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EXTREMAL PROPERTIES OF ONE-DIMENSIONAL CAUCHY-TYPE MEASURES*

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Abstract. In the paper we investigate some extremal properties of intervals for the one-dimensional Cauchy measure, related to isoperimetric inequalities. We also consider the analogous properties for the one-dimensional sections of the multidimensional isotropic Cauchy measure. In particular, among intervals with the fixed measure we find the ones with the extremal measure of the boundary. It turns out that, contrary to the Gaussian case, the type of extremal set depends on the value of the measure.

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1. INTRODUCTION

The classical isoperimetric theorem on the plane states that among all Borel sets with the fixed measure the circle has the smallest perimeter. The multidimensional version of the theorem states that in any dimension there exists a set with the smallest measure of the boundary and this minimum is attained for a ball.

When considering a probability measure μ on $\mathbb{R}^n$ we restrict our consideration to convex Borel sets and for such a set A we put $A^h = \{x \in \mathbb{R}^n : \text{dist}(x, A) < h\}$ and introduce its *perimeter* by the formula

$$per(A) = \limsup_{h \to 0^+} \frac{\mu(A^h) - \mu(A)}{h}.$$

The isoperimetric theorems for Gaussian distributions were investigated first in the papers [4] and [1]. It turns out that among all convex Borel sets in $\mathbb{R}^n$ with the same fixed measure the half-space, i.e. $\{x \in \mathbb{R}^n : x_n > a\}$, has the smallest Gaussian perimeter. The most complete approach to the Gaussian isoperimetric theory was presented in the paper by Ehrhard [2]. He adapted Steiner's symmetrization

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method [3] by constructing a family of the so-called (Gaussian) k-symmetrizations in the $\mathbb{R}^n$ space, $1 \leqslant k \leqslant n$, equipped with the standard n-dimensional Gaussian distribution γ_n. To define such a symmetrization, let T be a vector subspace of dimension $n-k$, $1 \leqslant k \leqslant n$, and let $\mathbf{u} \in \mathbb{R}^n$, $\|\mathbf{u}\| = 1$, be orthogonal to T. By $H(\mathbf{u}, a) = \{\mathbf{x} \in \mathbb{R}^n : (\mathbf{u}, \mathbf{x}) > a\}$ we denote an open half-space in $\mathbb{R}^n$. Let us denote by B a Borel subset of $\mathbb{R}^n$. For every $\mathbf{x} \in T$ we put $a = a(\mathbf{x})$ such that $\gamma_k\big(B \cap (\mathbf{x} + T^\perp)\big) = \gamma_k\big(H\big(\mathbf{u}, a(\mathbf{x})\big) \cap (\mathbf{x} + T^\perp)\big)$. If the measure is zero, we put $a(\mathbf{x}) = \infty$; if the measure is one, we put $a(\mathbf{x}) = -\infty$. Here γ_k denotes the k-dimensional standard Gaussian distribution in $\mathbb{R}^k$. The Gaussian k-symmetrization of the set B (in the direction $\mathbf{u}$) is now defined as

$$S^{(k)}(A) = \bigcup_{\mathbf{x} \in T} \Big(H\big(\mathbf{u}, a(\mathbf{x})\big) \cap (\{\mathbf{x}\} + T^\perp) \Big).$$

From the construction it follows that the operation $S^{(k)}$ preserves the measure γ_n and does not increase the perimeter.

Another important distribution in $\mathbb{R}^n$ is the rotationally invariant Cauchy measure with the density function

$$f_n(x_1, \ldots, x_n) = \frac{A_n}{(1 + x_1^2 + x_2^2 + \ldots + x_n^2)^{(n+1)/2}}, \quad (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n,$$

where $A_n = \Gamma\big(\frac{n+1}{2}\big)/\pi^{(n+1)/2}$. When extending the symmetrization procedure for this distribution we have to consider its k-dimensional sections, that is, the following family of k-dimensional densities, $1 \leqslant k \leqslant n-1$:

$$f_{\alpha,n}(x_{n-k+1}, \ldots, x_n) = \frac{A_{\alpha,n,k}}{(1 + \alpha^2 + x_{n-k+1}^2 + \ldots + x_n^2)^{(n+1)/2}},$$

where $\alpha \geqslant 0$, $x_i \in \mathbb{R}$ and $A_{\alpha,n,k}$ is an appropriate constant. We call these distributions *Cauchy-type measures*. Note that for $k = 1$ we deal with the so-called *Student's t-distribution*. For $n = 1$ and $\alpha = 0$ we obtain the standard one-dimensional Cauchy distribution.

In the present paper we deal only with the case $k = 1$, that is, with the distributions which appear in the case of one-symmetrization. The purpose of the paper is to examine what kind of one-dimensional convex sets (that is, intervals or half-lines) has minimal perimeter when the measure is fixed. And it turns out that the situation is different than that in the case of Gaussian distribution, namely the answer does depend on the measure of the set.

Section 2 is devoted to the case of $n = 1$, that is, when we consider the standard Cauchy distribution on the real line. The situation is then the most transparent one: for intervals (a, b) with the measure greater than $1/2$ the minimal perimeter is attained by the symmetric interval, while for the measure less than $1/2$ it is the half-line which has the minimal perimeter.

In Section 3, to avoid repetitions, we provide a very short proof, by the Lagrange multipliers, that only symmetric intervals $(-c, c)$ or of kind $(-1/d, d)$ and half-lines $(-\infty, g)$ may be extremal (maximal or minimal) among all convex sets with a fixed measure with the density $f_{\alpha,n}$ for every n.

In Section 4 we show that indeed, for $n = 2$, we encounter all three kinds of sets as extremal (minimal or maximal) depending on the measure. At the same time, dimension two is quite easily accessible in computational sense: there are relatively simple formulas for the measure of intervals or half-lines as well as for their perimeters. All essential quantities can be directly computed. In particular, we know exactly when (for what value of the measure) the extremal character of symmetric intervals and half-lines interchanges.

In Section 5, for $n > 2$, we have the same situation regarding convex extremal sets but this time computations are much more complicated. We also have three kinds of extremal intervals (including half-lines), but this time the critical parameter, when the interchange of extremal character of symmetric intervals and half-lines takes place, is given only in a form of a solution of an equation.

2. CONVEX EXTREMAL SETS FOR THE CAUCHY MEASURE ON THE REAL LINE

Let μ be the standard Cauchy measure on the real line. When dealing with intervals (a, b) we adopt the convention that $-\infty < a < b < \infty$. Let us compute the perimeter of the interval (a, b). We have

$$per(a,\, b) = \limsup_{h\to 0^+} \frac{\mu\big((a-h,\, b+h)\big) - \mu\big((a,\, b)\big)}{h} =$$

$$\limsup_{h\to 0^+} \frac{1}{h}\left(\int\limits_{a-h}^{a} \frac{1}{\pi(1+x^2)}\, dx + \int\limits_{b}^{b+h} \frac{1}{\pi(1+x^2)}\, dx \right) = \frac{1}{\pi(1+a^2)} + \frac{1}{\pi(1+b^2)}.$$

For the half-line $(-\infty,\, g)$ we get

$$per(-\infty, g) = \limsup_{h\to 0^+} \frac{1}{h} \int\limits_{g}^{g+h} \frac{1}{\pi(1+x^2)}\, dx = \frac{1}{\pi(1+g^2)}.$$

Define $g := g(a, b)$ by the equality

$$\mu(-\infty, g) = \mu(a, b); \tag{2.1}$$

$g^* := g^*(a, b)$ is defined by the similar identity, i.e.,

$$\mu(-g^*, g^*) = \mu(a, b). \tag{2.2}$$

We have the following

THEOREM 2.1. *Let* $-\infty < a < b < \infty$. *Then*:
- *If* $\mu(a,b) > 1/2$, *then* $per(-g^*, g^*) < per(a,b) < per(-\infty, g)$.
- *If* $\mu(a,b) < 1/2$, *then* $per(-\infty, g) < per(a,b) < per(-g^*, g^*)$.
- *If* $\mu(a,b) = 1/2$, *then* $per(-\infty, 0) = per(-1/b, b) = per(-1,1) = 1/\pi$.

We will prove Theorem 2.1 in a series of lemmas. First we compute the Cauchy measure of $(a,\, b)$:

$$\mu\big((a,b)\big) = \int_a^b \frac{1}{\pi(1+x^2)}\,dx = \frac{1}{\pi}(\arctan b - \arctan a). \tag{2.3}$$

Let us compare the perimeters of intervals (a,b) and half-lines $(-\infty, g)$ with the fixed measure $A \in (0,1)$.

LEMMA 2.1. *Let*

$$\mathcal{F}_A = \big\{(a,\, b),\ a < b,\ (-\infty,\, g),\ g \in \mathbb{R} : \mu\big((a,b)\big) = \mu\big((-\infty, g)\big) = A\big\}.$$

If an interval $(a,b) \in \mathcal{F}_A$ *has an extremal perimeter*, *then* $a = -b$ *or* $a = -1/b$.

P r o o f. We use the Lagrange method to find extrema of the function $h(a,b) = 1/(1+a^2) + 1/(1+b^2)$, $a < b$, under the condition $\arctan b - \arctan a = \pi A$. Let

$$F(a,b) = \frac{1}{1+a^2} + \frac{1}{1+b^2} + \lambda(\arctan b - \arctan a).$$

Then the following conditions for extrema must be fulfilled:

$$\frac{\partial F}{da} = \frac{-2a}{(1+a^2)^2} - \frac{\lambda}{1+a^2} = 0 \quad \text{and} \quad \frac{\partial F}{db} = \frac{-2b}{(1+b^2)^2} + \frac{\lambda}{1+b^2} = 0.$$

Consequently, $-a/(1+a^2) = b/(1+b^2)$ and this immediately gives $a = -b$ or $a = -1/b$. ■

To compute g as a function of a and b we apply the trigonometric identity

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$$

to both sides of the equality $\arctan b - \arctan a = \frac{\pi}{2} + \arctan g$, and obtain

$$g(a,b) = -\frac{1+ab}{b-a}. \tag{2.4}$$

Now, the measures of the interval (a,b) and of the half-line $\big(-\infty, -\frac{1+ab}{b-a}\big)$ are equal and their perimeters are as follows:

$$per(a,b) = \frac{1}{\pi(1+a^2)} + \frac{1}{\pi(1+b^2)} \quad \text{and} \quad per(-\infty, g) = \frac{1}{\pi\big(1+g^2(a,b)\big)}.$$

LEMMA 2.2. *For $a < b$ the inequality*

$$\frac{1}{1+a^2} + \frac{1}{1+b^2} \leqslant \frac{1}{1+\left(\frac{1+ab}{b-a}\right)^2} \tag{2.5}$$

holds if and only if $ab \leqslant -1$.

P r o o f. The left-hand side of the above inequality is equal to

$$\frac{(1+a^2)+(1+b^2)}{(1+a^2)(1+b^2)} = \frac{2+a^2+b^2}{(1+a^2)(1+b^2)},$$

while the right-hand side equals

$$\frac{(b-a)^2}{1+a^2+b^2+a^2b^2} = \frac{(b-a)^2}{(1+a^2)(1+b^2)}.$$

Hence (2.5) is equivalent to the inequality $2+a^2+b^2 \leqslant (b-a)^2$, which holds if and only if $ab \leqslant -1$. Observe also that the equality in (2.5) holds if and only if $ab = -1$. ■

LEMMA 2.3. *The following condition holds*:

$$\mu\big((a,b)\big) > \frac{1}{2} \quad \textit{if and only if} \quad ab < -1. \tag{2.6}$$

P r o o f. We know that

$$\tan\big(\pi\mu(a,b)\big) = \tan(\arctan b - \arctan a) = \frac{b-a}{1+ab}.$$

Hence the condition $ab < -1$ is equivalent to $\tan\big(\pi\mu(a,b)\big) < 0$, and this means that $\mu(a,b) > \frac{1}{2}$. The condition $ab > -1$ is equivalent to $\mu(a,b) < \frac{1}{2}$. By continuity, the condition $ab = -1$ implies $\mu(a,b) = \frac{1}{2}$. ■

LEMMA 2.4. *Let $g^* := g^*(a,b)$ be defined by* (2.2). *Then*

$$g^* = \sqrt{g^2+1} + g, \tag{2.7}$$

where $g := g(a,b) = -(1+ab)/(b-a)$.

P r o o f. From the condition $\mu(-g^*, g^*) = \mu(a,b) = \mu(-\infty, g)$ we obtain $2\arctan g^* = \frac{\pi}{2} + \arctan g$. Thus we get the equality

$$\frac{2g^*}{1-(g^*)^2} = -\frac{1}{g},$$

which gives the assertion. ■

LEMMA 2.5. *Let* $g^* := g^*(a,b)$ *be given by* (2.2). *The inequality* $per(a,b) > per(-g^*, g^*)$ *holds if and only if* $ab < -1$.

P r o o f. We have to check when the inequality

$$\frac{1}{1+a^2} + \frac{1}{1+b^2} > \frac{2}{1+(g^*)^2}$$

holds. Taking into account the form of $g = g(a,b)$ and $g^* = g^*(a,b)$ we obtain the following equivalent form of the above inequality:

$$\frac{(1+a^2)+(1+b^2)}{(1+a^2)(1+b^2)} > \frac{(b-a)^2}{\sqrt{1+a^2}\sqrt{1+b^2}}\,\frac{1}{\sqrt{1+a^2}\sqrt{1+b^2}-1-ab}.$$

This is equivalent to

$$2(1+ab)\sqrt{1+a^2}\sqrt{1+b^2} > (1+ab)(2+a^2+b^2),$$

which finally gives the conclusion. ■

P r o o f o f T h e o r e m 2.1. Lemmas 2.1–2.5 provide the proof of Theorem 2.1. ■

3. THREE TYPES OF CONVEX EXTREMAL SETS FOR $\nu_{\alpha,n}$

Let $\nu_{\alpha,n}$ be the probability measure with the density $f_{\alpha,n}$. Let (a,b) be any interval and $(-\infty,\, g)$ any half-line. Computing exactly as in the case when $n=1$, we get perimeters of the interval (a,b) in the form

$$per(a,\, b) = \frac{A_{\alpha,n}}{(1+\alpha^2+a^2)^{(n+1)/2}} + \frac{A_{\alpha,n}}{(1+\alpha^2+b^2)^{(n+1)/2}}$$

and those of the half-line $(-\infty,\, g)$ in the form

$$per(-\infty,\, g) = \frac{A_{\alpha,n}}{(1+\alpha^2+g^2)^{(n+1)/2}}.$$

First, as for $n=1$, we will compute the possible form of extremal sets. For simplicity, we drop the constant $A_{\alpha,n}$ in the density $f_{\alpha,n}$ and use the notation $\Gamma(a,b)$ instead of $per(a,b)$.

LEMMA 3.1. *Consider a family of intervals* $(a,\, b)$, $a<b$, *with a fixed measure* ν, *that is*, *with*

$$\int_a^b \frac{1}{(1+\alpha^2+x^2)^{(n+1)/2}}\,dx = C.$$

Then for $a < b$ the function

$$\Gamma(a,b) = \frac{1}{(1+\alpha^2+a^2)^{(n+1)/2}} + \frac{1}{(1+\alpha^2+b^2)^{(n+1)/2}}$$

can attain its extrema only if $a+b=0$ or if $ab = -(1+\alpha^2)$.

P r o o f. We use again the Lagrange method and for $a < b$ consider

$$F(a,b) =$$
$$= \frac{1}{(1+\alpha^2+a^2)^{(n+1)/2}} + \frac{1}{(1+\alpha^2+b^2)^{(n+1)/2}} + \lambda \int\limits_a^b \frac{dx}{(1+\alpha^2+x^2)^{(n+1)/2}}.$$

We have

$$\frac{\partial F}{\partial a}(a,b) = \frac{-(n+1)a}{(1+\alpha^2+a^2)^{(n+3)/2}} - \frac{\lambda}{(1+\alpha^2+a^2)^{(n+1)/2}} = 0$$

and

$$\frac{\partial F}{\partial b}(a,b) = \frac{-(n+1)b}{(1+\alpha^2+b^2)^{(n+3)/2}} + \frac{\lambda}{(1+\alpha^2+b^2)^{(n+1)/2}} = 0,$$

which, after simplification, gives the following system:

$$\lambda(1+\alpha^2+a^2) = -(n+1)a,$$
$$\lambda(1+\alpha^2+b^2) = (n+1)b.$$

Dividing the first equation by the second, we get the equality

$$\frac{1+\alpha^2+a^2}{1+\alpha^2+b^2} = -\frac{a}{b}.$$

Cross multiplication gives $(1+\alpha^2)b + a^2 b = -(1+\alpha^2)a - ab^2$, which is equivalent to $(1+\alpha^2)(a+b) = -ab(a+b)$. We see that $F(a,b)$ can attain extremal values within the family of intervals $\{(a,b) : a < b\}$ only if $a+b=0$ or if $ab = -(1+\alpha^2)$. These conditions describe two types of intervals: $(-c,\, c)$, $c > 0$, and $\big(-(1+\alpha^2)/d, d\big)$ with $d > 0$. Observe that for $c = d = \sqrt{1+\alpha^2}$ both intervals $(-c, c)$ and $\big(-(1+\alpha^2)/d,\, d\big)$ are equal. ∎

COROLLARY 3.1. *Since the function $F(a,b)$ is continuously differentiable, its maximal and minimal values can be attained only at stationary points or at the boundaries of its domain.*

4. CONVEX EXTREMAL SETS FOR THE MEASURE WITH DENSITY $f_{0,2}$

Before resolving the general case we consider the measure $\nu = \nu_{0,2}$ with the density $f_{0,2}(x) = 1/\big(2(1+x^2)^{3/2}\big)$ and find sets with the smallest perimeter. In the case of $n > 2$ we encounter the same kind of behavior but in the present case we are still able to perform direct calculations and find out the critical value at which intervals change their extremal character.

Fix a number $A \in (0,1)$ and consider, as before, a family of intervals $(a,\, b)$ and half-lines $(-\infty,\, g)$, all with the measure equal to A:

$$\mathcal{F}_A = \{(a,\, b),\ a < b,\ (-\infty,\, g),\ g \in \mathbb{R} :\ \nu(a,b) = \nu(-\infty,\, g) = A\}.$$

By Corollary 3.1, the smallest perimeter can be attained only for intervals $(-c,\, c)$ or $(-1/d,\, d)$ or for the half-line $(-\infty,\, g)$. Observe that

$$\int \frac{1}{(1+x^2)^{3/2}}\, dx = \frac{x}{\sqrt{1+x^2}}\,,$$

so, from the equality $\nu(-c,\, c) = \nu(-\infty,\, g)$ we obtain

$$\int_{-c}^{c} \frac{1}{2(1+x^2)^{3/2}}\, dx = \frac{c}{\sqrt{1+c^2}} = \int_{-\infty}^{g} \frac{1}{2(1+x^2)^{3/2}}\, dx = \frac{1}{2}\left(1 - \frac{g}{\sqrt{1+g^2}}\right).$$

After elementary calculations this gives $per(-c,c) < per(-\infty,\, g)$ if and only if $2\big(c+\sqrt{1+c^2}\big)^{3/2} < (4c)^{3/2}$, which is equivalent to the following: $c > c^* = 1/\sqrt{(2^{4/3}-1)^2-1} \approx 0.873731$. It is easy to compute the measure of $(-c^*,\, c^*)$:

$$\int_{-c^*}^{c^*} \frac{1}{2(1+x^2)^{3/2}}\, dx = \frac{1}{2^{4/3}-1} \approx 0.657963.$$

We have just proved the following:

LEMMA 4.1. *Let* $\nu(-c,\, c) = \nu(-\infty,\, g)$ *and* $c^* = 1/\sqrt{(2^{4/3}-1)^2-1}$. *Then*

$$per(-c,c) < per(-\infty,\, g) \quad \textit{iff} \quad c > c^* \quad \textit{iff} \quad \nu(-c,c) > \frac{1}{2^{4/3}-1}.$$

Now we will compare perimeters of the half-line $(-\infty,\, g)$ and of the interval $(-1/d,\, d)$. Observe that by the symmetry of ν it is enough to consider only $d \in (0,\, 1)$. We have

$$\nu(-1/d,\, d) = \int_{-1/d}^{d} \frac{1}{2(1+x^2)^{3/2}}\, dx = \frac{1+d}{2\sqrt{1+d^2}} = A$$

and notice that for $d \in (0,\, 1)$

$$\left(\frac{1+d}{\sqrt{1+d^2}}\right)^2 = 1 + \frac{2d}{1+d^2} \in (1,\, 2), \tag{4.1}$$

so that the values of $(1+d)/(2\sqrt{1+d^2})$ are contained in the interval $\bigl(1/2,\, 1/\sqrt{2}\bigr)$. In the case of a half-line, we have

$$\nu(-\infty,\, g) = \int_{-\infty}^{g} \frac{1}{2(1+x^2)^{3/2}}\, dx = \frac{1}{2}\left(1 - \frac{g}{\sqrt{1+g^2}}\right) = A,$$

and hence $h(d) := g(-1/d, d)$ solves the equation

$$\frac{1+d}{\sqrt{1+d^2}} = 1 - \frac{h}{\sqrt{1+h^2}}.$$

We obtain

$$\frac{1}{1+h^2} = 1 - \frac{h^2}{1+h^2} = 1 - \left(1 - \frac{1+d}{\sqrt{1+d^2}}\right)^2 = \frac{1+d}{1+d^2}\bigl(2\sqrt{1+d^2} - (1+d)\bigr). \tag{4.2}$$

LEMMA 4.2. *Let* $A \in \bigl(1/2,\, 1/\sqrt{2}\bigr)$ *and* $\nu(-1/d,\, d) = \nu\bigl(-\infty,\, h(d)\bigr) = A$. *Then for all* $d \in (0,\, 1)$

$$per(-1/d,\, d) < per\bigl(-\infty,\, h(d)\bigr).$$

Proof. We compare the following perimeters:

$$per(-1/d,\, d) = \frac{1+d^3}{2(1+d^2)^{3/2}} \quad \text{and} \quad per\bigl(-\infty,\, h(d)\bigr) = \frac{1}{2\bigl(1+h^2(d)\bigr)^{3/2}}.$$

Taking into account (4.2), we show that for all $d \in (0,1)$

$$\frac{1+d^3}{(1+d^2)^{3/2}} < \frac{1}{(1+h^2)^{3/2}} = \frac{(1+d)^{3/2}}{(1+d^2)^{3/2}}\bigl(2\sqrt{1+d^2} - (1+d)\bigr)^{3/2}.$$

Simplifying, we obtain the following equivalent form of the above inequality:

$$\frac{(1+d^3)^{2/3}}{(1+d)^2} < 2\frac{\sqrt{1+d^2}}{1+d} - 1. \tag{4.3}$$

Denote by f and k, respectively, the left-hand side and the right-hand side of the inequality (4.3). We infer that the inequality $f'(d) < k'(d)$ holds for $0 < d < 1$ if and only if the inequality

$$(1+d^3)^{1/3} < (1+d^2)^{1/2}$$

holds, which is obviously true for $0 < d < 1$. This together with the fact that $f(0) = 1 = k(0)$ completes the proof of (4.3). ∎

We now complete the results of this section. Let us recall that we have $c^* = 1/\sqrt{(2^{4/3}-1)^2-1}$ and $h(c)$ is determined by $\nu\big(-\infty, h(c)\big) = \nu(-c, c)$.

THEOREM 4.1. *Assume that* ν *has the density* $f_{0,2}(x) = 1/\big(2(1+x^2)^{3/2}\big)$. *Then*:

- *Among all convex sets with fixed measure* ν *only intervals* $(-c, c)$, $(-1/d, d)$ *or half-lines* $(-\infty, g)$ *are of extremal perimeter.*
- *For all* $d > 0$ *we have* $1/2 < \nu(-1/d, d) \leqslant 1/\sqrt{2}$ *and the interval* $(-1/d, d)$ *is of minimal perimeter among all intervals* (a, b) *and half-lines* $(-\infty, g)$ *with fixed* ν*-measure* $A \in (1/2,\ 1/\sqrt{2}]$.
- *If* $\nu(-c, c) = \nu\big(-\infty, h(c)\big)$, *then for* $0 < c < c^* = 1/\sqrt{(2^{4/3}-1)^2-1}$ *we have*

$$per\big(-\infty, h(c)\big) < per(-c, c),$$

while for $c > c^*$ *we have the reverse inequality*

$$per\big(-\infty, h(c)\big) > per(-c, c).$$

P r o o f. The first statement is a direct consequence of the results in Section 3. The first part of the next statement is justified in calculations below Lemma 4.1 – see (4.1). Furthermore, Lemma 4.2 gives $per(-1/d, d) < per\big(-\infty, h(d)\big)$ for all $d > 0$.

Assume now that $\nu(-1/d, d) = \nu(-c, c)$. Then we have

$$\nu(-c, c) = \frac{c}{\sqrt{1+c^2}} = \frac{1+d}{2\sqrt{1+d^2}} = \nu(-1/d, d),$$

which, in turn, shows that

$$1 + c^2 = \frac{4(1+d^2)}{3(1-d)^2 + 4d}\,.$$

We show that the assumption $\nu(-1/d, d) = \nu(-c, c)$ implies

$$per(-1/d,\ d) = \frac{1+d^3}{2(1+d^2)^{3/2}} \leqslant per(-c,\ c) = \frac{1}{(1+c^2)^{3/2}}\,. \tag{4.4}$$

Indeed, taking into account the previous formula, we see that (4.4) reduces to the following inequality:

$$(3d^2 - 2d + 3)^3 \geqslant 4^2(1+d^3)^2.$$

But $(3d^2 - 2d + 3)^3 - 4^2(1+d^3)^2 = (d-1)^4(11d^2 - 10d + 11) \geqslant 0$, which justifies our claim (4.4) and completes the proof of the second statement of the theorem.

The last statement of the theorem is contained in Lemma 4.1. Thus the proof of Theorem 4.1 is complete. ■

Now, we can deduce the following corollary.

COROLLARY 4.1. *We have the following four cases*:

• *Let* $\nu(-c,c) = \nu\big(-\infty,\, h(c)\big)$. *If* $c < 1/\sqrt{3}$, *then* $\nu\big(-\infty,\, h(c)\big) < 1/2$ *and* $per\big(-\infty,\, h(c)\big) < per(-c,c)$.

• *If we have* $1/\sqrt{3} < c < c^* = 1/\sqrt{(2^{4/3}-1)^2 - 1}$ *and* $d \in (0,1)$ *is such that* $\nu(-1/d,d) = \nu(-c,c)$, *then* $\nu\big(-\infty,\, h(c)\big) < 1/(2^{4/3}-1)$ *and*

$$per(-1/d,d) < per\big(-\infty,\, h(c)\big) < per(-c,c).$$

• *If* $c^* < c < 1$ *and* $d \in (0,1)$ *is such that* $\nu(-1/d,d) = \nu(-c,c)$, *then we have* $1/(2^{4/3}-1) < \nu(-c,c) = \nu\big(-\infty,\, h(c)\big) < 1/\sqrt{2}$ *and*

$$per(-1/d,d) < per(-c,c) < per\big(-\infty,\, h(c)\big).$$

• *Finally*, *for* $c > 1$ *we have* $1/\sqrt{2} < \nu(-c,c) = \nu\big(-\infty,\, h(c)\big)$ *and*

$$per(-c,c) < per\big(-\infty,\, h(c)\big).$$

P r o o f. Almost all assertions follow from Theorem 4.1. We only remark that if $d \in (0,\, 1]$, then $\nu(-1/d,\, d) = (1+d)/(2\sqrt{1+d^2}) \in \big(1/2,\, 1/\sqrt{2}\big]$ and there exists a unique interval $(-c,\, c)$ with $\nu(-c,\, c) = \nu(-1/d,\, d)$. Since

$$\nu(-c,c) = \frac{c}{\sqrt{1+c^2}},$$

we have

$$\frac{1}{2} = \nu(-c,c) = \frac{c}{\sqrt{1+c^2}} \quad \text{iff} \quad c = \frac{1}{\sqrt{3}}$$

and

$$\frac{1}{\sqrt{2}} = \nu(-c,c) = \frac{c}{\sqrt{1+c^2}} \quad \text{iff} \quad c = 1.$$

Therefore, for $c \in \big(1/\sqrt{3},\, c^*\big)$ we have three types of intervals with the same measure ν to check their perimeters: $(-c,\, c)$, $(-1/d,\, d)$, and $\big(-\infty,\, h(c)\big)$. Otherwise, we have only intervals $(-c,\, c)$ and $\big(-\infty,\, h(c)\big)$. What happens in each case is settled by Theorem 4.1. ■

REMARK 4.1. *The above theorem and corollary hold*, *after obvious modifications*, *for measures with densities* $f_{\alpha,2}$. *Indeed*, *changing variables in the integral we obtain*

$$\nu_{\alpha,2}(-c,\, c) = \int_{-c}^{c} \frac{(1+\alpha^2)/2}{(1+\alpha^2+x^2)^{3/2}}\,dx = \int_{-c/\sqrt{1+\alpha^2}}^{c/\sqrt{1+\alpha^2}} \frac{1/2}{(1+u^2)^{3/2}}\,du.$$

This shows a kind of homogeneity of such integrals with respect to the transformation $c \to c/\sqrt{1+\alpha^2}$, *which gives appropriate versions of all the above results for* $\alpha > 0$.

5. EXTREMAL CONVEX SETS FOR THE MEASURE WITH DENSITY $f_{\alpha,n}$

In this general case the situation is similar to that for $f_{0,2}$ but computations are much more difficult. We also use the above-mentioned homogeneity and the exact value of $A_{\alpha,n} = (1+\alpha^2)^{n/2}\Gamma\big(\frac{n+1}{2}\big)/\big(\sqrt{\pi}\Gamma\big(\frac{n}{2}\big)\big)$.

LEMMA 5.1. *For every $d > 0$ the following inequality holds*:

$$\int_{(-1/d)\sqrt{1+\alpha^2}}^{d\sqrt{1+\alpha^2}} \frac{A_{\alpha,n}}{(1+\alpha^2+x^2)^{(n+1)/2}}\,dx \geqslant \int_{-\infty}^{0} \frac{A_{\alpha,n}}{(1+\alpha^2+x^2)^{(n+1)/2}}\,dx = \frac{1}{2}$$

for all $\alpha \geqslant 0$ and $n = 1, 2, \ldots$

Proof. Let us start with some simplification:

$$\int_{(-1/d)\sqrt{1+\alpha^2}}^{d\sqrt{1+\alpha^2}} \frac{1}{(1+\alpha^2+x^2)^{(n+1)/2}}\,dx = \frac{1}{(1+\alpha^2)^{n/2}} \int_{-1/d}^{d} \frac{1}{(1+u^2)^{(n+1)/2}}\,du.$$

In the same way we obtain

$$\int_{-\infty}^{0} \frac{1}{(1+\alpha^2+x^2)^{(n+1)/2}}\,dx = \frac{1}{(1+\alpha^2)^{n/2}} \int_{-\infty}^{0} \frac{1}{(1+u^2)^{(n+1)/2}}\,du,$$

so that it is enough to prove the following: for all $d > 0$ and any $n = 1,\ 2,\ 3, \ldots$

$$\int_{-1/d}^{d} \frac{1}{(1+u^2)^{(n+1)/2}}\,du \geqslant \int_{-\infty}^{0} \frac{1}{(1+u^2)^{(n+1)/2}}\,du.$$

Consider the following function: for $x > 0$ and $n = 1,\ 2,\ 3, \ldots$

$$h_n(x) = \int_{-1/x}^{x} \frac{1}{(1+u^2)^{(n+1)/2}}\,du.$$

Observe that

$$\lim_{x\to 0^+} h_n(x) = \lim_{x\to\infty} h_n(x) = \int_{-\infty}^{0} \frac{1}{(1+u^2)^{(n+1)/2}}\,du.$$

We will show that $h_n'(x)$ is positive on $(0,\ 1)$ and negative on $(1,\ \infty)$, which implies that h is always greater than its limit at zero and infinity. Indeed,

$$h_n'(x) = \frac{1}{(1+x^2)^{(n+1)/2}} - \frac{1/x^2}{(1+1/x^2)^{(n+1)/2}} = \frac{1-x^{n-1}}{(1+x^2)^{(n+1)/2}},$$

and hence this derivative is positive on $(0,\ \infty)$ if and only if $x < 1$. This proves the lemma. ■

Observe that we have also proved that $h_n(x)$ has its maximum at $x = 1$, so the maximal value of $\nu_{\alpha,n}(-1/d,\, d)$ is attained for $d = \sqrt{1+\alpha^2}$ or, by the symmetry of the measure $\nu_{\alpha,n}$, for $d = 1/\sqrt{1+\alpha^2}$.

LEMMA 5.2. *For any* $\alpha \geqslant 0$ *and* $n = 1, 2, \ldots$ *let* $\nu_{\alpha,n}(-c, c) = \nu_{\alpha,n}(-\infty,\, g) = A \leqslant \frac{1}{2}$. *Then* $per(-\infty,\, g) \leqslant per(-c,\, c)$.

P r o o f. We must compare the perimeters

$$per(-c,\, c) = \frac{2A_{\alpha,n}}{(1+\alpha^2+c^2)^{(n+1)/2}} \quad \text{and} \quad per(g,\, \infty) = \frac{A_{\alpha,n}}{(1+\alpha^2+g^2)^{(n+1)/2}}.$$

We will show this in two steps. First we show that there exists $c_{\alpha,n}$ such that for all $c \in (0,\, c_{\alpha,n})$ it follows that

$$\frac{2A_{\alpha,n}}{(1+\alpha^2+c^2)^{(n+1)/2}} > \frac{A_{\alpha,n}}{(1+\alpha^2)^{(n+1)/2}},$$

where the right-hand side is the largest perimeter of any half-line. Next we will show that for $c > c_{\alpha,n}$ the measure of $(-c,\, c)$ is greater than or equal to $\frac{1}{2}$.

The first step is easy: we simply solve the equation

$$\frac{2}{(1+\alpha^2+c^2)^{(n+1)/2}} = \frac{1}{(1+\alpha^2)^{(n+1)/2}},$$

which gives $c^2 = (2^{2/(n+1)} - 1)(1+\alpha^2)$, and we put

$$c_{\alpha,n} = \sqrt{(2^{2/(n+1)} - 1)(1+\alpha^2)}.$$

Now we will show that

$$\int_{-c_{\alpha,n}}^{c_{\alpha,n}} \frac{A_{\alpha,n}}{(1+\alpha^2+x^2)^{(n+1)/2}}\, dx \geqslant \frac{1}{2}.$$

Since $A_{\alpha,n} = (1+\alpha^2)^{n/2}\Gamma\big(\frac{n+1}{2}\big)/\big(\sqrt{\pi}\,\Gamma\big(\frac{n}{2}\big)\big)$, we have

$$\int_{-c_{\alpha,n}}^{c_{\alpha,n}} \frac{A_{\alpha,n}}{(1+\alpha^2+x^2)^{(n+1)/2}}\, dx = \frac{\Gamma\big(\frac{n+1}{2}\big)}{\sqrt{\pi}\,\Gamma\big(\frac{n}{2}\big)} \int_{-\sqrt{2^{2/(n+1)}-1}}^{\sqrt{2^{2/(n+1)}-1}} \frac{1}{(1+u^2)^{(n+1)/2}}\, du.$$

Now it is enough to show that for all $n = 1, 2, 3, \ldots$

$$\frac{2\Gamma\big(\frac{n+1}{2}\big)}{\sqrt{\pi}\,\Gamma\big(\frac{n}{2}\big)} \int_{0}^{\sqrt{2^{2/(n+1)}-1}} \frac{1}{(1+u^2)^{(n+1)/2}}\, du \geqslant \frac{1}{2}. \tag{5.1}$$

One can easily compute the left-hand side of (5.1) for $n = 1, 2, 3, 4$. It is equal to, respectively, $\frac{1}{2}$, $\sqrt{1 - \frac{1}{2^{2/3}}}$, $\big(\sqrt{2(\sqrt{2}-1)} + 2\arctan\big(\sqrt{\sqrt{2}-1}\big)\big)/\pi$, $\sqrt{2^{2/5}-1}(2^{7/5}+1)/2^{8/5}$. All these numbers are greater than or equal to $\frac{1}{2}$. Thus we will prove (5.1) only for $n \geqslant 5$.

For $n \geqslant 1$ it follows that

$$\sqrt{2^{2/(n+1)} - 1} > \frac{1}{\sqrt{2+n}}.$$

Since $f(x) = 1/(1+x^2)^{(n+1)/2}$ is concave on $\big(0, \frac{1}{\sqrt{2+n}}\big)$, the integral of f over $\big(0, \sqrt{2^{2/(n+1)}-1}\big)$ is greater than the area of the trapezoid with vertices $(0,0)$, $(0,1)$, $\big(\frac{1}{\sqrt{2+n}}, 0\big)$, $\big(\frac{1}{\sqrt{2+n}}, f\big(\frac{1}{\sqrt{2+n}}\big)\big)$ and this area is equal to

$$\frac{1 + f\big(\frac{1}{\sqrt{2+n}}\big)}{2} \cdot \frac{1}{\sqrt{2+n}} = \frac{1 + \big(1 + \frac{1}{n+2}\big)^{-(n+1)/2}}{2} \cdot \frac{1}{\sqrt{2+n}}.$$

The sequence $\big(1 + \frac{1}{n+2}\big)^{(n+2)/2}$ is increasing to $\sqrt{e}$, so $\frac{1}{2}\big(1 + \big(1 + \frac{1}{n+2}\big)^{-(n+1)/2}\big)$ is decreasing and has the limit $\frac{1}{2}\big(1 + \frac{1}{\sqrt{e}}\big)$. This implies that for every $n \geqslant 1$

$$\begin{aligned}\int_0^{\sqrt{2^{2/(n+1)}-1}} \frac{1}{(1+u^2)^{(n+1)/2}}\,du &\geqslant \frac{1 + f\big(\frac{1}{\sqrt{2+n}}\big)}{2} \cdot \frac{1}{\sqrt{2+n}} \\ &\geqslant \frac{1}{2}\left(1 + \frac{1}{\sqrt{e}}\right) \cdot \frac{1}{\sqrt{2+n}}.\end{aligned}$$

We will show that for $n \geqslant 5$ the following inequality holds:

$$\frac{2\Gamma\big(\frac{n+1}{2}\big)}{\sqrt{\pi}\,\Gamma\big(\frac{n}{2}\big)} \frac{1}{\sqrt{n+2}} \frac{1 + \frac{1}{\sqrt{e}}}{2} \geqslant \frac{1}{2}.$$

If we put

$$B_n := \frac{2\Gamma\big(\frac{n+1}{2}\big)}{\sqrt{\pi}\,\Gamma\big(\frac{n}{2}\big)} \frac{1}{\sqrt{n+2}},$$

we only need to show that for $n \geqslant 5$

$$B_n \geqslant \frac{\sqrt{e}}{\sqrt{e}+1} \approx 0.622459.$$

But we can easily see that $B_{2k+2}/B_{2k} > 1$ and $B_{2k+1}/B_{2k-1} > 1$ for all $k \geqslant 1$, so it is enough to check that

$$B_5 = \frac{2\Gamma(3)}{\sqrt{\pi}\,\Gamma(5/2)} \frac{1}{\sqrt{7}} = \frac{16\sqrt{7}}{21\pi} \approx 0.64 > \frac{\sqrt{e}}{\sqrt{e}+1}$$

and
$$B_6 = \frac{2\Gamma(7/2)}{\sqrt{\pi}\Gamma(3)}\frac{1}{\sqrt{8}} = \frac{15\sqrt{2}}{32} \approx 0.66 > \frac{\sqrt{e}}{\sqrt{e}+1},$$
which completes the proof. ∎

Now we will show that, for $\nu = \nu_{0,n}$, if $\nu(-1/d,\, d) = \nu\big(-\infty,\, h(d)\big)$, then $per(-1/d,\, d) \leqslant per\big(-\infty,\, h(d)\big)$.

LEMMA 5.3. *For $d > 0$ we have the following inequality*:

$$per(-1/d,\, d) \leqslant per\big(-\infty,\, h(d)\big) \tag{5.2}$$

if

$$\int\limits_{-\infty}^{h(d)} \frac{dx}{(1+x^2)^{(n+1)/2}} = \int\limits_{-1/d}^{d} \frac{dx}{(1+x^2)^{(n+1)/2}}. \tag{5.3}$$

P r o o f. The inequality (5.2) reduces to the following:
$$\frac{1+d^{n+1}}{(1+d^2)^{(n+1)/2}} \leqslant \frac{1}{\big(1+h^2(d)\big)^{(n+1)/2}}.$$
The left-hand side of the above inequality is not greater than 1, so we may introduce $x(d) \geqslant 0$ such that
$$\frac{1+d^{n+1}}{(1+d^2)^{(n+1)/2}} = \frac{1}{\big(1+x^2(d)\big)^{(n+1)/2}}$$
or, equivalently, $(1+d^{n+1})\big(1+x^2(d)\big)^{(n+1)/2} = (1+d^2)^{(n+1)/2}$. Taking derivatives, we obtain for $n > 1$:
$$x'(d) = \frac{1+x^2(d)}{1+d^2}\,\frac{d}{x(d)}\,\frac{1-d^{n-1}}{1+d^{n+1}}.$$
Taking the derivatives of each of the sides of (5.3), we now obtain
$$\frac{h'(d)}{\big(1+h^2(d)\big)^{(n+1)/2}} = \frac{1-d^{n-1}}{(1+d^2)^{(n+1)/2}}.$$
We have
$$\begin{aligned}
&\left[\int\limits_{-\infty}^{h(d)} \frac{dx}{(1+x^2)^{(n+1)/2}} - \int\limits_{-\infty}^{x(d)} \frac{dx}{(1+x^2)^{(n+1)/2}}\right]' \\
&\qquad = \frac{h'(d)}{\big(1+h^2(d)\big)^{(n+1)/2}} - \frac{x'(d)}{\big(1+x^2(d)\big)^{(n+1)/2}} \\
&\qquad = \frac{1-d^{n-1}}{(1+d^2)^{(n+1)/2}} - \frac{1+d^{n+1}}{(1+d^2)^{(n+1)/2}}x'(d) \\
&\qquad = \frac{d(1-d^{n-1})\big(x(d)-d\big)\big(1-dx(d)\big)}{(1+d^2)(1+d^2)^{(n+1)/2}} < 0
\end{aligned}$$

for $d < 1$. Indeed, it follows that

$$\left(\frac{1+d^2}{1+x^2(d)}\right)^{(n+1)/2} = 1 + d^{n+1} > 1 \quad \text{implies} \quad x(d) < d.$$

Since $h(\infty) = h(0) = 0 = x(0) = x(\infty)$, we obtain

$$\int_{-\infty}^{h(d)} \frac{dx}{(1+x^2)^{(n+1)/2}} \leqslant \int_{-\infty}^{x(d)} \frac{dx}{(1+x^2)^{(n+1)/2}},$$

so that $h(d) \leqslant x(d)$ for $d < 1$. Moreover, from Lemma 5.1 we infer that $h(d) \geqslant 0$. Thus, we obtain finally

$$\frac{1+d^{n+1}}{(1+d^2)^{(n+1)/2}} = \frac{1}{\big(1+x^2(d)\big)^{(n+1)/2}} \leqslant \frac{1}{\big(1+h^2(d)\big)^{(n+1)/2}}.$$

Since our inequality is invariant under the transformation $d \to 1/d$, the proof is complete. ■

The next result is also of universal kind (that is, it does not depend on the value of the measure of intervals considered). We compare perimeters of $(-c, c)$ and $(-1/d, d)$ and show that if $\nu(-c, c) = \nu(-1/d, d)$, then

$$per(-1/d, d) \leqslant per(-c, c).$$

LEMMA 5.4. *For $d > 0$ we have*

$$per(-1/d, d) \leqslant per(-c, c), \tag{5.4}$$

where $c = c(d)$ is determined by the condition

$$2\int_{0}^{c(d)} \frac{dx}{(1+x^2)^{(n+1)/2}} = \int_{-1/d}^{d} \frac{dx}{(1+x^2)^{(n+1)/2}}.$$

Proof. We apply a similar technique to that in the previous proof. Let us put $y = y(d)$ such that

$$\frac{1+d^{n+1}}{(1+d^2)^{(n+1)/2}} = \frac{2}{\big(1+y^2(d)\big)^{(n+1)/2}}$$

with $d > 0$ and $y(d) > 0$. Equivalently,

$$(1+d^{n+1})\big(1+y^2(d)\big)^{(n+1)/2} = 2(1+d^2)^{(n+1)/2}.$$

Taking derivatives, we obtain

$$y(d)y'(d) = \frac{1+y^2(d)}{1+d^{n+1}} \frac{d(1-d^{n-1})}{1+d^2}.$$

From the definition of $c = c(d)$ we get

$$\frac{c'(d)}{\big(1+c^2(d)\big)^{(n+1)/2}} = \frac{1}{2} \frac{1-d^{n-1}}{(1+d^2)^{(n+1)/2}}.$$

For $0 < d < 1$ we have

$$\begin{aligned}
&\Bigg[\int\limits_0^{c(d)} \frac{dx}{(1+x^2)^{(n+1)/2}} - \int\limits_0^{y(d)} \frac{dx}{(1+x^2)^{(n+1)/2}}\Bigg]' \\
&= \frac{c'(d)}{\big(1+c^2(d)\big)^{(n+1)/2}} - \frac{y'(d)}{\big(1+y^2(d)\big)^{(n+1)/2}} \\
&= \frac{1-d^{n-1}}{2(1+d^2)^{(n+1)/2}} - \frac{1+d^{n+1}}{2(1+d^2)^{(n+1)/2}} y'(d) \\
&= \frac{1}{2} \frac{1}{(1+d^2)^{(n+1)/2}} [1-d^{n-1} - (1+d^{n+1})y'(d)] \\
&= \frac{1}{2} \frac{1}{(1+d^2)^{(n+1)/2}} \left[1-d^{n-1} - (1+d^{n+1}) \frac{1+y^2(d)}{1+d^{n+1}} \frac{d}{y(d)} \frac{1-d^{n-1}}{1+d^2}\right] \\
&= \frac{1}{2} \frac{1-d^{n-1}}{(1+d^2)^{(n+1)/2}} \left[1 - \frac{1+y^2(d)}{1+d^2} \frac{d}{y(d)}\right] \\
&= \frac{1}{2} \frac{(1-d^{n-1})\big(y(d)-d\big)\big(1-dy(d)\big)}{(1+d^2)^{(n+1)/2} y(d)(1+d^2)} > 0
\end{aligned}$$

since $y(0) = 2^{2/(n+1)} - 1 < 1$ for $n > 1$, $y(1) = 1$, and for $0 < d < 1$ the function $y(d)$ is increasing, so that $y(d) < 1$ for $0 < d < 1$. Moreover, for $0 < d < 1$ we have

$$\frac{1+d^{n+1}}{(1+d^2)^{(n+1)/2}} = \frac{1+1}{\big(1+y^2(d)\big)^{(n+1)/2}},$$

and hence

$$\frac{1+d^{n+1}}{1+1} = \left(\frac{1+d^2}{1+y(d)^2}\right)^{(n+1)/2} < 1,$$

which implies that $y(d) > d$ for $0 < d < 1$, and this justifies the claim that the derivative above is positive for $0 < d < 1$. Since $c(1) = y(1) = 1$, we obtain

$$\int\limits_0^{c(d)} \frac{dx}{(1+x^2)^{(n+1)/2}} \leqslant \int\limits_0^{y(d)} \frac{dx}{(1+x^2)^{(n+1)/2}},$$

which implies that for $d \in (0,\, 1)$ the inequality $c(d) \leqslant y(d)$ holds true. This, however, means that for $0 \leqslant d \leqslant 1$

$$\frac{1+d^{n+1}}{(1+d^2)^{(n+1)/2}} = \frac{2}{\big(1+y^2(d)\big)^{(n+1)/2}} \leqslant \frac{2}{\big(1+c^2(d)\big)^{(n+1)/2}},$$

which completes the proof. ■

The comparison of the perimeters of $(-c,\, c)$ and $\big(-\infty, g(-c,c)\big)$ is a much more difficult problem. We state the corresponding result in the next lemma.

LEMMA 5.5. *Suppose that* $\nu(-c,\, c) = \nu\big(-\infty,\, h(c)\big)$ *with* $h(c) = g(-c,c)$. *Then for* $c < c_0 = \sqrt{2^{2/(n+1)} - 1}$ *we have*

$$per\big(-\infty,\, h(c)\big) < per(-c,\, c), \tag{5.5}$$

while for $c > c_1 = 2^{1/(n+1)}$ *the reverse inequality holds. The change of the relation between perimeters occurs at the unique point* $c^* \in [c_0,\, c_1)$, *being a solution of the following equation*:

$$h^2(c) + 1 = 2^{-2/(n+1)}\,(c^2+1). \tag{5.6}$$

P r o o f. Observe that if $c < c_0$ with $c_0^2 + 1 = 2^{2/(n+1)}$, then for $c < c_0$ we obtain

$$\frac{1}{\big(1+h^2(c)\big)^{(n+1)/2}} \;<\; 1 \;<\; \frac{2}{(1+c^2)^{(n+1)/2}}. \tag{5.7}$$

Thus, for $c < c_0$ the relation (5.5) is satisfied.

We now define for $c \geqslant c_0$ an auxiliary function $z := z(c) \geqslant 0$ by the equation

$$\frac{2}{(1+c^2)^{(n+1)/2}} \;=\; \frac{1}{\big(1+z(c)^2\big)^{(n+1)/2}}. \tag{5.8}$$

We obtain

$$z'(c) = \frac{c}{z(c)}\,\frac{1+z^2(c)}{1+c^2}, \qquad \frac{h'(c)}{\big(1+h^2(c)\big)^{(n+1)/2}} = \frac{2}{(1+c^2)^{(n+1)/2}}. \tag{5.9}$$

Define

$$\kappa(c) = \int\limits_{-\infty}^{h(c)} \frac{dx}{(1+x^2)^{(n+1)/2}} - \int\limits_{-\infty}^{z(c)} \frac{dx}{(1+x^2)^{(n+1)/2}}.$$

We have

$$\kappa'(c) = \frac{2}{(1+c^2)^{(n+1)/2}} - \frac{2}{(1+c^2)^{(n+1)/2}}\, z'(c)$$
$$= \frac{2}{(1+c^2)^{(n+1)/2}}\, \frac{\big(z(c)-c\big)\big(1-cz(c)\big)}{z(c)(1+c^2)}$$

and the above expression is positive iff $c > c_1 = 2^{1/(n+1)}$. Indeed, by the definition of the function $z(c)$ we obtain

$$z^2(c) - c^2 = -(2^{2/(n+1)} - 1)\big(z^2(c)+1\big) < 0,$$

so $z = z(c) < c$. Furthermore,

$$2^{2/(n+1)}\big(z^2(c)c^2 - 1\big) = c^4 + (1 - 2^{2/(n+1)})c^2 - 2^{2/(n+1)},$$

so $z^2(c)c^2 - 1 > 0$ iff $c > c_1 = 2^{1/(n+1)}$ and the function $\kappa(c)$ is increasing for $c > c_1$ and decreasing for $c_0 < c < c_1$. Since $\lim_{c\to\infty} h(c) = \infty = \lim_{c\to\infty} z(c)$, we have $\lim_{c\to\infty} \kappa(c) = 0$, so $\kappa(c) < 0$ for $c \geqslant c_1$ and $h(c) < z(c)$ for $c \geqslant c_1$. Since $c_1 > 1$, by Lemma 5.1 we have $h(c_1) > 0$, so we obtain for $c > c_1$:

$$\frac{1}{\big(1+h(c)^2\big)^{(n+1)/2}} > \frac{1}{\big(1+z(c)^2\big)^{(n+1)/2}} = \frac{2}{(1+c^2)^{(n+1)/2}}.$$

We have to consider two cases. Assume first that $h(c_0) \geqslant 0$. Since $z(c_0) = 0$, we have $\kappa(c_0) \geqslant 0$. On the other hand, we have shown that $\kappa(c_1) < 0$. Since the function $\kappa(c)$ is monotonic on the interval (c_0, c_1), there is a unique point $c^* \in [c_0, c_1)$ at which $\kappa(c^*) = 0$. But this means that $h(c^*) = z(c^*)$ and this point is unique in $[c_0, \infty)$ since κ is negative in (c_1, ∞). The equation $h(c^*) = z(c^*)$ shows that c^* is a solution of the equation (5.6), so the proof of this case is complete.

Now, let $h(c_0) < 0$ and let $\tilde{c}$ be such that $h(\tilde{c}) = 0$. Then $|h(\tilde{c})|$ decreases monotonically to zero as $c \to \tilde{c}$, $c \in (c_0, \tilde{c})$. Therefore, $\big(1+h^2(c)\big)^{-(n+1)/2}$ increases to one while the expression $\big(1+z^2(c)\big)^{-(n+1)/2}$ decreases with the value at $c = c_0$ equal to one. Hence, there is a unique $c^* \in (c_0, \tilde{c})$ such that $|h(c^*)| = z(c^*)$. For $c \in (c_0, c^*)$ we obtain

$$\frac{1}{\big(1+h^2(c)\big)^{(n+1)/2}} < \frac{1}{\big(1+z^2(c)\big)^{(n+1)/2}} = \frac{2}{(1+c^2)^{(n+1)/2}},$$

while for $c \in (c^*, \tilde{c})$ we obtain the reverse inequality. Moreover, $\kappa(\tilde{c}) < 0$ and $\kappa'(c) < 0$ for $c \in (c_0, c_1)$, so $\kappa(c) < 0$ for all $c \in (\tilde{c}, c_1)$, hence also for $c \in (\tilde{c}, \infty)$. But this proves that the point c^* is unique and that it has all the required properties, which completes the proof. ■

We now summarize the results of this section in the following theorem:

THEOREM 5.1. *Let* ν *have the density* $f(x) = A_{\alpha,n}/(1+\alpha^2+x^2)^{(n+1)/2}$. *Then*:

• *Among all convex sets with fixed measure* ν *only intervals* $(-c, c)$, $(-1/d, d)$ *or half-lines* $(-\infty, g)$ *are of extremal perimeter.*

• *We have* $\nu(-1/d, d) > 1/2$ *for all* $d > 0$ *and the interval* $(-1/d, d)$ *is of minimal perimeter among all intervals* (a, b) *and half-lines* $(-\infty, g)$ *with fixed measure* ν *greater than* $1/2$.

• *If* $\nu(-c, c) = \nu\big(-\infty, h(c)\big)$, *then for* $0 < c < c^*$, *with* c^* *defined by the equation* (5.6), *we have*

$$per\big(-\infty, h(c)\big) < per(-c, c),$$

while for $c > c^*$ *we have the reverse inequality*

$$per\big(-\infty, h(c)\big) > per(-c, c).$$

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