PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 35, Fasc. 2 (2015), pp. 191–200

HCM PROPERTY AND THE HALF-CAUCHY DISTRIBUTION

BY

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Abstract. Let Z_α and $\tilde{Z}_\alpha$ be two independent positive α-stable random variables. It is known that $(Z_\alpha/\tilde{Z}_\alpha)^\alpha$ is distributed as the positive branch of a Cauchy random variable with drift. We show that the density of the power transformation $(Z_\alpha/\tilde{Z}_\alpha)^\beta$ is hyperbolically completely monotone in the sense of Thorin and Bondesson if and only if $\alpha \leqslant 1/2$ and $|\beta| \geqslant \alpha/(1-\alpha)$. This clarifies a conjecture of Bondesson (1992) on positive stable densities.

2010 AMS Mathematics Subject Classification: Primary: 60E07; Secondary: 60E05.

Key words and phrases: Half-Cauchy distribution, complete monotonicity, generalized gamma convolution, hyperbolically completely monotone function, Pick function, positive stable density.

1. INTRODUCTION

A function $f : (0, \infty) \to (0, \infty)$ is said to be *hyperbolically completely monotone* (HCM) if for every $u > 0$ the function $f(uv)f(u/v)$ is completely monotone (CM) as a function of the variable $w = v + v^{-1}$. This class coincides with that of functions of the form

$$c\, x^a \prod_{i=1}^{n} (1 + c_i x)^{-b_i} \tag{1.1}$$

with $a \in \mathbb{R}$ and $c, c_i, b_i > 0$, or pointwise limits thereof. A positive distribution is called *HCM* if it has a density which is HCM. The HCM class is closed with respect to multiplication and division of independent random variables. Moreover, if X has an HCM density, so has X^β for every $|\beta| \geqslant 1$. This class was introduced by L. Bondesson by using an idea of O. Thorin from 1977 (Bondesson introduced the HCM-condition in 1990 and the HCM-name in 1992). It is closely connected with the class of generalized gamma convolutions (GGC). We say that the distribution of a positive random variable X is a *GGC* if its Laplace transform reads

$$\mathbf{E}(e^{-\lambda X}) = \exp\left(-a\lambda - \int_0^\infty \log\left(1 + \frac{\lambda}{x}\right) \nu(dx)\right) \tag{1.2}$$

for some $a \geqslant 0$ called the *drift coefficient* and some positive measure ν called the *Thorin measure*, which is such that

$$\int_0^1 |\log(x)|\nu(dx) < \infty \quad \text{and} \quad \int_1^\infty x^{-1}\nu(dx) < \infty.$$

The GGC class is a subclass of the positive self-decomposable distributions, and in particular all GGC distributions are infinitely divisible. In [3] Bondesson proved the inclusion $\mathrm{HCM} \subset \mathrm{GGC}$, which allowed for showing the GGC property, and hence the infinite divisibility of many positive distributions whose Laplace transforms are not explicit enough. As a genuine example, Thorin proved in [14] the infinite divisibility of powers of a gamma random variable at order ξ with $|\xi| \geqslant 1$. This is actually a consequence to the fact that gamma densities are HCM.

Another link between the two above classes is that a probability distribution is a GGC if and only if its Laplace transform is HCM. This characterization, which is also due to Bondesson, can be used to show both GGC and HCM properties, and it will play some role in the proof of the main result of this paper. We refer to the monograph [4] for an account on these topics, including the proof of all above properties.

Let Z_α be a positive α-stable random variable, $\alpha \in (0,1)$, normalized so that its Laplace transform reads

$$\mathbf{E}(e^{-\lambda Z_\alpha}) = \exp\left(-\lambda^\alpha\right) = \exp\left(-\frac{\alpha\sin(\alpha\pi)}{\pi}\int_0^\infty \log\left(1+\frac{\lambda}{x}\right)x^{\alpha-1}dx\right).$$

Observe that this Laplace tranform is of the form (1.2), so that all positive α-stable distributions are GGC. In this paper we are concerned with the following

CONJECTURE 1.1 (Bondesson [5]). *The density of Z_α is HCM if and only if $\alpha \leqslant 1/2$.*

This problem is stated in [5], where the easy "only if" part is also obtained. If $\alpha = 1/n$ for some integer $n \geqslant 2$, the HCM property for the density of $Z_{1/n}$ is a direct consequence of the factorization (see Example 5.6.2 in [4])

$$Z_{1/n}^{-1} \stackrel{\mathrm{d}}{=} n^n \gamma_{1/n} \times \ldots \times \gamma_{(n-1)/n}, \tag{1.3}$$

where the factors are supposed to be independent and, here and throughout, γ_t denotes a gamma random variable with shape parameter t and explicit density

$$\frac{x^{t-1}e^{-x}}{\Gamma(t)}.$$

The "if" part of this conjecture is however still open when $\alpha \neq 1/n$. In [13], it is shown that Z_α has a hyperbolically monotone density (viz. its density f is such that $f(uv)f(u/v)$ is non-increasing in the variable $v+1/v)$ if and only if $\alpha \leqslant 1/2$. Proposition 4 of [9] shows that the quotient $Z_\alpha/\tilde{Z}_\alpha$ (with independent components) has an HCM density if and only if $\alpha \leqslant 1/2$. We refer to the whole article [9] for other partial results on Bondesson's conjecture. Lastly, a positive answer to the "if" part for $\alpha \in (0,1/4] \cup [1/3,1/2]$ has been recently announced in [8].

In this paper we consider the random variable

$$\left(\frac{Z_\alpha}{\tilde{Z}_\alpha}\right)^\beta,$$

where $Z_\alpha \perp \tilde{Z}_\alpha$ and $\beta \in \mathbb{R}$. It is well known – see (3.3.18) in [16] or Exercise 4.21 in [6] – that $(Z_\alpha/\tilde{Z}_\alpha)^\alpha$ has an explicit density which is that of an affine transformation of a Cauchy random variable conditioned to be positive:

THEOREM 1.1 (Zolotarev [16]).

$$\left(\frac{Z_\alpha}{\tilde{Z}_\alpha}\right)^\alpha \sim \frac{\sin(\pi\alpha)}{\pi\alpha\big(x^2+2\cos(\pi\alpha)x+1\big)}.$$

When $\alpha = 1/2$, the above distribution is the half-Cauchy distribution whose infinite divisibility has been obtained in [2]. This result has been refined into self-decomposability in [7]. On the other hand, we know from (ix), p. 68, in [4] that $(Z_\alpha/\tilde{Z}_\alpha)^\alpha$ has never an HCM density. Our main result shows that this property holds when taking sufficiently high power transformations:

THEOREM 1.2. *The power transformation* $(Z_\alpha/\tilde{Z}_\alpha)^\beta$ *has an HCM density if and only if* $\alpha \leqslant 1/2$ *and* $|\beta| \geqslant \alpha/(1-\alpha)$.

Whereas the "only if" part of this theorem is a direct consequence of known analytical properties of HCM functions, the "if" part is more involved and relies on a Pick function characterization. This result shows that the explicit density of $(Z_\alpha/\tilde{Z}_\alpha)^\beta$ is the pointwise limit of functions of the type (1.1) as soon as $\alpha \leqslant 1/2$ and $|\beta| \geqslant \alpha/(1-\alpha)$, but we could not find any constructive argument for that.

The main interest of our theorem is to propose a refined version of Bondesson's conjecture from the point of view of power transformations. It is indeed natural to raise the further

CONJECTURE 1.2. *The density of* Z_α^β *is HCM if and only if* $\alpha \leqslant 1/2$ *and* $|\beta| \geqslant \alpha/(1-\alpha)$.

Observe that our result shows already the "only if" part. Some partial results for the "if" part are also given in [9], where it is shown that the distribution of Z_α^β is self-decomposable when $\alpha \leqslant 1/2$ and $\beta \leqslant -\alpha/(1-\alpha)$ – see Proposition 1 in [9] and the whole Section 3 therein where the critical power exponent $\alpha/(1-\alpha)$ appears naturally. In general, this conjecture on the power transformations of Z_α seems hard to solve even when α is the reciprocal of an integer. In this paper we briefly handle the explicit case $\alpha = 1/2$, which is immediate, and the case $\alpha = 1/3$, which relies on the HCM property for the modified Bessel function K_ν. This latter property leads to the following

THEOREM 1.3. *Let γ_t and γ_s be two independent gamma random variables. The density of $\sqrt{\gamma_t \gamma_s}$ is HCM if and only if $|t-s| \leqslant 1/2$.*

In particular, the square root of the product of two independent unit exponential random variables has an infinitely divisible distribution, a fact which seems unnoticed in the literature. Recall that $\sqrt{\gamma_t}$ does not have an infinitely divisible distribution because of the superexponential tails of its distribution function (see Theorem 26.1 in [11]), and hence it is not HCM either.

2. PROOF OF THEOREM 1.2

Let us consider the function

$$f_{\alpha,t}(x) = \frac{1}{x^{2t} + 2\cos(\pi\alpha)x^t + 1} \tag{2.1}$$

with $\alpha \in (0,1)$ and $t \geqslant 0$. The function $f_{\alpha,t}$ is, up to the multiplication by the function $\sin(\pi\alpha)tx^{t-1}/\pi\alpha$, the density of $(Z_\alpha/\tilde{Z}_\alpha)^{\alpha/t}$. Hence Theorem 1.2 amounts to show that $f_{\alpha,t}$ is HCM if and only if $\alpha \leqslant 1/2$ and $t \leqslant 1-\alpha$.

2.1. Proof of the "only if" part. The condition $\alpha \leqslant 1/2$ is necessary because otherwise the function $f_{\alpha,t}$ would be locally increasing at $0+$ (see (x), p. 68, in [4]). To show the necessity of $t \leqslant 1-\alpha$ it suffices to invoke the fact that an HCM function can be extended to an analytic function on $\mathbb{C}\setminus\mathbb{R}_-$ (see (ix), p. 68, in [4]). More precisely, let

$$P_\alpha(z) = z^2 + 2\cos(\pi\alpha)z + 1. \tag{2.2}$$

Note that this polynomial has two zeroes $e^{\pm i(1-\alpha)\pi}$ and that the power function z^t is surjective from $\mathbb{C}\setminus\mathbb{R}_-$ onto the cone $\{\rho e^{i\theta}; (\rho,\theta) \in (0,\infty)\times(-\pi t, \pi t)\}$, so that the function $P_\alpha(z^t)$ vanishes on $\mathbb{C}\setminus\mathbb{R}_-$ if and only if $t > 1-\alpha$. Hence, $f_{\alpha,t}$ has an analytic continuation on $\mathbb{C}\setminus\mathbb{R}_-$ if and only if $t \leqslant 1-\alpha$.

2.2. Proof of the "if" part. By formula (iv), p. 68, in [4] it is enough to prove that $f_{\alpha,1-\alpha}$ is HCM. Observe first that

$$f_{\alpha,1-\alpha}(x) = \frac{1}{x^{2(1-\alpha)} + 2\cos(\pi\alpha)x^{1-\alpha}x^{1-\alpha} + 1} \to \frac{1}{(x+1)^2}$$

as $\alpha \to 0$, and that the limit is of the form (1.1).

Formula (iv), p. 68, and Theorem 5.4.1 in [4] show that it is enough to prove that $G = -\log(f_{\alpha,1-\alpha})$ is a Thorin–Bernstein function. By Theorem 8.2 (ii) in [12], this is equivalent to G' being a Stieltjes function in the sense of Definition 2.1 in [12]. We will use the Pick characterization of Stieltjes transform given by Corollary 7.4 in [12]. In fact, since the family of all Stieltjes functions is closed under pointwise limits (see Theorem 2.2 (iii) in [12]), it suffices to show that

$$G_\varepsilon = -\log(f_{\alpha,1-\alpha-\varepsilon})$$

is a Stieltjes function for $\varepsilon > 0$ small enough. Fixing $\varepsilon > 0$, we saw during the proof of the "only if" part that $f_{\alpha,1-\alpha-\varepsilon}$ has an analytic continuation on $\mathbb{C} \setminus \mathbb{R}_-$. Moreover, the function does not vanish on $\mathbb{C} \setminus \mathbb{R}_-$. All in all G_ε has also an analytic continuation on $\mathbb{C} \setminus \mathbb{R}_-$. Hence we need to check that

$$\mathrm{Im}(z) > 0 \;\Rightarrow\; \mathrm{Im}\left(G'_\varepsilon(z)\right) \leqslant 0.$$

The function $h = -\mathrm{Im}(G'_\varepsilon)$ defined on $\{z \in \mathbb{C}; \mathrm{Im}(z) > 0\}$ is a harmonic function as the imaginary part of the analytic function $-G'_\varepsilon$. Besides, h can be extended continuously to $\mathbb{H} = \{z \in \mathbb{C}; \mathrm{Im}(z) \geqslant 0\}$ and vanishes on $(0, \infty)$. Lastly, it is clear that $h(z) \to 0$ as $|z| \to \infty$ uniformly on $\mathbb{H}$. Hence, setting

$$m = \inf_{z \in \mathbb{H}} h(z),$$

we see that $m \in (-\infty, 0]$. We will now prove that $m = 0$. Applying the minimum principle to the harmonic function $h : \{z \in \mathbb{C}; \mathrm{Im}(z) > 0\} \to \mathbb{R}$, we see that the latter property follows as soon as $h(-x) \geqslant 0$ for all $x > 0$. First, we compute, for all $z \in \mathbb{H}$,

$$h(z) \;=\; -2(1-\alpha-\varepsilon)\,\mathrm{Im}\left(z^{-(\alpha+\varepsilon)}\frac{z^{1-\alpha-\varepsilon} + \cos(\pi\alpha)}{z^{2(1-\alpha-\varepsilon)} + 2\cos(\pi\alpha)z^{1-\alpha-\varepsilon} + 1}\right).$$

Hence, setting $-x = \rho^{1/(1-\alpha-\varepsilon)}e^{i\pi}$ for some $\rho > 0$ we find

$$\begin{aligned} h\,(-x) &= -A\ \mathrm{Im}\left[e^{-i(\alpha+\varepsilon)\pi}\left(\rho e^{i(1-\alpha-\varepsilon)\pi} + \cos(\pi\alpha)\right)\right. \\ &\qquad\qquad \left.\times\left(\rho^2 e^{-i2(1-\alpha-\varepsilon)\pi} + 2\cos(\pi\alpha)\rho e^{-i(1-\alpha-\varepsilon)\pi} + 1\right)\right] \\ &= A\cos(\pi\alpha)\sin\left((\alpha+\varepsilon)\pi\right)\left(\underbrace{\rho^2 - 2\frac{\cos\left((\alpha+\varepsilon)\pi\right)}{\cos(\pi\alpha)}\rho + 1}_{\geqslant 0}\right) \end{aligned}$$

with

$$A \;=\; \frac{2(1-\alpha-\varepsilon)\rho^{-(\alpha+\varepsilon)/(1-\alpha-\varepsilon)}}{|\rho^2 e^{i2(1-\alpha-\varepsilon)\pi} + 2\cos(\pi\alpha)\rho e^{i(1-\alpha-\varepsilon)\pi} + 1|^2} \geqslant 0.$$

This completes the proof.

In the following figure we give two plots of the function h along the lines $\{\mathrm{Im}(z) = 1\}$ and $\{\mathrm{Im}(z) = 0.1\}$ for $\alpha = 1/5$ and $\varepsilon = 1/10$.

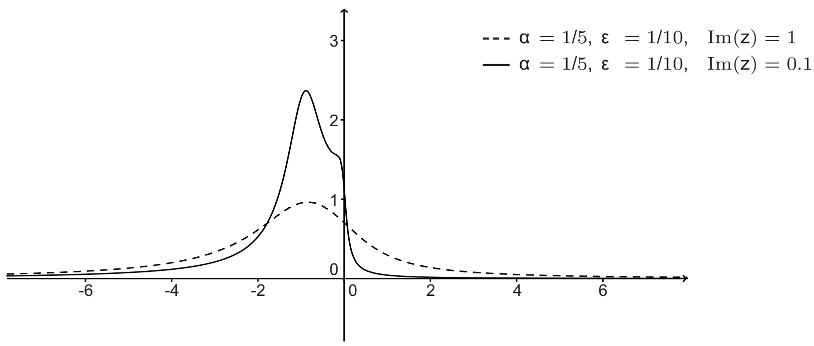

FIGURE 1. Plot of h along the lines $\mathrm{Im}(z) = c$

2.3. Remarks

2.3.1. *The cases* $\alpha = 1/2$ *and* $\alpha = 1/3$ Here we give another way to handle the two cases $\alpha = 1/2$ and $\alpha = 1/3$ proving directly the HCM property for $Z_{1/2}$ and $\sqrt{Z_{1/3}}$.

Using the identity (1.3) with $n = 2$, i.e.,

$$Z_{1/2} \overset{\mathrm{d}}{=} \frac{1}{4\gamma_{1/2}},$$

we infer that $Z_{1/2}$ has an HCM density.

The case $\alpha = 1/3$ is a little bit more involved. By using a change of variable and formula 2.8.31 in [16], the density of $\sqrt{Z_{1/3}}$ is given by

$$\frac{2}{3\pi x^2} K_{1/3}\left(\frac{2}{3\sqrt{3}x}\right),$$

where

$$K_\nu(x) \;=\; \int_0^\infty \cosh(\nu y) e^{-x\cosh(y)} dy, \qquad \nu \in \mathbb{R},$$

is a modified Bessel function. In Section 3 we will prove that this latter function is HCM if and only if $|\nu| \leqslant 1/2$.

2.3.2. Our result claims that the function $\log\big(x^{2t} + 2\cos(\pi\alpha)x^t + 1\big)$ is a Thorin–Bernstein function in the sense of Chapter 8 in [12] if and only if $\alpha \in [0, 1/2]$ and $t \in [0, 1-\alpha]$. In other words, the function

$$\frac{x^{2t} + 2\cos(\pi\alpha)x^t + 1}{x^{2t-1} + \cos(\pi\alpha)x^{t-1}}$$

is a complete Bernstein function if and only if $\alpha \in [0, 1/2]$ and $t \in [0, 1-\alpha]$.

2.3.3. A consequence of our result is that the function $f_{\alpha,t}$ is CM for all $\alpha \leqslant 1/2,\ t \in [0, 1-\alpha]$. When $t \leqslant 1/2$, this property follows also from the immediate fact that $f_{\alpha,t}$ is the reciprocal of a Bernstein function, hence the Laplace transform of the potential measure of some subordinator – see Chapter 1 in [12] for details and terminology. On the other hand, the function $x^{2(1-\alpha)} + 2\cos(\pi\alpha)x^{1-\alpha} + 1$ is not Bernstein for $\alpha < 1/2$.

One could ask if $f_{\alpha,1-\alpha}$ is also a Stieltjes function, viz. the double Laplace transform of a non-negative measure. The answer is however negative when $\alpha < 1/2$. Indeed, the Stieltjes inversion formula (see Chapter VIII, Theorem 7.a, in [15]) would entail

$$m(dx) = -\frac{1}{\pi} \lim_{\varepsilon \to 0^+} \operatorname{Im}\left[f_{\alpha,1-\alpha}(-x + i\varepsilon)\right] dx,$$

and we can check that the right-hand side is not non-negative. Another way to see this is to use again the fact that $1/f_{\alpha,1-\alpha}$ is not a Bernstein function, hence not a complete Bernstein function (see Chapter 6 in [12]).

2.3.4. Writing

$$f_{\alpha,t}(uv) f_{\alpha,t}(u/v) = \frac{1}{u^{4t} + c^2 u^{2t} + 1 + c(u^t + u^{3t}) w_t + u^{2t} w_{2t}}$$

with $w_a = v^a + v^{-a}$ for all $a \geqslant 0$ and using the fact that $w \mapsto w_a$ is a Bernstein function when $a \in [0, 1]$ (see p. 183 in [3]), we see that the right-hand side is CM in w for all $\alpha, t \leqslant 1/2$. But again, this argument does not work for $t = 1 - \alpha$.

3. PROOF OF THEOREM 1.3

3.1. The density of $\sqrt{\gamma_t \gamma_s}$ is given explicitly through the modified Bessel function K_{t-s} by

$$\frac{4x^{t+s-1}}{\Gamma(t)\Gamma(s)} K_{t-s}(2x)$$

for all $s, t > 0$. Indeed,

$$\begin{aligned}
\frac{d}{dx}\mathbf{P}(\sqrt{\gamma_t \gamma_s} \leqslant x) &= \frac{d}{dx} \int_0^\infty \mathbf{P}(\gamma_t \leqslant x^2/u) \frac{u^{s-1} e^{-u}}{\Gamma(s)} du \\
&= \frac{2x^{t+s-1}}{\Gamma(t)\Gamma(s)} \int_0^\infty v^{s-t-1}\, e^{-x(v+v^{-1})}\, dv \\
&= \frac{4x^{t+s-1}}{\Gamma(t)\Gamma(s)} \int_0^\infty \cosh\big((s-t)y\big)\, e^{-2x\cosh(y)}\, dy,
\end{aligned}$$

where we made the substitutions $u = vx$ and $v = e^y$. Then the density is HCM if and only if the function K_{t-s} is HCM. On the other hand, formula 9.6.23 in [1] states that for all $\nu > -1/2$ one has

$$K_\nu(x) = \frac{\sqrt{\pi}(x/2)^\nu}{\Gamma(\nu+1/2)} \int_1^\infty e^{-xy} \frac{dy}{(y^2-1)^{1/2-\nu}}, \tag{3.1}$$

and hence

$$K_\nu(x) = C_\nu\, x^\nu\, e^{-x} \int_0^\infty e^{-xy} \frac{dy}{\big(y(y+2)\big)^{1/2-\nu}}$$

for some constant $C_\nu > 0$. For $\nu \leqslant 1/2$ the function $\big(y(y+2)\big)^{\nu-1/2}$ is HCM, thus the function is the density of a widened GGC – we use the terminology of Chapter 3.5 in [4]. Therefore, its Laplace transform is HCM, and hence K_ν is HCM.

It remains to show that, for $\nu > 1/2$, K_ν is not HCM. Assume the contrary. Then it follows from (3.1) that the Laplace transform $\int_1^\infty e^{-xy}(y^2-1)^{\nu-1/2}dy$ is HCM, and hence the function $(y^2-1)^{\nu-1/2}$ is a widened GGC on $(1,\infty)$. But then $\big(y(y+2)\big)^{\nu-1/2}$ is a widened GGC on $(0,\infty)$. Applying Theorems 4.1.4 and 6.2.4 in [4], we see that $e^{-y}\big(y(y+2)\big)^{\nu-1/2}$ is proportional to the density of a GGC on $(0,\infty)$ whose Thorin mass is $\nu+1/2$. Theorem 4.1.1 in [4] then implies that $e^{-y}(y+2)^{\nu-1/2}$ is CM. However, it is easily shown that this latter function is not CM when $\nu > 1/2$.

3.2. Remark. It follows from the product formula 9.6.24 in [1] that for all $\nu \in (-1/2, 1/2)$, $x, y > 0$, one has

$$K_\nu(x)K_\nu(y) = 2\cos(\pi\nu) \int_0^\infty K_{2\nu}\big(2\sqrt{xy}\sinh(t)\big)e^{-(x+y)\cosh(t)}dt.$$

Hence, for all $u, v > 0$,

$$K_\nu(uv)K_\nu(u/v) = 2\cos(\pi\nu) \int_0^\infty \underbrace{K_{2\nu}\big(2u\sinh(t)\big)e^{-uw\cosh(t)}}_{\text{CM in } w}\, dt,$$

and since the CM class is closed under mixing (see Chapter 1 in [12]), all in all this shows that $K_\nu(uv)K_\nu(u/v)$ is CM in the variable $w = v + 1/v$, which entails the required HCM property for $\sqrt{\gamma_t\gamma_s}$ when $|t-s| \leqslant 1/2$. The converse cannot be retrieved with this formula which is less powerful than (3.1) in the HCM-context.

4. FURTHER REMARKS AND COMMENTS

4.1. Infinite divisibility of $(Z_\alpha/\tilde{Z}_\alpha)^\alpha$ distribution. The infinite divisibility of the half-Cauchy distribution was proved in [2]. As the author mentions in Remark 7.2, the half-Cauchy is a gamma mixture with shape parameter 2. In other words, the function

$$g(x) = \frac{1}{x(x^2+1)} = \int_0^\infty e^{-xy}\big(1-\cos(y)\big)dy$$

is CM. In [10], Kristiansen proved that a mixture with shape parameter smaller than 2 is infinitely divisible, and hence gave a new proof of the infinite divisibility of this distribution. It is then easy to see that the distribution of $(Z_\alpha/\tilde{Z}_\alpha)^\alpha$ is also infinitely divisible when $0 < \alpha < 1/2$. Indeed,

$$\frac{1}{x\big(x^2+2\cos(\pi\alpha)x+1\big)} = \big(\sin^{-3}(\pi\alpha)\big)\left(1+\frac{\cos(\pi\alpha)}{x}\right) g\left(\frac{x+\cos(\pi\alpha)}{\sin(\pi\alpha)}\right)$$

is a CM function.

4.2. Complete monotonicity of $f_{\alpha,t}$. Set $\alpha \leqslant 1/2$. We know that the function $f_{\alpha,t}$ is CM for all $0 \leqslant t \leqslant 1-\alpha$ (see Remark 2.3.3). Besides, this last constant $1-\alpha$ is optimal for the HCM property of $f_{\alpha,t}$ by our main result. Lastly, it is clear – see again Chapter 1 in [12] – that there exists some $t_\alpha \geqslant 1-\alpha$ such that $f_{\alpha,t}$ is CM if and only if $t \leqslant t_\alpha$, and it is a natural question whether $t_\alpha = 1-\alpha$ or not. The next proposition entails that $t_\alpha < 1$.

PROPOSITION 4.1. *The function $f_{\alpha,1}$ is not CM for any $\alpha \in (0,1)$.*

P r o o f. Using the Laplace inversion formula (see Chapter II, Theorem 7.4, in [15]), by a computation of the residues of the function $z \mapsto e^{\lambda z} f_{\alpha,1}(z)$ we obtain

$$\mathscr{L}^{-1} f(\lambda) = \frac{1}{2i\pi} \int_{\mathrm{Re}(z)=c} e^{\lambda z} f_{\alpha,1}(z) dz = e^{-\lambda a}\frac{\sin(\lambda b)}{b}$$

with $a = \cos(\pi\alpha)$ and $b = \sin(\pi\alpha)$. Therefore, $(2i\pi)^{-1}\int_{\mathrm{Re}(z)=c} e^{\lambda z} g(z)dz$ does not have a non-negative sign for all $\lambda > 0$, and hence $f_{\alpha,1}$ is not CM. ■

Recall that $x^{-\alpha} f_{\alpha,1-\alpha}(x)$ is up to a constant the density of $(Z_\alpha/\tilde{Z}_\alpha)^{\alpha/(1-\alpha)}$. Consequently, observe that the distribution is a gamma mixture with shape parameter $1-\alpha$. In other words, we have the factorization

$$\left(\frac{Z_\alpha}{\tilde{Z}_\alpha}\right)^{\alpha/(1-\alpha)} \overset{\mathrm{d}}{=} \gamma_{1-\alpha} \times Y_\alpha,$$

where Y_α is some positive random variable independent of $\gamma_{1-\alpha}$. More generally, it is easy to see that $f_{\alpha,t}$ is CM if and only if the distribution of $(Z_\alpha/\tilde{Z}_\alpha)^{\alpha/t}$ is a gamma mixture with shape parameter t, which means that the function

$$s \mapsto \frac{\Gamma\left(1-\frac{s}{t}\right)\Gamma\left(1+\frac{s}{t}\right)\Gamma(t)}{\Gamma\left(1-\frac{\alpha s}{t}\right)\Gamma\left(1+\frac{\alpha s}{t}\right)\Gamma(t+s)}$$

is the Mellin transform of some probability distribution. However, it is not easy to prove directly this latter property.

Acknowledgments. The author is grateful to his Ph.D. adviser Thomas Simon for his help during the preparation of this paper, to Lennart Bondesson for the interest he took in this work and some useful comments, and to the two referees for giving several enhancements.

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Received on 7.2.2014;
revised version on 8.10.2014