_{t^n_k}) + g(x^n_{t^n_k})(z_{t^n_{k+1}} - z_{t^n_k}), \\ x^n_{t^n_{k+1}} &= \max(x^n_{t^n_k} + \Delta y^n_{t^n_{k+1}}, l_{t^n_{k+1}}), \\ k^n_{t^n_{k+1}} &= k^n_{t^n_k} + (x^n_{t^n_{k+1}} - x^n_{t^n_k}) - \Delta y^n_{t^n_{k+1}}, \end{aligned}$$

and $x_t^n = x^n_{t^n_k}$, $k_t^n = k^n_{t^n_k}$, $t \in [t_k^n, t_{k+1}^n)$, $k \in \mathbb{N} \cup \{0\}$. *If* $\{(x^n, k^n)\}$ *is a sequence of solutions of* (3.3), *then*

$$\sup_{t \leqslant T}(|x_t^n - x_t| + |k_t^n - k_t|) \to 0, \quad T \in \mathbb{R}^+,$$

where (x, k) *is a unique solution of* (3.2).

P r o o f. Observe that $\{(x^n, k^n)\}$ is a sequence of solutions of (3.3) with $a_t^n = a_{t^n_k}$, $z_t^n = z_{t^n_k}$, $l_t^n = l_{t^n_k}$, $t \in [t_k^n, t_{k+1}^n)$, $k \in \mathbb{N} \cup \{0\}$, and that

$$\sup_n V_1(a^n)_T \leqslant V_1(a)_T < \infty, \qquad \sup_n V_p(z^n)_T \leqslant V_p(z)_T < \infty, \quad T \in \mathbb{R}^+.$$

Moreover, simple calculations show that (3.9) is satisfied. Therefore, the desired result follows from Corollary 3.3. ∎

4. SDEs WITH CONSTRAINTS

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space and let A be an $(\mathcal{F}_t)$-adapted process with trajectories in $\mathbb{D}(\mathbb{R}^+, \mathbb{R})$, Z, L be $(\mathcal{F}_t)$-adapted processes with trajectories in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ such that, for any $T \in \mathbb{R}^+$, $P(V_1(A)_T < \infty) = 1$ and $P(V_p(Z)_T < \infty) = 1$. Note that Z need not be a semimartingale. However, it is a p-semimartigale and a Dirichlet process in the sense considered in [17], [18] and [4].

DEFINITION 4.1. Let $X_0 \geqslant L_0$. We say that a pair (X, K) of $(\mathcal{F}_t)$-adapted processes with trajectories in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ such that $P\big(V_p(X)_T < \infty\big) = 1$ for $T \in \mathbb{R}^+$ is a *strong solution* of (1.1) if $(X, K) = SP_L(Y)$, where

$$Y_t = X_0 + \int_0^t f(X_{s-})\, dA_s + \int_0^t g(X_{s-})\, dZ_s, \quad t \in \mathbb{R}^+.$$

THEOREM 4.1. *Assume* (H1) *and* (H2) *are satisfied. If* $X_0 \geqslant L_0$, *then equation* (1.1) *has a unique strong solution* (X, K). *Moreover, if we define* (X^n, K^n) *so that*

$$X_t^n = X_{\tau_k^n}^n, \quad K_t^n = K_{\tau_k^n}^n, \quad t \in [\tau_k^n, \tau_{k+1}^n), \quad k \in \mathbb{N} \cup \{0\},$$

where $X_0^n = X_0$, $K_0^n = 0$, *and*

$$\begin{aligned}
\Delta Y_{\tau_{k+1}^n}^n &= f(X_{\tau_k^n}^n)(A_{\tau_{k+1}^n} - A_{\tau_k^n}) + g(X_{\tau_k^n}^n)(Z_{\tau_{k+1}^n} - Z_{\tau_k^n}),\\
X_{\tau_{k+1}^n}^n &= \max(X_{\tau_k^n}^n + \Delta Y_{\tau_{k+1}^n}^n, L_{\tau_{k+1}^n}),\\
K_{\tau_{k+1}^n}^n &= K_{\tau_k^n}^n + (X_{\tau_{k+1}^n}^n - X_{\tau_k^n}^n) - \Delta Y_{\tau_{k+1}^n}^n,
\end{aligned}$$

with $\tau_0^n = 0, \tau_k^n = \inf\big\{t > \tau_{k-1}^n; \max(|\Delta A_t|, |\Delta Z_t|, |\Delta L_t|) > \frac{1}{n}\big\} \wedge \big(\tau_{k-1}^n + \frac{1}{n}\big)$, $n, k \in \mathbb{N}$, *then, for any* $T \in \mathbb{R}^+$,

$$\sup_{t \leqslant T} |X_t^n - X_t| \to 0 \text{ } P\text{-a.s.}, \quad \sup_{t \leqslant T} |K_t^n - K_t| \to 0 \text{ } P\text{-a.s.}$$

P r o o f. From Theorem 3.2 we deduce that for every $\omega \in \Omega$ there exists a unique solution $\big(X(\omega), K(\omega)\big) = SP_{L(\omega)}\big(Y(\omega)\big)$. Moreover, by Corollary 3.4, for every $\omega \in \Omega$ and $T \in \mathbb{R}^+$,

$$\sup_{t \leqslant T} |X_t^n(\omega) - X_t(\omega)| \to 0, \quad \sup_{t \leqslant T} |K_t^n(\omega) - K_t(\omega)| \to 0.$$

Since for any $n \in \mathbb{N}$ the pair (X^n, K^n) is $(\mathcal{F}_t)$-adapted, the pair of limit processes (X, K) is $(\mathcal{F}_t)$-adapted as well, which completes the proof. ∎

COROLLARY 4.1. *Under the assumptions of Theorem* 4.1 *with random sequences of partitions* $\{\tau_n^k\}$ *replaced by constant sequences* $\{k/n\}$, $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, *we have*

$$(X^n, K^n) \to (X, K) \quad P\text{-a.s. in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}),$$

where (X, K) *is a unique strong solution of* (1.1).

P r o o f. It suffices to apply Corollary 3.2. ∎

Let B^H be a fractional Brownian motion with Hurst index $H > 1/2$, i.e., a continuous centered Gaussian process with covariance

$$EB^H_{t_2}B^H_{t_1} = \frac{1}{2}(t_2^{2H} + t_1^{2H} - |t_2 - t_1|^{2H}), \quad t_1, t_2 \in \mathbb{R}^+.$$

Let $Z^H = \int_0^{\cdot} \sigma_s \, dB^H_s$, where $\sigma : \mathbb{R}^+ \to \mathbb{R}$ is a measurable function such that $\|\sigma\|_{\mathbb{L}^{1/H}_{[0,T]}} := \big(\int_0^T |\sigma_s|^{1/H} ds\big)^H < \infty, T \in \mathbb{R}^+$. Then Z^H is also a continuous centered Gaussian process with continuous trajectories. Moreover, if $p > 1/H$, then

$$P\big(V_p(Z^H)_T < \infty\big) = 1, \quad T \in \mathbb{R}^+ \tag{4.1}$$

(see, e.g., [12], Proposition 2.1). Note also that Z^H is a Dirichlet process from the class $\mathcal{D}^{1/H}$ studied in [5].

We now show how to apply our results to fractional SDEs with constraints of the form (1.4). Let $B^H = (B^{H,1}, \ldots, B^{H,d})$, where $B^{H,1}, \ldots, B^{H,d}$ are independent fractional Brownian motions, and let $Z^H = (Z^{H,1}, \ldots, Z^{H,d})$, where $Z^{H,i} = \int_0^{\cdot} \sigma^i_s \, dB^{H,i}_s$ with $\sigma^i : \mathbb{R}^+ \to \mathbb{R}$ such that $\|\sigma^i\|_{\mathbb{L}^{1/H}_{[0,T]}} < \infty, T > 0, i = 1, \ldots, d$.

COROLLARY 4.2. *Assume* (H1) *and* (H2) *are satisfied. If* $X_0 \geqslant L_0$, *then equation* (1.4) *has a unique strong solution* (X, K). *Moreover*, *if*

$$X^n_t = X^n_{k/n}, \quad K^n_t = K^n_{k/n}, \quad t \in \big[k/n, (k+1)/n\big), \quad k \in \mathbb{N} \cup \{0\}, \, n \in \mathbb{N},$$

where $X^n_0 = X_0$, $K^n_0 = 0$, *and*

$$\begin{aligned} \Delta Y^n_{(k+1)/n} &= f(X^n_{k/n})(a_{(k+1)/n} - a_{k/n}) + g(X^n_{k/n})(Z^H_{(k+1)/n} - Z^H_{k/n}), \\ X^n_{(k+1)/n} &= \max(X^n_{k/n} + \Delta Y^n_{(k+1)/n}, L_{(k+1)/n}), \\ K^n_{(k+1)/n} &= K^n_{k/n} + (X^n_{(k+1)/n} - X^n_{k/n}) - \Delta Y^n_{(k+1)/n}, \end{aligned}$$

then, *for any* $T \in \mathbb{R}^+$,

$$\sup_{t \leqslant T} |X^n_t - X_t| \to 0 \, P\text{-}a.s., \qquad \sup_{t \leqslant T} |K^n_t - K_t| \to 0 \, P\text{-}a.s.$$

P r o o f. It suffices to apply Corollary 4.1 and use the facts that a is a continuous function and Z^H has continuous trajectories. ■

REMARK 4.1. *To approximate* Z^H *one can use the methods developed in* [12] *and* [32].

5. APPENDIX

PROPOSITION 5.1. *Let* $\{x^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{M}^d)$, $\{z^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, *and*

$$\sup_n \bar{V}_q(x^n)_T < \infty, \quad \sup_n V_p(z^n)_T < \infty, \quad T \in \mathbb{R}^+,$$

where $1/p + 1/q > 1$, $p, q \geqslant 1$. *If* $\{z^n\}$ *is relatively compact in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, *then*

$$\Big\{ \Big(\int_0^{\cdot} x^n_{s-}\, dz^n_s, z^n \Big) \Big\} \text{ is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).$$

P r o o f. Without loss of generality we may and will assume that

$$z^n \to z \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d).$$

We follow the arguments from the proof of Proposition 3.3 in [21] and Proposition 2 in [27]. Let $t^n_{k,0} = 0$, $t^n_{k,i+1} = \min(t^n_{k,i} + \delta_{k,i}, \inf\{t > t^n_{k,i}; |\Delta z^n_t| > \delta_k\})$ and $t_{k,0} = 0$, $t_{k,i+1} = \min(t_{k,i} + \delta_{k,i}, \inf\{t > t_{k,i}; |\Delta z_t| > \delta_k\})$, where $\{\delta_k\}$, $\big\{\{\delta_{k,i}\}\big\}$ are families of constants such that $\delta_k \downarrow 0$, $|\Delta z_t| \neq \delta_k, t \in \mathbb{R}^+$, $\delta_k/2 \leqslant \delta_{k,i} \leqslant \delta_k$ and $|\Delta z_{t_{k,i}+\delta_{k,i}}| = 0$, $i \in \mathbb{N} \cup \{0\}$, $k, n \in \mathbb{N}$. Define

$$z^{n,(k)}_t = z^n_{t^n_{k,i}},\ t \in [t^n_{k,i}, t^n_{k,i+1}), \quad \text{and} \quad z^{(k)}_t = z_{t_{k,i}},\ t \in [t_{k,i}, t_{k,i+1}),$$

for $i \in \mathbb{N} \cup \{0\}$, $n, k \in \mathbb{N}$. Then $V_p(z^{n,(k)})_T \leqslant V_p(z^n)_T$, $n \in \mathbb{N}$, $V_p(z^{(k)})_T \leqslant V_p(z)_T$ for $T \in \mathbb{R}^+$ and

$$t^n_{k,i} \to t_{k,i}, \quad z^n_{t^n_{k,i}} \to z_{t_{k,i}}, \quad i \in \mathbb{N} \cup \{0\}, k \in \mathbb{N}. \tag{5.1}$$

Consequently,

$$\sup_{n,k} V_p(z^n - z^{n,(k)})_T < \infty, \tag{5.2}$$

and

$$(z^{n,(k)}, z^n) \to (z^{(k)}, z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}),\ k \in \mathbb{N}. \tag{5.3}$$

Moreover,

$$\sup_{t \leqslant T} |z^{(k)}_t - z_t| \to 0, \quad T \in \mathbb{R}^+, \tag{5.4}$$

which together with (5.3) implies that

$$\lim_{k\to\infty} \limsup_{n\to\infty} \sup_{t \leqslant T} |z^{n,(k)}_t - z^n_t| = 0, \quad T \in \mathbb{R}^+. \tag{5.5}$$

On the other hand, $\int_0^t x^n_{s-}\, dz^{n,(k)}_s = \sum_{j\leqslant i} x^n_{t^n_{k,j}-}(z^n_{t^n_{k,j}} - z^n_{t^n_{k,j-1}})$, $t \in [t^n_{k,i}, t^n_{k,i+1})$. Using (5.1), (5.3) and the fact that $\sup_n \bar{V}_q(x^n)_T < \infty$ implies that $\{\sup_{t\leqslant T}|x^n_t|\}$ is bounded, we conclude that, for any $k \in \mathbb{N}$,

(5.6) $$\Big\{\Big(\int_0^{\cdot} x^n_{s-}\, dz^{n,(k)}_s, z^n\Big)\Big\} \text{ is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).$$

Let $p' > p$ be such that $1/p' + 1/q > 1$. By (3.1),

$$\sup_{t\leqslant T}\Big|\int_0^t x^n_{s-}\, d(z^n - z^{n,(k)})_s\Big| \leqslant C_{p',q}\bar{V}_q(x^n)_T V_{p'}(z^n - z^{n,(k)})_T.$$

Moreover,

$$V_{p'}(z^n - z^{n,(k)})_T \leqslant \mathrm{Osc}(z^n - z^{n,(k)})_T^{1-p/p'} V_p(z^n - z^{n,(k)})_T^{p/p'},$$

where $\mathrm{Osc}(x)_T = \sup_{s,t\leqslant T}|x_t - x_s|$. Since

$$\mathrm{Osc}(z^n - z^{n,(k)})_T \leqslant 2\sup_{t\leqslant T}|z^n - z^{n,(k)}_t| \to 0,$$

we deduce from the above that

(5.7) $$\lim_{k\to\infty}\limsup_{n\to\infty}\sup_{t\leqslant T}\Big|\int_0^t x^n_{s-}\, d(z^n - z^{n,(k)})_s\Big| = 0, \quad T \in \mathbb{R}^+.$$

Combining (5.6) with (5.7), we get the desired result. ■

COROLLARY 5.1. *Let* $\{a^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R})$, $\{z^n\}, \{y^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ *and* $\{x^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{M}^d)$ *be sequences of functions such that*

$$\sup_n \max\big(V_1(a^n)_T, V_p(z^n)_T, \bar{V}_q(y^n)_T, \bar{V}_q(x^n)_T\big) < \infty, \quad T \in \mathbb{R}^+,$$

where $1/p + 1/q > 1$, $p, q \geqslant 1$. *If*

$$\{(a^n,\, z^n)\} \textit{ is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d+1}),$$

then

$$\Big\{\Big(\int_0^{\cdot} y^n_{s-}\, da^n_s,\, a^n,\, \int_0^{\cdot} x^n_{s-}\, dz^n_s,\, z^n\Big)\Big\} \textit{ is relatively compact in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1}).$$

Proof. Set $\bar{d} = 2d$ and for every $n \in \mathbb{N}$ define $\bar{z}^n \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{\bar{d}})$ and $\bar{x}^n \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{\bar{d}^2})$ by the formulas

$$\bar{z}^{n,i} = \begin{cases} a^n, & i = 1, \ldots, d,\\ z^{n,i-d}, & i = d+1, \ldots, 2d, \end{cases}$$

and

$$(\bar{x}^n)_{i,j} = \begin{cases} y^{n,i}, & i = j = 1,\dots,d, \\ (x^n)_{i-d,j-d}, & i = d+1,\dots,2d,\ j = d+1,\dots,2d, \\ 0, & \text{otherwise.} \end{cases}$$

By Proposition 5.1,

$$\Big\{\Big(\int_0^{\cdot} \bar{x}^n_{s-}\,d\bar{z}^n_s,\ \bar{z}^n\Big)\Big\} \text{ is relatively compact in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2\bar{d}}),$$

from which one can deduce the corollary. ∎

PROPOSITION 5.2. *Let* $\{x^n\} \subset \mathbb{D}(\mathbb{R}^+,\mathbb{M}^d)$, $\{z^n\} \subset \mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$, *and*

$$\sup_n \bar{V}_q(x^n)_T < \infty, \quad \sup_n V_p(z^n)_T < \infty, \quad T \in \mathbb{R}^+,$$

where $1/p + 1/q > 1$, $p,q \geqslant 1$. *If* $(x^n,z^n) \to (x,z)$ *in* $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{d^2+d})$, *then*

$$\Big(x^n, z^n, \int_0^{\cdot} x^n_{s-}\,dz^n_s\Big) \to \Big(x, z, \int_0^{\cdot} x_{s-}\,dz_s\Big) \quad \textit{in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^{d^2+2d}).$$

Proof. Since $V_q(x)_T \leqslant \liminf_{n\to\infty} V_q(x^n)_T < \infty$ and, similarly, $V_p(z)_T \leqslant \liminf_{n\to\infty} V_p(z^n)_T < \infty$ for $T \in \mathbb{R}^+$, the integral $\int_0^{\cdot} x_{s-}dz_s$ is well defined. Let $\big\{\{z^{n,(k)}\}\big\}$, $\{z^{(k)}\}$ be families of functions defined in the proof of Proposition 5.1 with families of constants $\{\delta_k\}$, $\big\{\{\delta_{k,i}\}\big\}$ such that $\delta_k \downarrow 0$, $|\triangle z_t| \neq \delta_k$, $t \in \mathbb{R}^+$, $\delta_k/2 \leqslant \delta_{k,i} \leqslant \delta_k$ and $|\triangle z_{t_{k,i}+\delta_{k,i}}| + |\triangle x_{t_{k,i}+\delta_{k,i}}| = 0$, $i \in \mathbb{N}\cup\{0\}$, $k,n \in \mathbb{N}$. Then

$$t^n_{k,i} \to t_{k,i}, \quad z^n_{t^n_{k,i}} \to z_{t_{k,i}}, \quad x^n_{t^n_{k,i}} \to x_{t_{k,i}}, \quad i \in \mathbb{N}\cup\{0\}, k \in \mathbb{N},$$

and

$$(x^n, z^{n,(k)}, z^n) \to (x, z^{(k)}, z) \quad \text{in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^{d^2+2d}).$$

Hence and from the equality $\int_0^t x_{s-}dz^{(k)}_s = \sum_{j\leqslant i} x_{t_{k,j}-}(z_{t_{k,j}} - z_{t_{k,j-1}})$, where $t \in [t_{k,i}, t_{k,i+1})$, for any $k \in \mathbb{N}$,

$$\Big(x^n, z^n, \int_0^{\cdot} x^n_{s-}\,dz^{n,(k)}_s\Big) \to \Big(x, z, \int_0^{\cdot} x_{s-}\,dz^{(k)}_s\Big) \quad \text{in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^{d^2+2d}). \tag{5.8}$$

As in the proof of (5.7) we check that $\lim_{k\to\infty} \sup_{t\leqslant T} \big|\int_0^t x_{s-}d(z - z^{(k)})_s\big| = 0$, $T \in \mathbb{R}^+$. From this and (5.7), (5.8) the result follows. ∎

Using arguments from the proof of Corollary 5.1 it is easy to check that Proposition 5.2 implies the following corollary.

COROLLARY 5.2. *Let* $\{a^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R})$, $\{z^n\}, \{y^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ *and* $\{x^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{M}^d)$ *be sequences of functions such that*

$$\sup_n \max\big(V_1(a^n)_T, V_p(z^n)_T, \bar{V}_q(y^n)_T, \bar{V}_q(x^n)_T\big) < \infty, \quad T \in \mathbb{R}^+,$$

where $1/p + 1/q > 1$, $p, q \geqslant 1$. *If*

$$(y^n,\, a^n,\, x^n,\, z^n) \to (y,\, a,\, x,\, z) \quad \textit{in}\, \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d^2+2d+1}),$$

then

$$\Big(\int_0^{\cdot} y^n_{s-}\, da^n_s,\, a^n,\, \int_0^{\cdot} x^n_{s-}\, dz^n_s,\, z^n\Big) \to \Big(\int_0^{\cdot} y_{s-}\, da_s,\, a,\, \int_0^{\cdot} x_{s-}\, dz_s,\, z\Big)$$

in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d+1})$.

Acknowledgements. The authors thank the referee for careful reading of the paper and valuable remarks.

REFERENCES

[1] S. Asmussen, *Queueing simulation in heavy traffic*, Math. Oper. Res. 17 (1992), pp. 84–111.

[2] M. Besalu and C. Rovira, *Stochastic delay equations with non-negativity constraints driven by fractional Brownian motion*, Bernoulli 18 (1) (2012), pp. 24–45.

[3] R. J. Chitashvili and N. L. Lazrieva, *Strong solutions of stochastic differential equations with boundary conditions*, Stochastics 5 (1981), pp. 225–309.

[4] F. Coquet, J. Mémin, and L. Słomiński, *On non-continuous Dirichlet processes*, J. Theoret. Probab. 16 (2003), pp. 197–216.

[5] F. Coquet and L. Słomiński, *On the convergence of Dirichlet processes*, Bernoulli 5 (1999), pp. 615–639.

[6] R. M. Dudley, *Picard iteration and p-variation: The work of Lyons (1994)*, in: *Mini-proceedings: Workshop on Product Integrals and Pathwise Integration*, MaPhySto 1999.

[7] R. M. Dudley and R. Norvaiša, *An Introduction to p-variation and Young Integrals*, Lecture Note No. 1, Aarhus University, 1999.

[8] R. M. Dudley and R. Norvaiša, *Concrete Functional Calculus*, Springer Science+Business Media 2011.

[9] P. Dupuis and H. Ishii, *On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications*, Stoch. Stoch. Rep. 35 (1991), pp. 31–62.

[10] P. Dupuis and K. Ramanan, *A multiclass feedback queueing network with a regular Skorokhod problem*, Queueing Syst. 36 (2000), pp. 327–349.

[11] S. Ethier and T. Kurtz, *Markov Processes. Characterization and Convergence*, Wiley, New York 1986.

[12] A. Falkowski, L. Słomiński, and B. Ziemkiewicz, *On weak and strong discrete-time approximations of fractional SDEs*, Lithuanian Math. J. 54 (2014), pp. 409–428.

[13] M. Ferrante and C. Rovira, *Stochastic differential equations with non-negativity constraints driven by fractional Brownian motion*, J. Evol. Equ. 13 (2013), pp. 617–632.

[14] J. Jacod and A. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer, Berlin 1987.

[15] A. Jakubowski, J. Mémin, and G. Pagès, *Convergence en loi des suites d'intégrales stochastiques sur l'espace* D^1 *de Skorokhod*, Probab. Theory Related Fields 81 (1989), pp. 111–137.

[16] P. Krée and C. Soize, *Mathematics and Random Phenomena: Random Vibrations of Mechanical Structures*, Reidel, Dordrecht 1986.
[17] K. Kubilius, *The existence and uniqueness of the solution of an integral equation driven by a p-semimartingale of special type*, Stochastic Process. Appl. 98 (2002), pp. 289–315.
[18] K. Kubilius, *On weak and strong solutions of an integral equation driven by a continuous p-semimartingale*, Lithuanian Math. J. 43 (2003), pp. 34–50.
[19] P. L. Lions and A. S. Sznitman, *Stochastic differential equations with reflecting boundary conditions*, Comm. Pure Appl. Math. 37 (1983), pp. 511–537.
[20] T. J. Lyons, *Differential equations driven by rough signals (I): An extension of an inequality of L. C. Young*, Math. Res. Lett. 1 (1994), pp. 451–464.
[21] J. Mémin and L. Słomiński, *Condition UT et stabilité en loi des solutions d'équations différentielles stochastiques*, in: *Séminaire de Probabilités XXV*, Lecture Notes in Math., Vol. 1485, Springer, 1991, pp. 162–177.
[22] D. Nualart and A. Răşcanu, *Differential equations driven by fractional Brownian motion*, Collect. Math. 53 (2002), pp. 55–81.
[23] A. A. Ruzmaikina, *Stieltjes integrals of Hölder continuous functions with applications to fractional Brownian motion*, J. Statist. Phys. 100 (2000), pp. 1049–1069.
[24] M. A. Shashiashvili, *On the variation of the difference of singular components in the Skorokhod problem and on stochastic differential systems in a half-space*, Stochastics 24 (1988), pp. 151–169.
[25] L. A. Shepp and A. N. Shiryaev, *A new look at pricing of the "Russian option"*, Theory Probab. Appl. 39 (1994), pp. 103–119.
[26] A. V. Skorokhod, *Stochastic equations for diffusion processes in a bounded region 1,2*, Theory Probab. Appl. 6 (1961), pp. 264–274; 7 (1962), pp. 3–23.
[27] L. Słomiński, *Stability of strong solutions of stochastic differential equations*, Stochastic Process. Appl. 31 (1989), pp. 173–202.
[28] L. Słomiński, *On existence, uniqueness and stability of solutions of multidimensional SDE's with reflecting boundary conditions*, Ann. Inst. H. Poincaré 29 (1993), pp. 163–198.
[29] L. Słomiński, *Stability of stochastic differential equations driven by general semimartingales*, Dissertationes Math. 349 (1996), pp. 1–113.
[30] L. Słomiński, *Euler's approximations of solutions of SDEs with reflecting boundary*, Stochastic Process. Appl. 94 (2001), pp. 317–337.
[31] L. Słomiński and T. Wojciechowski, *Stochastic differential equations with time-dependent reflecting barriers*, Stochastics 85 (2013), pp. 27–47.
[32] L. Słomiński and B. Ziemkiewicz, *On weak approximations of integrals with respect to fractional Brownian motion*, Statist. Probab. Lett. 79 (2009), pp. 543–552.
[33] T. Tanaka, *Stochastic differential equations with reflecting boundary condition in convex regions*, Hiroshima Math. J. 9 (1979), pp. 163–177.
[34] L. C. Young, *An inequality of the Hölder type, connected with Stieltjes integration*, Acta Math. 67 (1936), pp. 251–282.

Adrian Falkowski
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
ul. Chopina 12/18
87-100 Toruń, Poland
E-mail: izinizik@mat.umk.pl

Leszek Słomiński
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
ul. Chopina 12/18
87-100 Toruń, Poland
E-mail: leszeks@mat.umk.pl

Received on 25.9.2014;
revised version on 18.2.2015