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SUPREMUM DISTRIBUTION OF BESSEL PROCESS OF DRIFTING BROWNIAN MOTION

BY

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Abstract. Let us assume that $(B_t^{(1)}, B_t^{(2)}, B_t^{(3)} + \mu t)$ is a three-dimensional Brownian motion with drift μ, starting at the origin. Then $X_t = \|(B_t^{(1)}, B_t^{(2)}, B_t^{(3)} + \mu t)\|$, its distance from the starting point, is a diffusion with many applications. We investigate the supremum of (X_t), give an infinite-series formula for its distribution function and an exact estimate of the density of this distribution in terms of elementary functions.

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1. INTRODUCTION

In his famous paper [10] (see also [9], p. 436) David Williams showed how one can decompose the paths of a transient one-dimensional diffusion at its maximum (or minimum). One of the best-known examples of such decomposition is that of $B(t) + \mu t$, a Brownian motion with a positive drift μ, as a Brownian motion with a negative drift $B(t) - \mu t$ and a diffusion with generator Δ_μ,

$$\Delta_\mu = \frac{1}{2}\frac{d^2}{dx^2} + \mu \coth(\mu x)\frac{d}{dx}. \tag{1.1}$$

Namely, for $\mu > 0$ let $B(t) - \mu t$ be a Brownian motion with constant drift $-\mu$, Z_t a diffusion with generator (1.1), started at zero, and let γ be a random variable with exponential distribution with parameter $\frac{1}{2\mu}$. Assume that B_t, Z_t and γ, defined on the same probability space, are independent. Put $\tau = \inf\{t > 0 :\ B_t - \mu t = -\gamma\}$. Then the process

$$X_t = \begin{cases} B_t - \mu t, & 0 \leqslant t \leqslant \tau, \\ Z_{t-\tau} - \gamma, & t > \tau, \end{cases}$$

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has the same law as $B_t + \mu t$, a Brownian motion with a positive drift μ. Williams [10] also showed that Z_t can be viewed as a Brownian motion with drift conditioned to stay positive forever. Later on a process with generator (1.1), known as a hyperbolic Bessel process ([8], p. 357) or as a Bessel process of drifting Brownian motion and denoted by $\mathrm{BES}(3,\mu)$ (see [9], [5]), appeared also in many papers as a drifting Brownian motion conditioned to not to hit zero. Recently, it was proved in [1] that it can be obtained as a deterministic involution of Brownian motion with drift μ. If $\mu = 1$ then (Z_t) is a radial part of a hyperbolic Brownian motion in three-dimensional hyperbolic space.

The transition density function of (Z_t) is well known (cf. [7] or [5]) but to our best knowledge, the distribution of different functionals of this process has not been investigated yet. In this paper we investigate the process (Z_t) killed on exiting interval $(0,\, r_0)$ and give a formula describing the distribution of $M_t = \sup_{s\leqslant t} Z_s$, the supremum of the process (Z_t). Because the formula is given as an infinite series, we obtain its exact estimate using elementary functions. Moreover, our method of estimation applied to a theta function $ss_y(v,t)$, used in a handbook by Borodin and Salminen [2], gives a very precise estimate of this function (cf. Remark 5.1 after Theorem 5.3).

2. TRANSITION DENSITY OF A BESSEL PROCESS OF DRIFTING BROWNIAN MOTION, KILLED ON EXITING INTERVAL $(0,\, r_0)$

Let $Z(t)$ be a diffusion on $(0,\infty)$ generated by the operator (1.1) with $\mu > 0$. The speed measure of this diffusion is equal to $m(dy) = \sinh^2(\mu y)\, dy$, and with respect to $m(dy)$ the transition density of $Z(t)$ has the following form (cf. [7]):

$$p(t;x,y) = \frac{e^{-\mu^2 t/2}\big(e^{-(y-x)^2/(2t)} - e^{-(y+x)^2/(2t)}\big)}{\sqrt{2\pi t}\,\sinh(\mu x)\sinh(\mu y)}. \tag{2.1}$$

By $(Z_t^{r_0})$ we will denote the process killed on exiting $(0,\, r_0)$. For positive μ, the process starting from $x > 0$ cannot reach zero and drifts to infinity so that almost all trajectories will be killed at r_0. Even if (Z_t) starts from zero, with probability one it will never visit zero again.

The transition density $p^{r_0}(t;x,y)$ of $(Z_t^{r_0})$, with respect to $m(dy)$, is a solution of the following Dirichlet problem:

$$\begin{cases} \frac{\partial}{\partial t} p^{r_0}(t;x,y) = \Delta_\mu p^{r_0}(t;x,y), & t>0,\ x\in(0,r_0),\ y\in(0,r_0),\\ p^{r_0}(t;x,r_0) = 0, & t>0,\ x\in(0,r_0),\\ \lim_{t\to 0} p^{r_0}(t;x,y)m(dy) = \delta_x(dy), & x\in(0,r_0),\ y\in(0,r_0), \end{cases} \tag{2.2}$$

where $\delta_x(dy)$ is the Dirac delta function. Because of killing, $p^{r_0}(t;x,r_0) = 0$ for $t > 0$. Moreover, if $\mu > 0$ then by (2.1) we have $\limsup_{y\to 0} p(t;x,y) < \infty$ for $t,\, x > 0$, and $p^{r_0}(t;x,y) \leqslant p(t;x,y)$ implies $\limsup_{y\to 0} p^{r_0}(t;x,y) < \infty$.

We will use the separation variable technique, which is well known in mathematical physics. Suppose that $p^{r_0}(t;x,y) = Y(y)T(t)$. Then, by (1.1), the first equation of the system (2.2) takes on the form

$$Y(y)T'(t) = \left(\frac{1}{2}Y''(y) + \mu\coth(\mu y)Y'(y)\right)T(t). \tag{2.3}$$

We add $\frac{1}{2}(\lambda^2+\mu^2)T(t)Y(y)$ to both sides of (2.3) and get two separate differential equations. The solution of the first equation, $T'(t) + \frac{1}{2}(\lambda^2+\mu^2)T(t) = 0$, is $T(t) = c_1 e^{-(\lambda^2+\mu^2)t/2}$. If in the second equation, $Y''(y) + 2\mu\coth(\mu y)Y'(y) + (\lambda^2+\mu^2)Y(y) = 0$, we substitute $Y(y) = u(y)/\sinh(\mu y)$ (see [6], Chapter XI, formula (1.9)), we get

$$\frac{1}{\sinh(\mu y)}\big(u''(y) + \lambda^2 u(y)\big) = 0.$$

By the above discussion, $\limsup_{y\to 0} Y(y) < \infty$ and $Y(y) = u(y)/\sinh(\mu y)$, so $\lim_{y\to 0} u(y) = 0$. It is well known that this boundary problem has a solution if and only if $\lambda = \frac{n\pi}{r_0}$, $n = 1,2,\ldots$, and this solution (up to a multiplicative constant) is given by $u_n(y) = \sin(n\pi y/r_0)$. Thus we may expand $p^{r_0}(t;x,y)$ as

$$p^{r_0}(t;x,y) = \sum_{n=1}^{\infty}\left[\left(\frac{a_n(x)}{\sinh(\mu y)}\sin\left(\frac{n\pi y}{r_0}\right)\right)\exp\left(\frac{-(n^2\pi^2/r_0^2+\mu^2)t}{2}\right)\right]. \tag{2.4}$$

To determine the coefficient $a_n(x)$ let us multiply the equation (2.4) by

$$a_k(x)\frac{\sin(k\pi y/r_0)}{\sinh(\mu y)},$$

then integrate the product over $(0,r_0)$ with respect to the measure $\sinh^2(\mu y)dy$ and let $t\to 0$. Then we get

$$\frac{a_k(x)\sin(k\pi x/r_0)}{\sinh(\mu x)} = \frac{r_0}{2}a_k^2(x);$$

hence $a_k(x) = 2\sin(k\pi x/r_0)/\big(r_0\sinh(\mu x)\big)$. To sum up, we have just proved the following theorem.

THEOREM 2.1. *Transition density* (*with respect to the measure* $\sinh^2(\mu y)\,dy$) *of the Bessel process of drifting Brownian motion*, *starting from* $x\in(0,r_0)$ *and killed at* r_0, *is given by the following formula*:

$$p^{r_0}(t;x,y) = \sum_{n=1}^{\infty}\left[\left(\frac{2\sin(n\pi x/r_0)\sin(n\pi y/r_0)}{r_0\sinh(\mu x)\sinh(\mu y)}\right)\exp\left(-\frac{(n^2\pi^2/r_0^2+\mu^2)t}{2}\right)\right]. \tag{2.5}$$

Observe that

$$p^{r_0}(t;x,y) = \\ = \left(\frac{2\exp(-\mu^2 t/2)}{r_0 \sinh(\mu x)\sinh(\mu y)}\right) \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{r_0}\right) \sin\left(\frac{n\pi y}{r_0}\right) e^{-(n^2\pi^2 t)/2}$$

and use the Jacobi-type identity (see Exercise 3, p. 67, in [3]). In this way we get

THEOREM 2.2. *Transition density* (*with respect to the measure* $\sinh^2(\mu y)\,dy$) *of the Bessel process of drifting Brownian motion*, *starting from* $x \in (0, r_0)$ *and killed at* r_0, *is given by the following formula*:

$$p^{r_0}(t;x,y) = \\ = \frac{e^{-\mu^2 t/2}}{\sqrt{2\pi t}\sinh(\mu x)\sinh(\mu y)} \sum_{k=-\infty}^{\infty} [e^{-(y-x+2kr_0)^2/(2t)} - e^{-(y+x+2kr_0)^2/(2t)}].$$

3. EXIT TIME

Let us consider $M_t = \sup_{s \leqslant t} Z_s$, the supremum of the Bessel process of drifting Brownian motion. The distribution of (M_t) is closely related to the distribution of the time when the process (Z_t) exits the interval $(0,\, r_0)$. Recall that for $\mu > 0$ the process (Z_t) exits $(0,\, r_0)$ at the point r_0. For $r_0 > 0$ let us define $\tau_{r_0} = \inf\{s : Z_s > r_0\}$. The distribution of (M_t) and the survival probability of the killed process $(Z_t^{r_0})$ are related by the following formula:

$$\mathbb{P}^x(M_t < r_0) = \mathbb{P}^x(\tau_{r_0} > t) = \int_0^{r_0} p^{r_0}(t;x,y)\sinh^2(\mu y)dy.$$

Integrating the n-th term of (2.5), we get (cf. [4], formula 2.671)

$$\int_0^{r_0} \sin\left(\frac{n\pi y}{r_0}\right)\sinh(\mu y)dy = \frac{(-1)^{n+1}\pi r_0 n \sinh(\mu r_0)}{n^2\pi^2 + \mu^2 r_0^2},$$

so that we may write the following:

THEOREM 3.1. *For* $t,\ r_0 > 0$ *and* $x \in (0,\, r_0)$ *the following formula holds*:

$$\mathbb{P}^x(M_t < r_0) = \mathbb{P}^x(\tau_{r_0} > t) \\ = \sum_{n=1}^{\infty} \left[\left(\frac{(-1)^{n+1} 2\pi n \sinh(\mu r_0)\sin(n\pi x/r_0)}{\sinh(\mu x)(n^2\pi^2 + \mu^2 r_0^2)}\right) \exp\left(-\frac{(n^2\pi^2/r_0^2 + \mu^2)t}{2}\right)\right]. \tag{3.1}$$

If we differentiate the above series term by term with respect to t we get

$$\frac{\pi\sinh(\mu r_0)}{\sinh(\mu x) r_0^2} \exp\left(-\frac{\mu^2 t}{2}\right) \sum_{n=1}^{\infty} (-1)^n n \sin(n\pi x/r_0) \exp\left(-\frac{n^2\pi^2}{2r_0^2}t\right).$$

Since $\mathbb{P}^x(\tau_{r_0} \in dt) = -\frac{\partial}{\partial t}\mathbb{P}^x(\tau_{r_0} > t)$, the exit time density is given by the following formula.

THEOREM 3.2. *For fixed* $r_0 > 0$, $0 < x < r_0$ *and any* $t > 0$

$$\frac{\mathbb{P}^x(\tau_{r_0} \in dt)}{dt} = \frac{\pi \sinh(\mu r_0)}{\sinh(\mu x) r_0^2} e^{-\mu^2 t/2} \sum_{n=1}^{\infty} (-1)^{n+1} n \sin(n\pi x/r_0) e^{-n^2\pi^2 t/(2r_0^2)}. \tag{3.2}$$

In order to obtain another representation of the exit time density, we will use the Poisson summation formula (cf. (13.4) in [11]): for any function g absolutely integrable on $(-\infty,\ \infty)$

$$\sum_{n=-\infty}^{\infty} g(n) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) e^{-2\pi i k x} dx. \tag{3.3}$$

Note that the series in formula (3.2) can be written in the following form:

$$\sum_{n=1}^{\infty} (-1)^{n+1} n \sin\left(\frac{n\pi x}{r_0}\right) e^{-n^2\pi^2 t/(2r_0^2)} = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^{n+1} n \sin\left(\frac{n\pi x}{r_0}\right) e^{-n^2\pi^2 t/(2r_0^2)}.$$

In order to use (3.3), we will compute the Fourier transform of its n-th term, taking z in place of n. First we take only the cosine of $\exp(-2\pi i k z)$, next we integrate by parts and finally we use formula 3.896.4 in [4] to get

$$\int_{-\infty}^{\infty} z \sin\left(\frac{z\pi y}{r_0}\right) \exp\left(\frac{-z^2\pi^2 t}{2r_0^2}\right) \exp(-2\pi i k z) dz = \frac{r_0^2}{\sqrt{2}(\pi t)^{3/2}} [(y + 2kr_0) e^{-(y+2kr_0)^2/(2t)} + (y - 2kr_0) e^{-(y-2kr_0)^2/(2t)}]. \tag{3.4}$$

Observe that

$$\sin\left(\frac{n\pi(r_0 - x)}{r_0}\right) = (-1)^{n+1} \sin(n\pi x/r_0);$$

hence putting $y = r_0 - x$ in (3.4) and using the Poisson formula, we get the second representation of the exit time density:

THEOREM 3.3. *For fixed* $r_0 > 0$, $0 < x < r_0$ *and any* $t > 0$

$$\frac{\mathbb{P}^x(\tau_{r_0} \in dt)}{dt} = \frac{\sinh(\mu r_0) e^{-\mu^2 t/2}}{\sinh(\mu x)\sqrt{2\pi} t^{3/2}} \sum_{k=-\infty}^{\infty} (r_0 - x + 2kr_0) e^{-(r_0 - x + 2kr_0)^2/(2t)}.$$

4. MEAN EXIT TIME

Now we want to compute $\mathbb{E}^x(\tau_{r_0})$, the mean exit time of (Z_t) from the interval $(0, r_0)$. We will use the formula $\mathbb{E}^x(\tau_{r_0}) = \int_0^\infty t\, \mathbb{P}^x(\tau_{r_0} \in dt)$, hence we need to compute the integral

$$\frac{\sinh(\mu r_0)}{\sqrt{2\pi}\sinh(\mu x)} \int\limits_0^\infty \frac{e^{-\mu^2 t/2}}{\sqrt{t}} \sum_{k=-\infty}^{\infty} (r_0 - x + 2kr_0) e^{-(r_0 - x + 2kr_0)^2/(2t)} dt.$$

Integrating a single term we use formula (3.471(15)) from [4] and get

$$\int\limits_0^\infty \frac{1}{\sqrt{t}} \exp\left(\frac{-(r_0 + 2kr_0 - x)^2}{2t} - \frac{\mu^2 t}{2}\right) dt = \frac{\sqrt{2\pi}}{\mu} \exp(-\mu|r_0 + 2kr_0 - x|)dt,$$

which gives

$$\text{(4.1)}\quad \mathbb{E}^x(\tau_{r_0}) = \frac{\sinh(\mu r_0)}{\mu \sinh(\mu x)} \sum_{k=-\infty}^{\infty} [(r_0 - x + 2kr_0) \exp(-\mu|r_0 - x + 2kr_0|)]$$

$$= \frac{\sinh(\mu r_0)}{\mu \sinh(\mu x)} \sum_{k=1}^{\infty} \big[(r_0 - x + 2kr_0) \exp\big(\mu(-r_0 + x - 2kr_0)\big)\big] + \frac{\sinh(\mu r_0)}{\mu \sinh(\mu x)}$$

$$\times \Big[\sum_{k=1}^{\infty} \big[(r_0 - x - 2kr_0) \exp\big(\mu(r_0 - x - 2kr_0)\big)\big] + (r_0 - x) \exp\big(-\mu(r_0 - x)\big)\Big].$$

But

$$\sum_{k=1}^{\infty} \exp(-2\mu k r_0) = 1/(e^{2\mu r_0} - 1)$$

and

$$\sum_{k=1}^{\infty} k \exp(-2\mu k r_0) = e^{2\mu r_0}/(e^{2\mu r_0} - 1)^2.$$

If we put them into (4.1), after some algebraic manipulation the formula for $\mathbb{E}^x(\tau_B)$ can be simplified a lot. Namely, we get the following:

THEOREM 4.1. *For any fixed* $r_0 > 0$ *and any starting point* $x \in (0, r_0)$

$$\mathbb{E}^x(\tau_{r_0}) = \frac{1}{\mu}\big(r_0 \coth(\mu r_0) - x \coth(\mu x)\big).$$

5. ESTIMATES

Except for the last one, all the above formulas are given as series so that they are not convenient for computations or applications. In this section we give exact approximations by elementary functions of the transition density of the killed process, of the killing time and of the density of the distribution of the supremum of the process. The notation $f \approx g$ means that there exist two absolute constants c_1 and c_2 such that for all possible values of variables and parameters it follows that $c_1 f < g < c_2 f$.

For simplicity let us put $\gamma^{r_0}(t;x)dt = \mathbb{P}^x(\tau_{r_0} \in dt)$. Recall that by Theorems 3.2 and 3.3 we have

$$(5.1)\quad \gamma^{r_0}(t,x) = \frac{\pi \sinh(\mu r_0) e^{-\mu^2 t/2}}{\sinh(\mu x) r_0^2} \sum_{n=1}^{\infty} (-1)^{n+1} n \sin(n\pi x/r_0) e^{-n^2\pi^2 t/(2r_0^2)},$$

$$(5.2)\quad \gamma^{r_0}(t,x) = \frac{\sinh(\mu r_0) e^{-\mu^2 t/2}}{\sinh(\mu x)\sqrt{2\pi} t^{3/2}} \sum_{n=-\infty}^{\infty} (r_0 - x + 2nr_0) e^{-(r_0 - x + 2nr_0)^2/(2t)}$$

and, by Theorem 2.2,

$$(5.3)\quad p^{r_0}(t;x,y) = \\ = \frac{e^{-\mu^2 t/2}}{\sinh(\mu x)\sinh(\mu y)\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} [e^{-(y-x+2nr_0)^2/(2t)} - e^{-(y+x+2nr_0)^2/(2t)}].$$

We will start with estimates in the particular case $r_0 = 1$. We also make separate calculations for $t \in \left(0, \frac{1}{4}\right]$ and for $t \in \left[\frac{1}{4}, \infty\right)$.

THEOREM 5.1. *Let* $r_0 = 1$. *For* $0 < t \leqslant \frac{1}{4}$ *and* $0 < x < 1$

$$0.25 \leqslant \gamma^1(t,x)\left(\frac{1}{\sqrt{2\pi} t^{3/2}} \frac{x(1-x)}{t+x} \frac{\sinh(\mu)}{\sinh(\mu x)} e^{-\mu^2 t/2 - (1-x)^2/(2t)}\right)^{-1} \leqslant 4.02.$$

Proof. First we consider the case $0 < x \leqslant \frac{1}{2}$ and use the formula (5.2). It is enough to estimate only the part of $\gamma^1(t,x)$ consisting of a series, multiplied by $e^{(1-x)^2/(2t)}/x$, that is, the following quantity:

$$I = \frac{e^{(1-x)^2/(2t)}}{x} \sum_{k=-\infty}^{\infty} (2k+1-x) e^{-(2k+1-x)^2/(2t)}.$$

Observe that we can group terms of the series: $k = 0$ with $k = -1$, $k = 1$ with $k = -2$, and so on. In this way we get

$$(5.4)\qquad I = \sum_{k=0}^{\infty} e^{-2k(k+1-x)/t} \frac{(2k+1-x) - (2k+1+x)e^{-2(2k+1)x/t}}{x}.$$

Now

$$(t+x)I = \sum_{k=0}^{\infty} (t+x) e^{-2k(k+1-x)/t} \frac{(2k+1-x) - (2k+1+x)e^{-2(2k+1)x/t}}{x}.$$

Below we will estimate separately the term for $k=0$ and terms for $k\geqslant 1$. The term for $k=0$ is equal to $g(x,t)=1-x-t+\frac{t}{x}(1-e^{-2x/t})-(1+x+t)e^{-2x/t}$. We will show that for $0<x\leqslant 1/2$ and $0<t<1/4$ it follows that $g(x,t)\leqslant 2$. Indeed, a derivative

$$\frac{\partial g(x,t)}{\partial x}=\frac{e^{-2x/t}}{tx^2}\Big(2x^2(1+x)-tx\big(x(e^{2x/t}-1)-2\big)-t^2(e^{2x/t}-1)\Big)$$

is negative for $x>0$ and $t>0$ because $e^{2x/t}-1>2x/t+(2x)^2/(2t^2)$ and this inequality implies that, in the formula for the derivative, the quantity in the brackets is negative. Hence $g(x,t)\leqslant \lim_{x\to 0} g(x,t)=2-2t$, and this is less than 2 for $t>0$.

In order to estimate terms for $k\geqslant 1$, we use the assumptions $0<t\leqslant \frac{1}{4}$, $0<x\leqslant \frac{1}{2}$ and the inequality $1-e^{-x}<x$, valid for $x>0$, and get

$$\begin{aligned}(t+x)I &\leqslant 2+\sum_{k=1}^{\infty} e^{-2k^2/t}\left(2k+1+\frac{t}{x}(2k+1)(1-e^{-2(2k+1)x/t})\right)\\ &\leqslant 2+\sum_{k=1}^{\infty} e^{-8k^2}(2k+1)(4k+3)\leqslant 2+21e^{-8}+\frac{14}{8}\sum_{k=2}^{\infty} 8k^2e^{-8k^2}\\ &\leqslant 2+21e^{-8}+\frac{14}{8}\sum_{n=32}^{\infty} ne^{-n}\leqslant 2.01.\end{aligned}$$

To get an estimate from below, we will use the inequality $1-e^{-x}\geqslant x/(x+1)$, valid for $x>-1$. Using formula (5.4), we obtain, for $0<t\leqslant\frac{1}{4}$ and $0<x\leqslant\frac{1}{2}$,

$$\begin{aligned}I &= \sum_{k=0}^{\infty} e^{-2k(k+1-x)/t}\,\frac{-2x+(2k+1+x)(1-e^{-2(2k+1)x/t})}{x}\\ &\geqslant \sum_{k=0}^{\infty} e^{-2k(k+1-x)/t}\,\frac{2(2k+1)(2k+1-x)-2t}{t+2(2k+1)x}\\ &\geqslant \sum_{k=0}^{\infty} e^{-2k(k+1-x)/t}\,\frac{2k+1-x-t}{t+x}\\ &\geqslant \frac{1}{t+x}\sum_{k=0}^{\infty} e^{-2k(k+1-x)/t}\left(2k+\frac{1}{4}\right)\geqslant \frac{1}{4(t+x)},\end{aligned}$$

so that $(t+x)I\geqslant\frac{1}{4}$. Both the above estimates imply the following:

$$0.25\leqslant \frac{t+x}{x}e^{(1-x)^2/(2t)}\sum_{k=-\infty}^{\infty}(2k+1-x)e^{-(2k+1-x)^2/(2t)}\leqslant 2.01.$$

If, for $0<x\leqslant\frac{1}{2}$, we divide the middle term of the above inequality by $(1-x)\in\left[\frac{1}{2},1\right)$, we must multiply its left- and right-hand sides by, respectively, 1 and 2. This proves the theorem in this case.

Now we prove the theorem in the case $\frac{1}{2} \leqslant x < 1$ and $0 < t \leqslant \frac{1}{4}$. As in the proof of the first case, we will examine the following quantity:

$$\frac{e^{(1-x)^2/(2t)}}{1-x} \sum_{k=-\infty}^{\infty} (2k+1-x)e^{-(2k+1-x)^2/(2t)} = \sum_{k=-\infty}^{\infty} e^{-(2k^2+2k(1-x))/t}$$
$$+ \frac{1}{1-x} \sum_{k=1}^{\infty} 2k(e^{-(2k^2+2k(1-x))/t} - e^{-(2k^2-2k(1-x))/t}) = A + \frac{1}{1-x} B.$$

First we estimate the series denoted by A: for $0 < 1-x \leqslant \frac{1}{2}$ and $0 < t \leqslant \frac{1}{4}$ we get

$$\begin{aligned} A - 1 &= \sum_{k=1}^{\infty} (e^{-(2k^2+2k(1-x))/t} + e^{-(2k^2-2k(1-x))/t}) \\ &\leqslant \sum_{k=1}^{\infty} (e^{-2k^2/t} + e^{-(2k^2-k)/t}) \\ &\leqslant \sum_{k=1}^{\infty} (e^{-8k^2} + e^{-4k^2}) \leqslant \frac{1}{e^8} + \frac{1}{e^4} + \sum_{k=16}^{\infty} 2e^{-k} < 0.02, \end{aligned}$$

and hence $1 < A < 1.02$.

Now we have to estimate $\frac{1}{1-x}B$. Observe that B is negative for $\frac{1}{2} < x < 1$ because all terms of the series are negative. We will estimate the following positive quantity:

$$\begin{aligned} \frac{-B}{1-x} &= \sum_{k=1}^{\infty} 2k \left(\frac{e^{-(2k^2-2k(1-x))/t} - e^{-(2k^2+2k(1-x))/t}}{1-x} \right) \\ &= \sum_{k=1}^{\infty} 2k e^{-2k^2/t} \left(\frac{e^{2k(1-x)/t} - e^{-2k(1-x)/t}}{1-x} \right) \\ &= 4 \sum_{k=1}^{\infty} \frac{2k^2}{t} e^{-2k^2/t} \frac{\sinh\big(2k(1-x)/t\big)}{2k(1-x)/t}. \end{aligned}$$

For fixed $t > 0$ and $k = 1, 2, 3, \ldots$ the function

$$g(x) = \frac{\sinh\big(2k(1-x)/t\big)}{2k(1-x)/t}$$

is decreasing for $\frac{1}{2} \leqslant x < 1$, hence its maximal value is attained for $x = \frac{1}{2}$ and is equal to

$$\frac{\sinh(k/t)}{k/t} = \frac{e^{k/t} - e^{-k/t}}{2k/t}.$$

Thus

$$0 < \frac{-B}{1-x} = 4\sum_{k=1}^{\infty} \frac{2k^2}{t} e^{-2k^2/t} \frac{e^{k/t} - e^{-k/t}}{2k/t} \leqslant 8 \sum_{k=1}^{\infty} k e^{-(2k^2-k)/t}$$
$$\leqslant 8\sum_{k=1}^{\infty} k e^{-k^2/t} \leqslant 2\sum_{k=1}^{\infty} 4k^2 e^{-4k^2} \leqslant 2\Big(\frac{4}{e^4} + \sum_{n=16}^{\infty} n e^{-n}\Big) \leqslant 0.15.$$

Finally, using the estimates $-0.15 < B/(1-x) < 0$ and $1 < A < 1.02$, we get the desired result: for $0 < t \leqslant \frac{1}{2}$ and $\frac{1}{2} \leqslant x < 1$

$$0.85 < A + \frac{1}{1-x} B < 1.02,$$

which, for $\frac{1}{2} \leqslant x < 1$, implies the following:

$$0.85 \leqslant \gamma^1(t,x)\bigg(\frac{1}{\sqrt{2\pi} t^{3/2}}(1-x)\frac{\sinh(\mu)}{\sinh(\mu x)} e^{-\mu^2 t/2 - (1-x)^2/(2t)}\bigg)^{-1} \leqslant 1.02.$$

If we want to write the factor $x(1-x)/(t+x)$ instead of $(1-x)$ in the denominator, we need to multiply the right-hand side constant by 3/2 because for $0 < t \leqslant \frac{1}{4}$ and $\frac{1}{2} \leqslant x < 1$ we have the relation $\frac{2}{3} \leqslant x/(t+x) < 1$. This gives the estimate for $t \geqslant \frac{1}{4}$ with constants 0.85 and 1.53. Finally, taking into account estimates for $0 < x \leqslant \frac{1}{2}$ and for $\frac{1}{2} \leqslant x < 1$ we get the desired estimate, which completes the proof of the theorem. ∎

Now we will give the estimate for $t \geqslant \frac{1}{4}$.

THEOREM 5.2. *For* $t \geqslant \frac{1}{4}$ *and* $0 < x < 1$

$$0.8 \leqslant \gamma^1(t,x)\bigg(\pi \sin(\pi x)\frac{\sinh(\mu)}{\sinh(\mu x)} e^{-(\mu^2+\pi^2)t/2}\bigg)^{-1} \leqslant 1.2.$$

P r o o f. For $t \geqslant \frac{1}{4}$ we use (5.1) and the inequality $|\sin(k\pi x)| \leqslant k\sin(\pi x)$, valid for $0 < x < 1$. The first term of the series

$$\sum_{k=1}^{\infty} (-1)^{k+1} k \sin(k\pi x) e^{-k^2\pi^2 t/2}$$

is much larger than the sum of the absolute values of all the rest:

$$\Big|\sum_{k=2}^{\infty} (-1)^{k+1} k \sin(k\pi x) e^{-k^2\pi^2 t/2}\Big| \leqslant \sin(\pi x) e^{-\pi^2 t/2} \sum_{k=2}^{\infty} k^2 e^{-(k^2-1)\pi^2 t/2}$$
$$\leqslant \sin(\pi x) e^{-\pi^2 t/2} \sum_{n=4}^{\infty} n e^{-(n-1)\pi^2/8} \leqslant \frac{e^{-\pi^2/4}(4e^{\pi^2/8} - 3)}{(e^{\pi^2/8} - 1)^2} \sin(\pi x) e^{-\pi^2 t/2}$$
$$\leqslant 0.2 \sin(\pi x) e^{-\pi^2 t/2}.$$

This means that

$$0.8\sin(\pi x)e^{-\pi^2 t/2} \leqslant \sum_{k=1}^{\infty}(-1)^{k+1}k\sin(k\pi x)e^{-k^2\pi^2 t/2} \leqslant 1.2\sin(\pi x)e^{-\pi^2 t/2},$$

and hence for $t \geqslant 1/4$ and $0 < x < 1$

$$\begin{aligned}0.8\pi\sin(\pi x)\frac{\sinh(\mu)}{\sinh(\mu x)}e^{-\mu^2 t/2-\pi^2 t/2} &\leqslant \gamma^1(t,x)\\ &\leqslant 1.2\pi\sin(\pi x)\frac{\sinh(\mu)}{\sinh(\mu x)}e^{-\mu^2 t/2-\pi^2 t/2},\end{aligned}$$

which completes the proof. ■

Now, because for all $x \in (0,1)$ we have $\pi^2 < \pi\sin(\pi x)/\big(x(1-x)\big) < 4\pi$, the above inequality implies the following:

$$0.8\pi^2 \leqslant \gamma^1(t,x)\left(x(1-x)\frac{\sinh(\mu)}{\sinh(\mu x)}e^{-\mu^2 t/2-\pi^2 t/2}\right)^{-1} \leqslant 4.8\pi. \tag{5.5}$$

Observe that the above inequality differs from that given in Theorem 5.1 for the case $0 < x \leqslant \frac{1}{2}$: it does not contain the factor $1/(\sqrt{2\pi}t^{3/2})$ in the denominator and instead of the factor $e^{-(1-x)^2/(2t)}$ it has $e^{-\pi^2 t/2}$. If we want to have one estimate for all $t > 0$ and $0 < x < 1$, we must add such factors. Multiplying the estimating function from Theorem 5.1 by $e^{-\pi^2 t/2} \in (e^{-\pi^2/8}, 1)$, we must multiply the constant on the left-hand side by $e^{-\pi^2/8} \approx 0.29$. This operation changes the constant $\frac{1}{4}$ from Theorem 5.1 to $e^{-\pi^2/8}/4 \approx 0.0728 > 0.07$.

On the other hand, the estimating function from the denominator in (5.5) must be multiplied by $(1+t)^{5/2}e^{-(1-x)^2/(2t)}/\big(\sqrt{2\pi}(t+x)t^{3/2}\big)$. But for $t \geqslant \frac{1}{4}$ and $0 < x < 1$ the above function is greater than $1/\sqrt{2\pi} \approx 0.3989\ldots$ and less than $25\sqrt{5}e^{-1/8}/(4\sqrt{2\pi}) \approx 4.92\ldots < 5$. Hence the constant 4.8π from (5.5) must be changed to $5 \cdot 4.8\pi = 24\pi$, and the constant $0.8\pi^2$ from (5.5) must be changed to $0.8\pi^2/\sqrt{2\pi} > 3.1499 > \pi$.

In this way Theorems 5.1 and 5.2 together imply the following:

COROLLARY 5.1. *For all $t > 0$ and $0 < x < 1$ we have*

$$\begin{aligned}0.07 \leqslant \gamma^1(t,x)\left(\frac{x(1-x)}{t+x}\frac{\sinh(\mu)}{\sinh(\mu x)}\frac{(1+t)^{5/2}}{\sqrt{2\pi}t^{3/2}}\exp\left(-\frac{\mu^2+\pi^2}{2}t-\frac{(1-x)^2}{2t}\right)\right)^{-1}&\\ \leqslant 24\pi < 75.4.&\end{aligned}$$

It is easy to notice that by (5.2) the following scaling property holds:

$$\gamma^{r_0}(t,x) = \frac{\sinh(\mu x/r_0)}{\sinh(\mu)\sinh(\mu x)r_0^2}\exp\left(-\frac{\mu}{2}t\left(1-\frac{1}{r_0^2}\right)\right)\gamma^1\left(\frac{t}{r_0^2},\frac{x}{r_0}\right).$$

This together with the above corollary proves the following:

THEOREM 5.3. *For μ, r_0, $t > 0$ and $0 < x < r_0$ we have*

$$(5.6)\quad 0.07 \leqslant \gamma^{r_0}(t,x)\left(\frac{x(r_0-x)}{t+r_0x}\frac{\sinh(\mu r_0)}{\sinh(\mu x)}\frac{(r_0^2+t)^{5/2}}{\sqrt{2\pi}r_0^4t^{3/2}}\right)^{-1} \times\left(\exp\left(-\frac{(r_0\mu)^2+\pi^2}{2r_0^2}t-\frac{(r_0-x)^2}{2t}\right)\right)^{-1} \leqslant 75.4.$$

REMARK 5.1. *Many formulas in the book* [2] *are given in the language of a theta function of imaginary argument* (cf. [2], p. 641):

$$ss_y(v,t) = \frac{1}{\sqrt{2\pi}y^{3/2}}\sum_{k=-\infty}^{\infty}(t-v+2kt)e^{-(t-v+2kt)^2/(2y)}, \qquad v \leqslant t.$$

Observe that $ss_t(x,r_0)$ is precisely the series in our $\gamma^{r_0}(t,x)$ and was estimated above. Using our method we can give an estimate of $ss_t(x,r_0)$ for all possible values of variables (like we did it in Theorem 5.3 for $\gamma^{r_0}(t,x)$) but estimates for different sets of t and x, given in Theorems 5.1 and 5.2, are much more exact. For instance, the proof of Theorem 5.1 gives the following: for $r_0=1$, $0<t\leqslant\frac14$ and $0<x\leqslant\frac12$ we have

$$0.25 \leqslant ss_t(x,1)\left(\frac{x}{\sqrt{2\pi t^3}(t+x)}e^{-(1-x)^2/(2t)}\right)^{-1} \leqslant 2.01.$$

Now we estimate the density of the transition probability of the killed process.

THEOREM 5.4. *For fixed $r_0>0$, all $x,y\in(0,\,r_0)$ and $t>0$*

$$p^{r_0}(t;x,y) \approx \frac{(r_0^2+t)^{5/2}}{r_0^5\sinh(\mu x)\sinh(\mu y)\sqrt{t}}\left(1\wedge\frac{xy}{t}\right)\left(1\wedge\frac{(r_0-x)(r_0-y)}{t}\right) \times\exp\left(-\frac{(r_0\mu)^2+\pi^2}{2r_0^2}t-\frac{(x-y)^2}{2t}\right).$$

In the above estimate we can take the constants $c_1 = 0.0029$ and $c_2 = 2413$.

P r o o f. Observe that

$$(5.7)\quad p^{r_0}(t;x,y) = \frac{\sinh(\mu x/r_0)\sinh(\mu y/r_0)}{\sinh(\mu x)\sinh(\mu y)r_0}\exp\left(-\frac{\mu^2t}{2}\left(1-\frac{1}{r_0^2}\right)\right)p^1\left(\frac{t}{r_0^2};\frac{x}{r_0},\frac{y}{r_0}\right).$$

Thus it is sufficient to consider only the case when $r_0=1$. Define the following function:

$$\lambda(w) = \frac{1}{\sqrt{2\pi}t^{3/2}}\sum_{k=-\infty}^{\infty}(w+2k)\exp\left(-\frac{(w+2k)^2}{2t}\right), \qquad w\in\mathbb{R}.$$

Note that the function $ss_t(0, w)$ mentioned in Remark 5.1 is given by the same series, but the range of its argument w is different. By the formula (5.3) we have

$$p^1(t; x, y) = \frac{e^{-\mu^2 t/2}}{\sinh(\mu x)\sinh(\mu y)} \int\limits_{x-y}^{x+y} \lambda(w)dw.$$

From the definition of $\lambda(w)$ it follows that

$$\lambda(-w) = -\lambda(w), \qquad \lambda(1 + w) = -\lambda(1 - w),$$

which implies

$$p^1(t; x, y) = \frac{e^{-\mu^2 t/2}}{\sinh(\mu x)\sinh(\mu y)} \int\limits_{|x-y|}^{1-|1-x-y|} \lambda(w)dw.$$

Observe that $|x - y| < 1 - |1 - x - y|$ for $x, y \in (0, 1)$. Because, by (5.2),

$$\gamma^1(t, x) = \frac{\sinh(\mu)e^{-\mu^2 t/2}}{\sinh(\mu x)}\lambda(1 - x),$$

we get, by virtue of Corollary 5.1,

$$\lambda(1 - x) \approx \frac{x(1 - x)}{t + x}\frac{(1 + t)^{5/2}}{t^{3/2}} \exp\left(-\frac{\pi^2}{2}t - \frac{(1 - x)^2}{2t}\right). \tag{5.8}$$

Consequently,

$$p^1(t; x, y) \approx \frac{e^{-(\mu^2+\pi^2)t/2}(1 + t)^{5/2}}{\sinh(\mu x)\sinh(\mu y)t^{3/2}} \int\limits_{|x-y|}^{1-|1-x-y|} \frac{w(1 - w)}{t + (1 - w)}e^{-w^2/(2t)}dw.$$

Now, substituting $w = \sqrt{1 - vt}$, we get

$$1 - w = 1 - \sqrt{1 - vt} = \frac{vt}{1 + \sqrt{1 - vt}} \quad \text{and} \quad dw = \frac{-t}{\sqrt{1 - vt}},$$

and hence

$$\frac{1}{2}tv \leqslant 1 - w \leqslant tv. \tag{5.9}$$

Combining this with Lemma 6.1 from the Appendix, we obtain

$$\begin{aligned} p^1(t; x, y) &\approx \frac{e^{-(\mu^2+\pi^2)t/2}(1 + t)^{5/2}}{\sinh(\mu x)\sinh(\mu y)t^{1/2}}e^{-1/(2t)} \int\limits_{(1-(1-|1-x-y|)^2)/t}^{(1-(x-y)^2)/t} \frac{v}{1 + v}e^{v/2}dv \\ &\approx \frac{e^{-t(\mu^2+\pi^2)/2}(1 + t)^{5/2}}{\sinh(\mu x)\sinh(\mu y)t^{1/2}}e^{-(x-y)^2/(2t)} \\ &\quad \times \left(1 \wedge \frac{1 - (x - y)^2}{t}\right)\left(1 \wedge \frac{(1 - |1 - x - y|)^2 - (x - y)^2}{t}\right). \end{aligned}$$

We rewrite the expression in the last parentheses as follows:

$$(1-|1-x-y|)^2-(x-y)^2 = \begin{cases} 4xy, & y+x<1, \\ 4(1-x)(1-y), & y+x \geqslant 1, \end{cases}$$
$$= 4\big[xy \wedge \big((1-x)(1-y)\big)\big].$$

Moreover, it is easy to check that $1 \leqslant \big(1-(x-y)^2\big)/(1-x)(1-y) \leqslant 4$ for $0 \leqslant x \leqslant 1-y \leqslant 1$. Since $f(x,y) = 1-(x-y)^2 = f(1-x,1-y)$, we get for all $x,\ y \in (0,\ 1)$

$$1 \leqslant \frac{1-(x-y)^2}{(xy) \vee \big((1-x)(1-y)\big)} \leqslant 4.$$

Thus for all $x,\ y \in (0,\ 1)$ we obtain

$$\begin{aligned} &\left(1 \wedge \frac{1-(x-y)^2}{t}\right)\left(1 \wedge \frac{(1-|1-x-y|)^2-(x-y)^2}{t}\right) \\ &\leqslant 16\left(1 \wedge \frac{(xy) \vee \big((1-x)(1-y)\big)}{t}\right)\left(1 \wedge \frac{(xy) \wedge \big((1-x)(1-y)\big)}{t}\right) \\ &= 16\left(1 \wedge \frac{xy}{t}\right)\left(1 \wedge \frac{(1-x)(1-y)}{t}\right) \end{aligned}$$

and

$$\begin{aligned} &\left(1 \wedge \frac{1-(x-y)^2}{t}\right)\left(1 \wedge \frac{(1-|1-x-y|)^2-(x-y)^2}{t}\right) \\ &\qquad\qquad \geqslant \left(1 \wedge \frac{xy}{t}\right)\left(1 \wedge \frac{(1-x)(1-y)}{t}\right). \end{aligned}$$

Theorem 5.3 implies that in the estimate (5.8) of $\lambda(x)$ we have the constants $c_1 = 0.07$ and $c_2 = 75.4$. Using this, the above inequalities, constants from Lemma 6.1 in Appendix and inequality (5.9), we get the constants in the assertion. ■

Recall that $M_t = \sup_{0<s\leqslant t} Z_s$ and put $m(t,x,y)dy = \mathbb{P}^x(M_t \in dy)$. An estimate of this density is the most complicated. In the proof we will use three elementary lemmas, which will be proved in the Appendix.

THEOREM 5.5. *For all* $0 < x < y$ *and* $t > 0$ *we have*

$$\begin{aligned} m(t;x,y) \approx \frac{x(y-x)}{y^2 t}\frac{\sinh(\mu y)}{\sinh(\mu x)}\frac{(y^2+t)^{5/2}\left(1+\frac{t^{3/2}}{y^3}+\frac{\sqrt{t}}{y-x}+\sqrt{t}\mu\right)}{\big(y^2+t\big((y\mu)^2+1\big)\big)(t+yx)} \\ \times \frac{\exp\left(-(y-x)^2/(2t)-\frac{(y\mu)^2+\pi^2}{2y^2}t\right)}{\sqrt{1+\frac{(y-x)^2y^2}{t(y^2+t((y\mu)^2+1))}}}. \end{aligned}$$

P r o o f. By the strong Markov property we have for $y' > y > x$

$$\begin{aligned}\mathbb{P}^x\big(M_t \in (y,y')\big) &= \mathbb{P}^x(\tau_y < t, \tau_{y'} > t)\\ &= \mathbb{E}^x[\tau_y < t; \mathbb{E}^y(\tau_{y'} > t-\tau_y)] = \int_0^t \gamma^y(u,x) \int_{t-u}^\infty \gamma^{y'}(v,y)\,dv\,du.\end{aligned}$$

By (5.6) we obtain

$$\begin{aligned}\mathbb{P}^x\big(M_t \in (y,y')\big) &\approx \frac{x(y-x)y(y'-y)}{(y'y)^4}\frac{\sinh(\mu y')}{\sinh(\mu x)}\\ &\times \int_0^t \frac{(y^2+u)^{5/2}}{(u+yx)u^{3/2}}\exp\left(-\frac{(y\mu)^2+\pi^2}{2y^2}u - \frac{(y-x)^2}{2u}\right)\\ &\times \int_{t-u}^\infty \frac{(y'^2+v)^{5/2}}{(v+y'y)v^{3/2}}\exp\left(-\frac{(y'\mu)^2+\pi^2}{2y'^2}v - \frac{(y'-y)^2}{2v}\right)dv\,du\\ &:= F(t,x;y,y').\end{aligned}$$

Since $m(t;x,y)$ is continuous (it is obtained by differentiating formula (3.1) with respect to r_0), we have

$$m(t;x,y) = \lim_{y'\to y}\frac{\mathbb{P}^x\big(M_t\in(y,y')\big)}{y'-y}.$$

Hence

$$\begin{aligned}m(t;x,y) &\approx \lim_{y'\to y}\frac{F(t,x;y,y')}{y'-y}\\ &= \frac{x(y-x)}{y^7}\frac{\sinh(\mu y)}{\sinh(\mu x)}\int_0^t \frac{(y^2+u)^{5/2}}{(u+yx)u^{3/2}}\exp\left(-\frac{(y\mu)^2+\pi^2}{2y^2}u - \frac{(y-x)^2}{2u}\right)\\ &\times \int_{t-u}^\infty \frac{(y^2+v)^{3/2}}{v^{3/2}}\exp\left(-\frac{(y\mu)^2+\pi^2}{2y^2}v\right)dv\,du\\ &= \frac{x(y-x)}{y^7}\frac{\sinh(\mu y)}{\sinh(\mu x)}\exp\left(-\frac{(y\mu)^2+\pi^2}{2y^2}t\right)\int_0^t \frac{(y^2+u)^{5/2}}{(u+yx)u^{3/2}}\\ &\times \exp\left(-\frac{(y-x)^2}{2u}\right)\int_0^\infty \frac{(y^2+t-u+v)^{3/2}}{(t-u+v)^{3/2}}\exp\left(-\frac{(y\mu)^2+\pi^2}{2y^2}v\right)dv\,du.\end{aligned}$$

Now we apply Lemma 6.2 from the Appendix with $a = y^2$, $b = t-u$ and $c = \big((y\mu)^2+\pi^2\big)/(2y^2)$ to the inner integral and get

$$\begin{aligned}m(t;x,y) &\approx \frac{x(y-x)}{y^5}\frac{\sinh(\mu y)}{\sinh(\mu x)}\exp\left(-\frac{(y\mu)^2+\pi^2}{2y^2}t\right)\\ &\times \int_0^t \frac{(y^2+u)^{5/2}(y^2+t-u)^{3/2}e^{-(y-x)^2/(2u)}}{\sqrt{t-u}(u+yx)u^{3/2}\big(y^2+(t-u)\big((y\mu)^2+1\big)\big)}du.\end{aligned}$$

We split the above-given integral into two parts: $\int_0^{t/2} + \int_{t/2}^{t} = I_1 + I_2$, and substitute

$$w = \frac{(y-x)^2}{2}\left(\frac{1}{u} - \frac{2}{t}\right), \qquad w = \frac{(y-x)^2}{2}\left(\frac{1}{u} - \frac{1}{t}\right)$$

in I_1 and I_2, respectively. Before the substitution we use the following estimates: $t - u \approx t$ for $u \in \left(0, \frac{t}{2}\right)$ and $u \approx t$ for $u \in \left(\frac{t}{2}, t\right)$. Additionally, the second substitution gives $t - u = 2t^2 w / \big(2tw + (y-x)^2\big)$. Consequently,

$$\begin{aligned}
I_1 &\approx \frac{(y^2+t)^{3/2}}{\sqrt{t}\big(y^2 + t\big((y\mu)^2+1\big)\big)} \int\limits_0^{t/2} \frac{(y^2+u)^{5/2} e^{-(y-x)^2/(2u)}}{(u+yx)u^{3/2}} du \\
&= \frac{\sqrt{2}y^4(y^2+t)^{3/2} e^{-(y-x)^2/t}}{\sqrt{t}x(y-x)\big(y^2 + t\big((y\mu)^2+1\big)\big)} \\
&\quad \times \int\limits_0^{\infty} \frac{\left(w + (y-x)^2\left(\frac{1}{t} + \frac{1}{2y^2}\right)\right)^{5/2} e^{-w}}{\left(w + \frac{(y-x)^2}{t}\right)^2 \left(w + (y-x)^2\left(\frac{1}{t} + \frac{1}{2xy}\right)\right)} dw, \\
I_2 &\approx \frac{(y^2+t)^{5/2}}{(t+yx)t^{3/2}} \int\limits_{t/2}^{t} \frac{(y^2+t-u)^{3/2} e^{-(y-x)^2/(2u)}}{\sqrt{t-u}\big(y^2 + (t-u)\big((y\mu)^2+1\big)\big)} du \\
&= \frac{2(y^2+t)^4 e^{-(y-x)^2/(2t)}}{(y-x)^2(t+yx)\big(y^2+t(y\mu)^2+1\big)} \\
&\quad \times \int\limits_0^{(y-x)^2/(2t)} \frac{\left(w + \frac{(y-x)^2y^2}{2t(y^2+t)}\right)^{3/2} e^{-w}}{\sqrt{w}\left(w + \frac{(y-x)^2y^2}{2t(y^2+t((y\mu)^2+1))}\right)} dw.
\end{aligned}$$

Now we apply Lemma 6.3 to the integral in the estimate of I_1 with

$$a = (y-x)^2\left(\frac{1}{t} + \frac{1}{2y^2}\right), \quad b = \frac{(y-x)^2}{t}, \quad c = (y-x)^2\left(\frac{1}{t} + \frac{1}{2yx}\right)$$

and Lemma 6.4 to the integral in the estimate of I_2 with

$$a = \frac{(y-x)^2}{2t}, \quad b = \frac{(y-x)^2y^2}{2t(y^2+t)}, \quad c = \frac{(y-x)^2y^2}{2t\big(y^2+t\big((y\mu)^2+1\big)\big)}.$$

For $(y-x)^2/(2t) \geqslant 1$ we have $y^2 \geqslant t$, which implies

$$\frac{(y-x)^2y^2}{2t(y^2+t)} = \frac{(y-x)^2}{t}\frac{1}{2(1+t/y^2)} \approx \frac{(y-x)^2}{t},$$

so all the assumptions of Lemma 6.4 are satisfied. Hence

$$I_1 + I_2 \approx$$

$$\approx \frac{(y^2+t)^{3/2} y^4 e^{-(y-x)^2/t}}{\sqrt{t}\big(y^2+t\big((y\mu)^2+1\big)\big)x(y-x)} \left(\frac{t(y-x)\left(\frac{y^2+t}{ty^2}\right)^{5/2}}{\left(1+\frac{(y-x)^2}{t}\right)\frac{yx+t}{yxt}} + \frac{1}{1+(y-x)^2\frac{yx+t}{yxt}} \right)$$

$$+ \frac{(y^2+t)^4 e^{-(y-x)^2/(2t)}}{(y-x)^2(t+yx)\big(y^2+t\big((y\mu)^2+1\big)\big)}$$

$$\times \left(1 \wedge \frac{(y-x)^2}{2t} + \frac{\sqrt{t\big(y^2+t\big((y\mu)^2+1\big)\big)}(y-x)^2y^2}{\big((y^2+t)t\big)^{3/2}\sqrt{1+\frac{(y-x)^2y^2}{t(y^2+t((y\mu)^2+1))}}} \right)$$

$$= \frac{(y^2+t)^{3/2} y^4 e^{-(y-x)^2/(2t)}}{\sqrt{t}\big(y^2+t\big((y\mu)^2+1\big)\big)(t+yx)}$$

$$\times \left(\frac{(y^2+t)^{5/2}e^{-(y-x)^2/(2t)}}{\sqrt{t}y^4\left(1+\frac{(y-x)^2}{t}\right)} + \frac{\frac{t+yx}{x(y-x)}e^{-(y-x)^2/(2t)}}{1+(y-x)^2\frac{yx+t}{yxt}} \right.$$

$$\left. + \left(1 \wedge \frac{(y-x)^2}{2t}\right) \frac{(y^2+t)^{5/2}\sqrt{t}}{(y-x)^2y^4} + \frac{(y^2+t)\sqrt{y^2+t\big((y\mu)^2+1\big)}}{\sqrt{t}y^2\sqrt{1+\frac{(y-x)^2y^2}{t(y^2+t((y\mu)^2+1))}}} \right).$$

Let us denote the expression in the last brackets by $\mathcal{J}(y,x,t,\mu)$. To prove the theorem we need to show that

$$\mathcal{J}(y,x,t,\mu) \approx \frac{y^2+t}{y\sqrt{t}} \frac{1+\frac{t^{3/2}}{y^3}+\frac{\sqrt{t}}{y-x}+\sqrt{t}\mu}{\sqrt{1+\frac{(y-x)^2y^2}{t(y^2+t((y\mu)^2+1))}}}. \tag{5.10}$$

Assume $(y-x)^2/t > 1$. Then we have $y^2 > t$, which implies

$$\frac{(y^2+t)^{5/2}e^{-(y-x)^2/(2t)}}{\sqrt{t}y^4\left(1+\frac{(y-x)^2}{t}\right)} < 2\frac{(y^2+t)^{3/2}}{\sqrt{t}y^2\sqrt{1+\frac{(y-x)^2}{t}}} < 2\frac{(y^2+t)\sqrt{y^2+t\big((y\mu)^2+1\big)}}{\sqrt{t}y^2\sqrt{1+\frac{(y-x)^2y^2}{t(y^2+t((y\mu)^2+1))}}},$$

$$\frac{\frac{t+yx}{x(y-x)}e^{-(y-x)^2/(2t)}}{1+(y-x)^2\left(\frac{yx+t}{yxt}\right)} < \frac{\frac{\sqrt{t}}{y-x}y}{\sqrt{t}\frac{(y-x)^2}{t}}\left(\frac{y^2+t}{y^2}\right) < 2\frac{(y^2+t)\sqrt{y^2+t\big((y\mu)^2+1\big)}}{\sqrt{t}y^2\sqrt{1+\frac{(y-x)^2y^2}{t(y^2+t((y\mu)^2+1))}}},$$

$$\left(1 \wedge \frac{(y-x)^2}{2t}\right)\frac{(y^2+t)^{5/2}\sqrt{t}}{(y-x)^2y^4} < 2\frac{(y^2+t)^{3/2}}{\sqrt{t}y^2\frac{y-x}{\sqrt{t}}} < 2\frac{(y^2+t)\sqrt{y^2+t\big((y\mu)^2+1\big)}}{\sqrt{t}y^2\sqrt{1+\frac{(y-x)^2y^2}{t(y^2+t((y\mu)^2+1))}}},$$

so we can estimate the expression $\mathcal{J}(y,x,t,\mu)$ from above by its last component. But for $x,y,z>0$ we have $\sqrt{x^2+y^2+z^2}\approx x+y+z$ because l_2- and l_1-norms are equivalent in $\mathbb{R}^3$. Hence

$$\frac{\sqrt{y^2+t\big((y\mu)^2+1\big)}}{y}\approx 1+\sqrt{t}\mu+\frac{\sqrt{t}}{y},$$

and because $(y-x)^2>t$, the relation (5.10) holds true.

In the case $(y-x)^2/t\leqslant 1$ we have $t+xy\approx t+y^2$ and also (recall that $0<x<y$)

$$1+(y-x)^2\frac{yx+t}{yxt}=\frac{y}{x}+\frac{x}{y}+\frac{(y-x)^2}{t}-1\approx\frac{y}{x}.$$

Hence

$$\begin{aligned}\mathcal{J}(y,x,t,\mu)&\approx\frac{(y^2+t)^{5/2}}{\sqrt{t}y^4}+\frac{t+y^2}{y(y-x)}\\&\quad+\frac{(y^2+t)^{5/2}}{\sqrt{t}y^4}+\frac{(y^2+t)\sqrt{y^2+t\big((y\mu)^2+1\big)}}{\sqrt{t}y^2}\\&=\frac{y^2+t}{y\sqrt{t}}\left(\frac{2(y^2+t)^{3/2}}{y^3}+\frac{\sqrt{t}}{y-x}+\frac{\sqrt{y^2+t\big((y\mu)^2+1\big)}}{y}\right)\\&\approx\frac{y^2+t}{y\sqrt{t}}\left(1+\frac{t^{3/2}}{y^3}+\frac{\sqrt{t}}{y-x}+\sqrt{t}\mu+\frac{\sqrt{t}}{y}\right),\end{aligned}$$

which again is equivalent to (5.10). ■

6. APPENDIX

Here we gathered four lemmas which were used in the estimates carried out in the previous section.

LEMMA 6.1. *For* $0<a<b$ *we have*

$$\frac{1}{12}\leqslant\frac{\int_a^b\frac{w}{1+w}e^{w/2}dw}{e^{b/2}(1\wedge b)\big(1\wedge(b-a)\big)}\leqslant 2.$$

P r o o f. Let us put $\mathcal{I}(a,b)=\int_a^b\frac{w}{1+w}e^{w/2}dw$ and let $a,b\leqslant 1$. For $w\in(0,\,1)$ the function $f(w)=e^{w/2}/(1+w)$ is decreasing, and hence for such w it follows that $\sqrt{e}/2\leqslant f(w)\leqslant 1$. Using these inequalities we get

$$\frac{\sqrt{e}}{2}\int_a^b w\,dw\leqslant\int_a^b\frac{w}{1+w}e^{w/2}\,dw\leqslant\int_a^b w\,dw=b^2-a^2=(b-a)(b+a).$$

Since $0 < a < b \leqslant 1$, we have $b < b + a < 2b$, and this implies

$$\frac{1}{2}e^{b/2}b(b-a) \leqslant \frac{\sqrt{e}}{2}b(b-a) \leqslant \mathcal{I}(a,b) \leqslant \int_a^b w\,dw \leqslant 2b(b-a) \leqslant 2e^{b/2}b(b-a).$$

Now let $b > 1$. We use the inequality $1 - e^{-x} \leqslant 1 \wedge x$, $x \geqslant 0$, and get

$$\mathcal{I}(a,b) \leqslant \int_a^b e^{w/2}dw = e^{b/2}(1 - e^{-(b-a)/2}) \leqslant e^{b/2}\big(1 \wedge (b-a)\big).$$

For $x \geqslant 0$ it follows that $1 - e^{-x} \geqslant x/(1+x) \geqslant \frac{1}{2}(1 \wedge x)$. Using this, the fact that the function $x \to x/(1+x)$ is increasing and $(a+b)/2 \geqslant 1/2$, we obtain

$$\begin{aligned}\mathcal{I}(a,b) &\geqslant \int_{(a+b)/2}^{b} \frac{w}{1+w}e^{w/2}dw \geqslant \frac{(a+b)/2}{1+(a+b)/2} \int_{(a+b)/2}^{b} e^{w/2}dw \\ &\geqslant \frac{\frac{1}{2}}{1+\frac{1}{2}}2e^{b/2}(1-e^{-(b-a)/4}) \geqslant \frac{1}{12}e^{b/2}\big(1 \wedge (b-a)\big),\end{aligned}$$

which completes the proof. ■

LEMMA 6.2. *For $a, b, c > 0$ and $ac > 1$ we have*

$$\int_0^\infty \frac{(a+b+v)^{3/2}}{(b+v)^{3/2}}e^{-cv}\,dv \approx \frac{(a+b)^{3/2}}{\sqrt{b}(1+bc)}. \tag{6.1}$$

Proof. Let us denote the integral in the assertion by $I(a,b,c)$. Substituting $v = s/c$ we obtain

$$I(a,b,c) = \frac{1}{c}\int_0^\infty \frac{\big((a+b)c+s\big)^{3/2}}{(bc+s)^{3/2}}e^{-s}\,ds. \tag{6.2}$$

For $bc \geqslant 1$ we get

$$I(a,b,c) = \frac{1}{c}\left(\frac{a+b}{b}\right)^{3/2}\int_0^\infty \frac{\big(1+s/\big((a+b)c\big)\big)^{3/2}}{\big(1+s/(bc)\big)^{3/2}}e^{-s}\,ds \approx \frac{1}{c}\left(\frac{a+b}{b}\right)^{3/2},$$

which is equivalent to (6.1). For $bc < 1$ we split the integral in (6.2) as follows: $\frac{1}{c}\int_0^1 + \frac{1}{c}\int_1^\infty = I_1 + I_2$. Then

$$I_2 \approx \sqrt{c}(a+b)^{3/2}.$$

In I_1 we substitute $s = bct$ and get

$$I_1 \approx b\int_0^{1/(bc)} \frac{\left(\frac{a+b}{b}\right)^{3/2} + t^{3/2}}{(1+t)^{3/2}}e^{-bct}\,dt \approx b\left(\frac{a+b}{b}\right)^{3/2} + \frac{1}{c},$$

and hence

$$I_1 + I_2 \approx \frac{1}{c}\left(bc\left(\frac{a+b}{b}\right)^{3/2} + 1 + (bc)^{3/2}\left(\frac{a+b}{b}\right)^{3/2}\right).$$

From the assumptions $bc < 1$ and $ac > 1$ it follows that

$$bc\left(\frac{a+b}{b}\right)^{3/2} > (bc)^{3/2}\left(\frac{a+b}{b}\right)^{3/2} > 1,$$

which completes the proof of the lemma. ∎

LEMMA 6.3. *Let $a, c > b > 0$ and $a < 2$ if $b < 1$. Then we have*

$$\int_0^\infty \frac{(w+a)^{5/2}}{(w+b)^2(w+c)} e^{-w}\,dw \approx \frac{a^{5/2}}{b(b+1)c} + \frac{1}{1+c}. \tag{6.3}$$

Proof. Denote the integral in the assertion by $F(a,b,c)$. It is clear that for $b \geqslant 1$ we have $F(a,b,c) \approx a^{5/2}/(b^2 c)$, which is equivalent to (6.3). For $b < 1$ we split the integral into two parts:

$$F(a,b,c) \approx a^{5/2}\int_0^\infty \frac{1}{(w+b)^2(w+c)} e^{-w}\,dw + \int_0^\infty \frac{w^{5/2}}{(w+b)^2(w+c)} e^{-w}\,dw.$$

The last integral can be approximated by $1/(1+c)$ as follows:

$$\begin{aligned}\frac{1}{1+c} \approx \int_0^\infty \frac{w^{3/2}}{(w+1)^2}\frac{w}{w+c} e^{-w}\,dw &\leqslant \int_0^\infty \frac{w^{5/2}}{(w+b)^2(w+c)} e^{-w}\,dw \\ &\leqslant \int_0^\infty \frac{1}{\sqrt{w}}\frac{w}{w+c} e^{-w}\,dw \approx \frac{1}{1+c}.\end{aligned}$$

Moreover,

$$\begin{aligned}&\int_0^\infty \frac{1}{(w+b)^2(w+c)} e^{-w}\,dw \\ &\quad = \int_0^1 + \int_1^\infty = \frac{1}{bc}\int_0^{1/b} \frac{e^{-bu}\,du}{(1+u)^2\left(1+\frac{b}{c}u\right)} + \int_0^\infty \frac{e^{-(w+1)}\,dw}{(w+b+1)^2(w+c+1)} \\ &\quad \approx \frac{1}{bc} + \frac{1}{1+c}.\end{aligned}$$

Hence for $b < 1$ (in this case $a < 2$ by assumption) we obtain

$$F(a,b,c) \approx a^{5/2}\left(\frac{1}{bc} + \frac{1}{1+c}\right) + \frac{1}{1+c} \approx \frac{a^{5/2}}{bc} + \frac{1}{1+c},$$

which is again equivalent to (6.3). ∎

LEMMA 6.4. *Let $a > b > c > 0$. Assume also that $a \approx b$ if $a \geqslant 1$. Then we have*

$$\int_0^a \frac{(w+b)^{3/2} e^{-w} dw}{\sqrt{w}(w+c)} \approx (1 \wedge a) + \frac{b^{3/2}}{\sqrt{c(1+c)}}. \tag{6.4}$$

Proof. Let us denote the integral in (6.4) by $I(a,b,c)$. Assume now that $c \geqslant 1$. Then we have $a, b \geqslant 1$ and it follows that

$$I(a,b,c) = \frac{b^{3/2}}{c} \int_0^a \frac{(w/b+1)^{3/2} e^{-w} dw}{\sqrt{w}(w/c+1)} \approx \frac{b^{3/2}}{c}.$$

It is equivalent to the assertion because

$$\frac{\sqrt{2} b^{3/2}}{\sqrt{c(1+c)}} > \frac{b^{3/2}}{c} > b^{1/2} > 1.$$

In the case $c < 1$ we substitute $w = cu$. Then

$$I(a,b,c) = c \int_0^{a/c} \frac{(u+b/c)^{3/2} \, e^{-cu}}{\sqrt{u}(u+1)} du. \tag{6.5}$$

For $a \geqslant 1$ we use the assumption that $a \approx b$ if $a \geqslant 1$. It follows that $u + b/c \approx b/c$ for $u \in [0, a/c]$. Hence

$$I(a,b,c) \approx c \left(\frac{b}{c}\right)^{3/2} \int_0^{a/c} \frac{e^{-uc}}{\sqrt{u}(u+1)} du \approx \frac{b^{3/2}}{\sqrt{c}},$$

which, as before, is equivalent to the assertion. For $a < 1$ the formula (6.5) gives us the estimate

$$I(a,b,c) \approx c \int_0^{a/c} \frac{u du}{u+1} + \frac{b^{3/2}}{c} \int_0^{a/c} \frac{e^{-uc} du}{\sqrt{u}(u+1)}.$$

The last integral is bounded by $\int_0^1 \frac{e^{-u} du}{\sqrt{u}(u+1)}$ from below and by $\int_0^\infty \frac{du}{\sqrt{u}(u+1)}$ from above. Moreover,

$$c \int_0^{a/c} \frac{u du}{u+1} = c \left(\frac{a}{c} - \ln\left(1 + \frac{a}{c}\right)\right) \approx a,$$

which completes the proof. ∎

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