ASYMPTOTIC PROPERTIES OF GPH ESTIMATORS OF THE MEMORY PARAMETERS OF THE FRACTIONALLY INTEGRATED SEPARABLE SPATIAL ARMA (FISSARMA) MODELS

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Abstract. In this article, we first extend Theorem 2 of Robinson [11] from one dimension to two dimensions. Then the theoretical asymptotic properties of the means, variances, covariance and MSEs of the regression/GPH (GPH states for Geweke and Porter-Hudak’s) estimators of the memory parameters of the FISSARMA model are established. We also performed simulations to study MSE and covariances for finite sample sizes. We found that through the simulation study the MSE values of the memory parameters tend to the theoretical MSE values as the sample size increases. It is also found that \( m^{1/2}(d_1 - d_1) \) and \( m^{1/2}(d_2 - d_2) \) are independent and identically distributed as \( N(0, \pi^2/24) \), when \( m = o(n^{4/5}) \) and \( \ln^2 n = o(m) \).


Key words and phrases: Spatial processes, FISSARMA models, asymptotic properties, GPH estimators, long-memory parameters.

1. INTRODUCTION

Many random phenomena are observed over a region. For instance, air temperature, rainfall, and fertility of soil, to name just a few. Whenever observations are made over a region, they may display spatial correlation, and it is therefore important to take this fact into consideration when analyzing spatial data. More importantly, spatial modelling becomes significant, and in this respect various models have been introduced from time to time. Spatial models on lattice are like the SAR, CAR, MA, spatial ARMA models, etc. These models take into consideration the spatial correlation in one way or the other. At times the spatial correlation structure might exhibit long-memory patterns, and by including an index parameter into existing spatial models (FISSAR, GENSSAR), various types of correlation structures can be produced. This in turn would assist a data analyst to model spatial data with numerous types of correlation structure.

The autocorrelation function of the long-memory processes decays rather slowly. The long-memory processes in area of time series are modelled by fractionally
integrated ARMA (ARFIMA) models (see [3] and [4]). Boissy et al. [2] extended
the long-memory concept from time series to the spatial context and introduced
the fractional autoregressive model and established the strong consistency of Whittle’s
estimator for the parameters of the model. Independently, Shitan [13] considered
the same model and termed it “Fractionally Integrated Separable Spatial Autore-
gressive” (FISSAR) model and proposed a regression estimation method for esti-
timation of the memory parameters in terms of the log-periodogram. Ghodsi and
Shitan [6] compared the regression and Whittle’s estimations of memory param-
eters by simulation study. For the values considered in that study, they found that
the regression method of estimation was better when compared with the Whittle
estimator in the sense that it had smaller root mean squared errors (RMSE) values.
Beran et al. [1] introduced the FISSARMA($p_1$, $d_1$, $q_1$)$\times$(p2, $d_2$, q2) model and de-
rived the asymptotic distribution of the least squares estimators of its parameters.
Guo et al. [8] showed that the Whittle estimators of the memory parameters of the
general spatial fractional ARMA model are consistent and asymptotically normal.

The regression method of estimating memory parameters seems to be useful
because it does not require any prior knowledge of other model parameters. The
asymptotic properties of regression estimator for the memory parameter of a long-
memory ARFIMA models in one dimension were extensively explored by Robin-
son [lli] and [12] and Hurvich et al. [H]. The study of log-periodogram regression
for general long-memory spatial processes seems to be lacking in the literature.
Wang [14] derived the asymptotic properties of the mean and variance of Geweke
and Porter-Hudak’s (GPH) estimator of the memory parameter of d-dimensional
isotropic long-memory random fields with spectral density function as

$$f(\omega_1, \omega_2) = \left( \sum_{k=1}^{d} |1 - e^{-i \omega_k}|^2 \right)^{-\alpha} f^*(\omega_1, \ldots, \omega_d),$$

where $\alpha$ is the memory parameter. In this paper we derive some asymptotic prop-
ties of log-periodogram regression of FISSARMA models in two dimensions as
defined in (1.1). Note that, in the model considered by Wang, the long memory in
all directions is the same, but in our model is not. It is also obvious that the spectral
function of Wang’s model is different from the spectral function of the FISSARMA
models defined in (1.2).

The stationary fractionally integrated separable spatial ARMA processes
(FISSARMA($p_1$, $d_1$, $q_1$)$\times$(p2, $d_2$, q2)) on a two-dimensional regular lattice {$X_{ij}$,
i, j $\in$ $\mathbb{Z}$} are defined as follows:

$$\Phi(B_1, B_2)(1 - B_1)^{d_1}(1 - B_2)^{d_2}X_{ij} = \Theta(B_1, B_2)Z_{ij},$$

where $B_1$ and $B_2$ are the usual backward shift operators acting in the $i$th and $j$th
directions, respectively, i.e., $B_1^X X_{ij} = X_{i-k,j}, B_2^X X_{ij} = X_{i,j-l}, -0.5 < d_1, d_2 <
0.5, and \{Z_{ij}\} is a two-dimensional Gaussian white noise process with mean zero
and variance $\sigma^2$, and

\[
\Phi(B_1, B_2) = \Phi_1(B_1)\Phi_2(B_2), \\
\Theta(B_1, B_2) = \Theta_1(B_1)\Theta_2(B_2),
\]

where

\[
\Phi_1(z) = 1 - \sum_{j=1}^{p_1} \phi_{1j} z^j, \\
\Phi_2(z) = 1 - \sum_{j=1}^{p_2} \phi_{2j} z^j, \\
\Theta_1(z) = 1 + \sum_{j=1}^{q_1} \theta_{1j} z^j, \\
\Theta_2(z) = 1 + \sum_{j=1}^{q_2} \theta_{2j} z^j,
\]

and the roots of the polynomials $\Phi_i$ and $\Theta_i$ ($i = 1, 2$) are outside the unit circle.

The spectral function of this model is given by

\[
|1 - e^{-i\omega_1}|^{-2d_1} |1 - e^{-i\omega_2}|^{-2d_2} f^*(\omega_1, \omega_2),
\]

where $\omega_1, \omega_2 \in [-\pi, \pi] \setminus \{0\}$ and $f^*$ is the spectral function of the standard separable spatial ARMA (SSARMA) model determined by

\[
f^* (\omega_1, \omega_2) = \frac{\sigma^2}{4\pi^2} \left| \frac{\Theta_1(e^{-i\omega_1})}{\Phi_1(e^{-i\omega_1})} \right|^2 \left| \frac{\Theta_2(e^{-i\omega_2})}{\Phi_2(e^{-i\omega_2})} \right|^2,
\]

which can be rewritten as

\[
f^*(\omega_1, \omega_2) = f_1^*(\omega_1) f_2^*(\omega_2) / \sigma^2,
\]

where $f_1^*$ and $f_2^*$ are spectral functions of the ARMA($p_1, q_1$) and ARMA($p_2, q_2$) models in time series, respectively. $f_1^*$ and $f_2^*$ are even, positive, continuous on $[-\pi, \pi]$, bounded above and bounded away from zero with $f_1^*(0) = f_2^*(0) = 0$, and second and third derivatives of $f_1^*$ and $f_2^*$ are bounded in a neighborhood of zero.

Let $X_{1,1}, \ldots, X_{1,n_2}, X_{2,1}, \ldots, X_{2,n_2}, \ldots, X_{n_1,1}, \ldots, X_{n_1,n_2}$ be the random sample on a regular lattice. The periodogram in the two-dimensional case is given by the formula

\[
I_{n_1,n_2}(\omega_1, \omega_2) = \frac{1}{4\pi^2 n_1 n_2} \left| \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} X_{k,l} e^{i(k\omega_1 + l\omega_2)} \right|^2.
\]

The article is organized as follows. First we extend Theorem 2 of Robinson [11] for the FISSARMA models, then we establish the theoretical asymptotic properties of the means, variances, covariance and MSEs of the regression or GPH estimators of $\hat{d}_1$ and $\hat{d}_2$ for the FISSARMA models (1.1) with spectral function as in (1.2). In Section 3, we assess the accuracy of our asymptotic theory on the MSE for small sample sizes by simulation, and finally, in Section 4, the conclusions are drawn.
2. MAIN RESULTS

Let $\omega_{1,j_1} = 2\pi j_1/n_1$ and $\omega_{2,j_2} = 2\pi j_2/n_2$, where $j_k = -m_k, \ldots, m_k$ for $k = 1, 2$ and $m_k$ is a positive integer which tends to infinity slower than $n_k$ (where $m_k$ can be equal to $\sqrt{n_k}$ as suggested by Geweke and Porter-Hudak [3]), and suppose $I_{j_1,j_2}$ and $f^*_{j_1,j_2}$ denote $I_{n_1,n_2}(\omega_1, \omega_2)$ and $f^*(\omega_1, \omega_2)$ evaluated at $\omega_1 = \omega_{1,j_1}$ and $\omega_2 = \omega_{2,j_2}$, respectively.

Taking the logarithm of the spectral function of the FISSARMA model defined in equation (2.2) and evaluating at the points $\omega_1 = \omega_{1,j_1}$ and $\omega_2 = \omega_{2,j_2}$, after some algebraic manipulation, we obtain the multiple regression equation

\begin{equation}
\ln I_{j_1,j_2} = \ln f^*(0, 0) - \gamma - 2d_1 x_{1,j_1} - 2d_2 x_{2,j_2} + \ln \frac{f_{j_1,j_2}^*}{f_{0,0}^*} + \varepsilon_{j_1,j_2},
\end{equation}

where $x_{1,j_1} = \ln|1 - e^{-i\omega_{1,j_1}}|$, $x_{2,j_2} = \ln|1 - e^{-i\omega_{2,j_2}}|$, $\varepsilon_{j_1,j_2} = \ln(I_{j_1,j_2}/f_{j_1,j_2}^*) + \gamma$, $f_{j_1,j_2}^* = f(\omega_{1,j_1}, \omega_{2,j_2})$ defined in (1.2), and $\gamma = 0.577216 \ldots$ is Euler’s constant. Ghodsi and Shitan [7] showed that the ‘errors’, $\varepsilon_{j_1,j_2}$’s, are not independent and identically distributed and $\lim_{n \to \infty} E(\varepsilon_{j_1,j_2})$ depends on $j_1, j_2$.

The regression (or GPH) estimators of $d_1$ and $d_2$ can be obtained as follows by using the least squares method:

\begin{equation}
\hat{d}_1 = -\frac{\sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} (x_{1,j_1} - \bar{x}_1) \ln I_{j_1,j_2}}{2m_2 \sum_{j_1=1}^{m_1} (x_{1,j_1} - \bar{x}_1)^2},
\end{equation}

\begin{equation}
\hat{d}_2 = -\frac{\sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} (x_{2,j_2} - \bar{x}_2) \ln I_{j_1,j_2}}{2m_1 \sum_{j_2=1}^{m_2} (x_{2,j_2} - \bar{x}_2)^2},
\end{equation}

where $\bar{x}_1 = \frac{1}{m_1} \sum_{j_1=1}^{m_1} x_{1,j_1}$ and $\bar{x}_2 = \frac{1}{m_2} \sum_{j_2=1}^{m_2} x_{2,j_2}$. Since $f(-\omega) = f(\omega)$, we have $f(-\omega_1, -\omega_2) = f(\omega_1, \omega_2) = f(\omega_1, \omega_2)$. So, we consider only the positive values for $j_1$ and $j_2$, i.e., $j_k = 1, 2, \ldots, m_k$ for $k = 1, 2$.

Using (2.1), we can obtain

\begin{equation}
\ln I_{j_1,j_2} = -2d_1 x_{1,j_1} - 2d_2 x_{2,j_2} + \ln \frac{f_{j_1,j_2}^*}{f_{0,0}^*} + \varepsilon_{j_1,j_2} - \gamma;
\end{equation}

putting (2.3) into (2.2), defining $a_{1,j_1} = x_{1,j_1} - \bar{x}_1$ and $a_{2,j_2} = x_{2,j_2} - \bar{x}_2$ and noting that $\sum_{j_1=1}^{m_1} a_{1,j_1} = 0$ and $\sum_{j_2=1}^{m_2} a_{2,j_2} = 0$, we get

\begin{equation}
\hat{d}_1 - d_1 = -\frac{1}{2S_{x_1}} \sum_{j_1=1}^{m_1} a_{1,j_1} \ln f_{j_1,j_1}^* - \frac{1}{2m_2 S_{x_1}} \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} a_{1,j_1} \varepsilon_{j_1,j_2},
\end{equation}

\begin{equation}
\hat{d}_2 - d_2 = -\frac{1}{2S_{x_2}} \sum_{j_2=1}^{m_2} a_{2,j_2} \ln f_{j_2,j_2}^* - \frac{1}{2m_1 S_{x_2}} \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} a_{2,j_2} \varepsilon_{j_1,j_2},
\end{equation}

where $S_{x_1} = \sum_{j_1=1}^{m_1} a_{1,j_1}^2$ and $S_{x_2} = \sum_{j_2=1}^{m_2} a_{2,j_2}^2$. As $m_1, m_2 \to \infty$, the entries of the estimation bias approach zero.
2.1 Since we need the following lemmas. In Lemma 2.1, we extend Theorem 2 in [11] for the ARFIMA model in one dimension to the FISSARMA model defined in (1.1). Under Condition A, when \( n_1 = n_2 = n \) and \( m_1 = m_2 = m \), we have

\[
S_{x_1,x_1} = \sum_{j_1=1}^{m_1} a_{1,j_1}^2 = \sum_{j_1=1}^{m_1} x_{1,j_1} a_{1,j_1}, \quad S_{x_2,x_2} = \sum_{j_2=1}^{m_2} a_{2,j_2}^2 = \sum_{j_2=1}^{m_2} x_{2,j_2} a_{2,j_2},
\]

and \( f_{1,j_1}^* = f_1^*(\omega_{1,j_1}) \) and \( f_{2,j_2}^* = f_2^*(\omega_{2,j_2}) \).

To derive the asymptotic properties of GPH estimators in Theorem 2.1 below, we assume that the process (1.1) is Gaussian and that the following condition holds true.

**CONDITION A.** We have:
\[
m_1/n_1 \to 0 \text{ and } (m_1 \ln m_1)/n_1 \to 0 \text{ as } m_1, n_1 \to \infty,
\]
\[
m_2/n_2 \to 0 \text{ and } (m_2 \ln m_2)/n_2 \to 0 \text{ as } m_2, n_2 \to \infty.
\]

**THEOREM 2.1.** Suppose that \( \hat{d}_1 \) and \( \hat{d}_2 \) are the regression (GPH) estimators of memory parameters \( \alpha_1 \) and \( \alpha_2 \) of the FISSARMA model defined in (1.1). Under Condition A, when \( n_1 = n_2 = n \) and \( m_1 = m_2 = m \), we have

\[
\begin{align*}
\text{(a)} & \quad \mathbb{E}(\hat{d}_1 - d_1) = \frac{-2\pi^2 f_1''(0) m^2}{9 f_1'(0) n^2} + o\left( \frac{m^2}{n^2} \right) + O\left( \frac{\ln^3 m}{m} \right), \\
\text{(b)} & \quad \mathbb{E}(\hat{d}_2 - d_2) = \frac{-2\pi^2 f_2''(0) m^2}{9 f_2'(0) n^2} + o\left( \frac{m^2}{n^2} \right) + O\left( \frac{\ln^3 m}{m} \right), \\
\text{(c)} & \quad \text{Var}(\hat{d}_1) = \text{Var}(\hat{d}_2) = \frac{\pi^2}{24m^2} + o\left( \frac{1}{m^2} \right) + O\left( \frac{\ln^{14} m}{m^2} \right), \\
\text{(d)} & \quad \text{Cov}(\hat{d}_1, \hat{d}_2) = o\left( \frac{1}{m^2} \right) + O\left( \frac{\ln^{14} m}{m^2} \right).
\end{align*}
\]

**COROLLARY 2.1.** Since \( f_1''(0) \), \( f_1'(0) \) and \( f_2''(0) \), \( f_2'(0) \) depend on the parameters of the ARMA \((p_1, q_1)\) and ARMA \((p_2, q_2)\) models, respectively, \( \mathbb{E}(\hat{d}_1 - d_1) \) and \( \mathbb{E}(\hat{d}_2 - d_2) \) also depend on them, respectively.

To prove Theorem 2.1 we need the following lemmas. In Lemma 2.1, we will extend Theorem 2 in [11] for the ARFIMA model in one dimension to the FISSARMA model in two dimensions.

**LEMMA 2.1.** For the stationary FISSARMA model observed on a two-dimensional regular lattice \( \{X_{ij}\} \) of size \( n_1 \times n_2 \) defined in (1.1) we have

\[
\begin{align*}
\text{(a)} & \quad \mathbb{E}\left( \frac{I_{j_1,j_2}}{f_{j_1,j_2}} \right) = \mathbb{E}\left( \frac{J_{j_1,j_2} J_{j_1,j_2}}{f_{j_1,j_2} f_{j_1,j_2}} \right) = 1 + O\left( \max\left\{ \frac{\ln j_1}{j_1}, \frac{\ln j_2}{j_2} \right\} \right), \\
\text{(b)} & \quad \mathbb{E}\left( \frac{J_{j_1,j_2}^2}{f_{j_1,j_2}} \right) = O\left( \frac{\ln j_1 \ln j_2}{j_1 j_2} \right),
\end{align*}
\]
(c) \[ E \left( \frac{J_{j_1,j_2} J_{k_1,k_2}}{\sqrt{f_{j_1,j_2} f_{k_1,k_2}}} \right) = O \left( \frac{\ln j_1 \ln j_2}{k_1 k_2} \right), \]

(d) \[ E \left( \frac{J_{j_1,j_2} J_{k_1,k_2}}{\sqrt{f_{j_1,j_2} f_{k_1,k_2}}} \right) = O \left( \frac{\ln j_1 \ln j_2}{k_1 k_2} \right), \]

where

\[ J_{j_1,j_2} = \frac{1}{2\pi n_1 n_2} \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} X_{kl} \exp \left( i(k \omega_{1,j_1} + l \omega_{2,j_2}) \right), \]

\[ j_1 = j_1(n_1), \quad j_2 = j_2(n_2), \quad j_1 > k_1, \quad j_2 > k_2, \quad \text{and} \quad j_1/n_1, j_2/n_2 \to 0 \text{ as } n_1, n_2 \to \infty. \]

**Proof.** Using properties of the spectral representation of \{X_{ij}\}, we can show that (see [11])

\[ (2.6) \quad E \left( \frac{J_{j_1,j_2} J_{j_1,j_2}}{f_{j_1,j_2}} \right) = \frac{\pi}{\sqrt{n_1 n_2}} \int_{-\pi}^{\pi} E_{j,k} (\lambda_1) f_1(\lambda_1) d\lambda_1 \int_{-\pi}^{\pi} E_{j,k} (\lambda_2) f_2(\lambda_2) d\lambda_2, \]

where

\[ E_{j,k} (\lambda) = \frac{1}{2\pi n} D_n(\omega_j - \lambda) D_n(\lambda - \omega_k), \]

and \[ D_n(\lambda) = \sum_{s=1}^{n} e^{i\lambda s} \] is the Dirichlet kernel. Note that

\[ E_{j,j} (\lambda) = \frac{1}{2\pi n} |D_n(\lambda - \omega_j)|^2 = K_n(\lambda - \omega_j), \]

where \( K_n(\cdot) \) is the Fejér kernel.

To prove part (a), replacing \( k_1 \) and \( k_2 \) by \( j_1 \) and \( j_2 \), respectively, in (2.6) and using part (a) of Theorem 2 in [11], we obtain

\[ E \left( \frac{J_{j_1,j_2} J_{j_1,j_2}}{f_{j_1,j_2}} \right) = E \left( \frac{J_{j_1} J_{j_1}}{f_{j_1}} \right) E \left( \frac{J_{j_2} J_{j_2}}{f_{j_2}} \right) = 1 + O \left( \frac{\ln j_1}{j_1} \right) \left\{ 1 + O \left( \frac{\ln j_2}{j_2} \right) \right\} = 1 + O \left( \max \left\{ \frac{\ln j_1}{j_1}, \frac{\ln j_2}{j_2} \right\} \right), \]

since \( \ln j < j \) for any \( j > 0 \) implies

\[ \frac{\ln j_1 \ln j_2}{j_1 j_2} < \max \left\{ \frac{\ln j_1}{j_1}, \frac{\ln j_2}{j_2} \right\}. \]

Parts (b), (c) and (d) can be proved similarly. \( \blacksquare \)
Under Condition A we have (see [9])

\begin{align}
(2.7) \quad S_{x_1,x_1} &= m_1 + o(m_1), \quad a_{1,j_1} = O(\ln m_1), \\
(2.8) \quad S_{x_2,x_2} &= m_2 + o(m_2), \quad a_{2,j_2} = O(\ln m_2).
\end{align}

Similarly to Lemma 1 in [10], we obtain the following:

**Lemma 2.2.** Under Condition A, we have

\begin{align}
(2.9) \quad -\frac{1}{2S_{x_1,x_1}} \sum_{j_1=1}^{m_1} a_{1,j_1} \ln f_{1,j_1}^* &= -\frac{2\pi^2}{9} f_1''(0) m_1^2 + o\left(\frac{m_1^2}{n_1^2}\right), \\
(2.10) \quad -\frac{1}{2S_{x_2,x_2}} \sum_{j_2=1}^{m_2} a_{2,j_2} \ln f_{2,j_2}^* &= -\frac{2\pi^2}{9} f_2''(0) m_2^2 + o\left(\frac{m_2^2}{n_2^2}\right).
\end{align}

Now, let \( \alpha_{j_1,j_2,k_1,k_2} = \max \{ |\sigma_{13}|, |\sigma_{14}|, |\sigma_{23}|, |\sigma_{24}| \} \), where \( \sigma_{ij} = \text{Cov}(\nu_i, \nu_j) \) for \( i, j = 1, 2, 3, 4 \), and

\[(\nu_1, \nu_2, \nu_3, \nu_4) = \left( A_{j_1,j_2} \sqrt{f_{j_1,j_2}}, B_{j_1,j_2} \sqrt{f_{j_1,j_2}}, A_{k_1,k_2} \sqrt{f_{k_1,k_2}}, B_{k_1,k_2} \sqrt{f_{k_1,k_2}} \right) \]

with

\[ A_{j_1,j_2} = \frac{1}{2\pi \sqrt{n_1 n_2}} \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} X_{kl} \cos(k \omega_{1,j_1} + l \omega_{2,j_2}), \]

\[ B_{j_1,j_2} = \frac{1}{2\pi \sqrt{n_1 n_2}} \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} X_{kl} \sin(k \omega_{1,j_1} + l \omega_{2,j_2}). \]

In the following lemma we give an asymptotic expression for \( \alpha_{j_1,j_2,k_1,k_2} \).

**Lemma 2.3.** We have

\[ \alpha_{j_1,j_2,k_1,k_2} = O\left( \frac{\ln j_1 \ln j_2}{k_1 \ln k_2} \right) \]

uniformly for \( 1 < k_1 < j_1 \leq m_1 \) and \( 1 < k_2 < j_2 \leq m_2 \).

**Proof.** From the proof of Proposition 3 in [7] we get

\[
\text{E}(J_{j_1,j_2} J_{k_1,k_2}) = \text{E}(A_{j_1,j_2} A_{k_1,k_2} - B_{j_1,j_2} B_{k_1,k_2}) + i \text{E}(A_{j_1,j_2} B_{k_1,k_2} + B_{j_1,j_2} A_{k_1,k_2}) \\
= \text{Cov}(A_{j_1,j_2}, A_{k_1,k_2}) - \text{Cov}(B_{j_1,j_2}, B_{k_1,k_2}) \\
+ i[\text{Cov}(A_{j_1,j_2}, B_{k_1,k_2}) + \text{Cov}(B_{j_1,j_2}, A_{k_1,k_2})].
\]
Then, by the definition of $\sigma_{ij}$, we obtain

$$\frac{1}{f_{j_1,j_2}f_{k_1,k_2}} |E(J_{j_1,j_2}J_{k_1,k_2})|^2$$

$$= \left[ \text{Cov} \left( \frac{A_{j_1,j_2}}{\sqrt{f_{j_1,j_2}}}, \frac{A_{k_1,k_2}}{\sqrt{f_{k_1,k_2}}} \right) - \text{Cov} \left( \frac{B_{j_1,j_2}}{\sqrt{f_{j_1,j_2}}}, \frac{B_{k_1,k_2}}{\sqrt{f_{k_1,k_2}}} \right) \right]^2$$

$$+ \left[ \text{Cov} \left( \frac{A_{j_1,j_2}}{\sqrt{f_{j_1,j_2}}}, \frac{B_{k_1,k_2}}{\sqrt{f_{k_1,k_2}}} \right) + \text{Cov} \left( \frac{B_{j_1,j_2}}{\sqrt{f_{j_1,j_2}}}, \frac{A_{k_1,k_2}}{\sqrt{f_{k_1,k_2}}} \right) \right]^2$$

$$= (\sigma_{13} - \sigma_{24})^2 + (\sigma_{14} + \sigma_{23})^2.$$

Similarly we can show that

$$\frac{1}{f_{j_1,j_2}f_{k_1,k_2}} |E(J_{j_1,j_2}J_{k_1,k_2})|^2 = (\sigma_{13} + \sigma_{24})^2 + (\sigma_{14} - \sigma_{23})^2.$$

Therefore, after some algebraic manipulations we get

$$\frac{1}{2f_{j_1,j_2}f_{k_1,k_2}} \left\{ |E(J_{j_1,j_2}J_{k_1,k_2})|^2 + |E(J_{j_1,j_2}J_{k_1,k_2})|^2 \right\}$$

$$= \sigma_{13}^2 + \sigma_{14}^2 + \sigma_{23}^2 + \sigma_{24}^2 \geq \max \{ |\sigma_{13}|, |\sigma_{14}|, |\sigma_{23}|, |\sigma_{24}| \}$$

$$= \alpha_{j_1,j_2,k_1,k_2}^2.$$

From Lemma 2.1 (parts (c) and (d)) we obtain

$$\frac{1}{2f_{j_1,j_2}f_{k_1,k_2}} \left\{ |E(J_{j_1,j_2}J_{k_1,k_2})|^2 + |E(J_{j_1,j_2}J_{k_1,k_2})|^2 \right\} = O \left( \frac{\ln^2 j_1 \ln^2 j_2}{k_1^2 k_2^2} \right),$$

which completes the proof. 

**Lemma 2.4.** We have $\text{Cov}(\varepsilon_{j_1,j_2}, \varepsilon_{k_1,k_2}) = O(\alpha_{j_1,j_2,k_1,k_2}^2)$ uniformly for $\ln^2 m_1 \leq k_1 < j_1 \leq m_1$ and $\ln^2 m_2 \leq k_2 < j_2 \leq m_2$.

**Proof.** The proof is similar to that of Lemma 2 in [10].

**Lemma 2.5.** We have

$$\lim_{n_1,n_2 \to \infty} \inf_{1 \leq j_1 \leq m_1, 1 \leq j_2 \leq m_2} E \left( \frac{I_{j_1,j_2}}{f_{j_1,j_2}} \right) > 0.$$

**Proof.** From the proof of Proposition 1 in [2] we know that

$$E \left( \frac{I_{j_1,j_2}}{f_{j_1,j_2}} \right) = E \left( \frac{I_{j_1}}{f_{j_1}} \right) E \left( \frac{I_{j_2}}{f_{j_2}} \right);$$

by taking $\lim_{n_1,n_2 \to \infty} \inf_{1 \leq j_1 \leq m_1, 1 \leq j_2 \leq m_2}$ of both sides of this equation and using Lemma 4 in [10] we get the desired result. 


Lemma 2.6. We have \( \lim_{n_1,n_2 \to \infty} \sup_{1 \leq j_1 \leq m_1, 1 \leq j_2 \leq m_2} \mathbb{E} \left( \ln^2 \frac{J_{j_1,j_2}}{J_{j_1,j_2}} \right) < \infty. \)

Proof. The proof is similar to that of Lemma 5 in [14]. \( \blacksquare \)

Corollary 2.2. From Lemmas 2.5 and 2.6 it follows that \( \mathbb{E}(\varepsilon_{j_1,j_2}^2) = O(1), \) and so \( \mathbb{E}(\varepsilon_{j_1,j_2}) = O(1) \) and \( \text{Var}(\varepsilon_{j_1,j_2}) = O(1). \)

Lemma 2.7. Letting \( \gamma_{j_1,j_2} = \max\{ (\ln j_1)/j_1, (\ln j_2)/j_2 \}, \) we have

\[
\mathbb{E}(\varepsilon_{j_1,j_2}) = O(\gamma_{j_1,j_2}) \quad \text{and} \quad \text{Var}(\varepsilon_{j_1,j_2}) = \frac{\pi^2}{6} + O(\gamma_{j_1,j_2})
\]

uniformly for \( \ln^2 m_i \leq j_i \leq m_i, i = 1, 2. \)

Proof. It can be easily shown that (see [7]) \( \varepsilon_{j_1,j_2} = \ln(I_{j_1,j_2}/f_{j_1,j_2}) + \gamma = \ln(\nu_1^2 + \nu_2^2) + \gamma, \) where \( \nu_1 \) and \( \nu_2 \) are defined as in Lemma 2.2. We also have \( J_{j_1,j_2}/f_{j_1,j_2} = \nu_1 + i\nu_2. \) From parts (a) and (b) of Lemma 2.1 we can obtain

\[
\mathbb{E}(\nu_1^2) = \frac{1}{2} + O(\gamma_{j_1,j_2}), \quad \mathbb{E}(\nu_2^2) = \frac{1}{2} + O(\gamma_{j_1,j_2}), \quad \mathbb{E}(\nu_1\nu_2) = O(\gamma_{j_1,j_2}),
\]

and, consequently,

\[
\Sigma^{-1} = 2I_2 + O(\gamma_{j_1,j_2})I_2,
\]

where \( I_2 \) and \( I_2 \) are \( 2 \times 2 \) identity and unit matrices, respectively. Therefore, the asymptotic joint distribution of \( \nu = (\nu_1, \nu_2)' \) is as follows:

\[
f(\nu_1, \nu_2) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp \left( -\frac{\nu'S^{-1}\nu}{2} \right) = \frac{1}{\pi} \exp \left( -(\nu_1^2 + \nu_2^2) - (\nu_1 + \nu_2)^2O(\gamma_{j_1,j_2}) \right) = \frac{1}{\pi} \exp \left( -(\nu_1^2 + \nu_2^2) \right) + O(\gamma_{j_1,j_2}).
\]

The remaining part of the proof is similar to the proof of Lemma 6 in [13]. \( \blacksquare \)

Now, let

\[
T_{i1}^{h(m)} = \sum_{j_1=1}^{h(m)} \sum_{j_2=1}^{h(m)} a_{i,j_1,j_2}, \quad T_{i2}^{h(m)} = \sum_{j_1=1}^{h(m)} \sum_{j_2=1}^{h(m)} a_{i,j_1,j_2}, \\
T_{i3}^{h(m)} = \sum_{j_1=1}^{h(m)} \sum_{j_2=1}^{h(m)} a_{i,j_1,j_2}, \quad T_{i4}^{h(m)} = \sum_{j_1=h(m) + 1}^{h(m) + 1} \sum_{j_2=h(m) + 1}^{h(m) + 1} a_{i,j_1,j_2},
\]

where \( h(m) \) is a function of \( m \) and \( i = 1, 2. \)
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Lemma 2.8. Under Condition A, when \( n_1 = n_2 = n \) and \( m_1 = m_2 = m \), we have

\[
- \frac{1}{2mS_{x_1,x_1}} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} a_{1,j_1} \mathbb{E}(\varepsilon_{j_1,j_2}) = O\left(\frac{\ln^3 m}{m}\right),
\]

(2.12)

\[
- \frac{1}{2mS_{x_2,x_2}} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} a_{2,j_2} \mathbb{E}(\varepsilon_{j_1,j_2}) = O\left(\frac{\ln^3 m}{m}\right).
\]

(2.13)

Proof. We first prove (2.12). By letting \( h(m) = \ln^2 m \) in (2.11) we can write

\[
\frac{1}{2m} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} a_{1,j_1} \mathbb{E}(\varepsilon_{j_1,j_2}) = \frac{4}{2m} \sum_{s=1}^{4} \mathbb{E}(T_{1s}^{(\ln^2 m)}) \leq \sum_{s=1}^{4} \mathbb{E}(T_{1s}^{(\ln^2 m)})
\]

\[
= O(\ln^5 m) + O((\ln^3 m)(m - \ln^2 m)) + O((m - \ln^2 m) \ln^3 m)
\]

\[
+ O((\ln m) \sum_{j_1=(\ln^2 m)+1}^{m} \sum_{j_2=(\ln^2 m)+1}^{m} o(\gamma_{j_1,j_2})),
\]

using (2.7), (2.8), Corollary 2.2 and Lemma 2.7. Since \( \sum_{j=(\ln^2 m)+1}^{m} (\ln j)/j = O(\ln^2 m) \), the last term is equal to

\[
O\left((\ln m)(m - \ln^2 m) \sum_{j_2=(\ln^2 m)+1}^{m} \frac{\ln j_2}{j_2}\right) = O((\ln^3 m)(m - \ln^2 m)) \text{ if } \frac{\ln j_2}{j_2} > \frac{\ln j_1}{j_1},
\]

and is equal to

\[
O\left((\ln m)(m - \ln^2 m) \sum_{j_1=(\ln^2 m)+1}^{m} \frac{\ln j_1}{j_1}\right) = O((\ln^3 m)(m - \ln^2 m)) \text{ if } \frac{\ln j_1}{j_1} > \frac{\ln j_2}{j_2}.
\]

Therefore,

\[
- \frac{1}{2mS_{x_1,x_1}} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} a_{1,j_1} \mathbb{E}(\varepsilon_{j_1,j_2})
\]

\[
\leq \frac{1}{2m^2(1 + o(1))}\{O(\ln^5 m) + O(m \ln^3 m)\} = O\left(\frac{\ln^5 m}{m^2}\right) + O\left(\frac{\ln^3 m}{m}\right) = O\left(\frac{\ln^3 m}{m}\right)
\]

because \( (\ln^5 m)/m < (\ln^2 m)/m \) for any \( m > 0 \). This completes the proof of (2.12). Similarly we can prove (2.13). ∎

Now, using the above lemmas, we can prove Theorem 2.1.
Proof of Theorem 2.1. Parts (a) and (b) follow directly from equations (2.4), (2.5) and Lemmas 2.2 and 2.8.

To prove part (c), by (2.4), (2.5) and (2.11) we can write

\[
\text{(2.14) } \text{Var}(\hat{d}_1) = \frac{1}{4m^2S_{x_1}^2} \text{Var}\left( \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} a_{1,j_1} \varepsilon_{j_1,j_2} \right) \\
= \frac{1}{4m^2S_{x_1}^2} \text{Var}\left( \sum_{s=1}^{4} T_{1s}^{(\ln^6 m)} \right) \\
= \frac{1}{4m^4(1 + o(1))} \left\{ \sum_{s=1}^{4} \text{Var}(T_{1s}^{(\ln^6 m)}) + 2 \sum_{s=1}^{4} \sum_{r=s+1}^{4} \text{Cov}(T_{1s}^{(\ln^6 m)}, T_{1r}^{(\ln^6 m)}) \right\}.
\]

Now, using Corollary 2.2, we have

\[
\text{Var}(T_{11}^{(\ln^6 m)}) = \sum_{j_1=1}^{\ln^6 m} \sum_{j_2=1}^{\ln^6 m} a_{1,j_1}^2 \text{Var}(\varepsilon_{j_1,j_2}) \\
+ \sum_{(j_1,j_2) \neq (k_1,k_2)} \sum_{a_1,j_1} a_{1,j_1} a_{1,k_1} \text{Cov}(\varepsilon_{j_1,j_2}, \varepsilon_{k_1,k_2}) \\
= O(\ln^{14} m) + O\left( \ln^{26} m \sup_{j_1,j_2} \sqrt{\text{Var}(\varepsilon_{j_1,j_2})} \sup_{k_1,k_2} \sqrt{\text{Var}(\varepsilon_{k_1,k_2})} \right) \\
= O(\ln^{26} m) = o(m^2).
\]

Similarly we can obtain

\[
\text{Var}(T_{12}^{(\ln^6 m)}) = O(m^2 \ln^{14} m)
\]

and

\[
\text{Var}(T_{13}^{(\ln^6 m)}) = (\ln^6 m)(m + o(m)) + O(\ln^{26} m) = o(m^2).
\]

Using Lemmas 2.3, 2.4 and 2.7 and noting that

\[
\sum_{j_1=1}^{\ln^6 m+1} \sum_{j_2=1}^{\ln^6 m+1} a_{1,j_1}^2 = (m - \ln^6 m)\left( \sum_{j_1=1}^{\ln^6 m} a_{1,j_1}^2 - \sum_{j_1=1}^{\ln^6 m} a_{1,j_1}^2 \right) \\
= (m - \ln^6 m)(m + o(m) + O(\ln^8 m)) \\
= m^2 + o(m^2) + O(m \ln^8 m) - m \ln^6 m + o(m \ln^6 m) + O(\ln^{14} m) \\
= m^2 + o(m^2),
\]
we get

\[
\text{Var}(T_{14}^{(\ln^6 m)}) = \sum_{j_1=(\ln^6 m)+1}^m \sum_{j_2=(\ln^6 m)+1}^m a_{1,j_1}^2 \text{Var}(\varepsilon_{j_1,j_2})
\]

\[
+ \sum_{(j_1,j_2) \neq (k_1,k_2)} \sum_{j_1=(\ln^6 m)+1}^m \sum_{j_2=(\ln^6 m)+1}^m a_{1,j_1} a_{1,k_1} \text{Cov}(\varepsilon_{j_1,j_2}, \varepsilon_{k_1,k_2})
\]

\[
= \sum_{j_1=(\ln^6 m)+1}^m \sum_{j_2=(\ln^6 m)+1}^m a_{1,j_1}^2 \left( \frac{\pi^2}{6} + O(\gamma_{j_1,j_2}) \right)
\]

\[
+ O((\ln^2 m) \sum_{(j_1,j_2) \neq (k_1,k_2)} \sum_{j_1=(\ln^2 m)+1}^m \sum_{j_2=(\ln^2 m)+1}^m O(\alpha_{j_1,j_2,k_1,k_2})
\]

\[
= \frac{\pi^2 m^2}{6} + o(m^2) + O((\ln^2 m) \sum_{j=(\ln^6 m)+1}^m \sum_{k=j+1}^m \frac{\ln^2 j}{k^2})
\]

\[
= \frac{\pi^2 m^2}{6} + o(m^2) + O((\ln^4 m)(m - \ln^2)) + O((\ln^6 m) \left( \sum_{j=(\ln^6 m)+1}^m \frac{m}{k^2} \right)^2)
\]

\[
= \frac{\pi^2 m^2}{6} + o(m^2) + O((\ln^4 m)(m - \ln^2) m) + O((\ln^6 m) \left( \frac{m}{\ln^6 m} \right)^2)
\]

\[
= \frac{\pi^2 m^2}{6} + o(m^2) + O((\ln^4 m)(m - \ln^2) m) + O((\ln^6 m) \left( \frac{m}{\ln^6 m} \right)^2)
\]

\[
= \frac{\pi^2 m^2}{6} + o(m^2).
\]

To find the covariances in \((\ref{eq:14})\) we note that

\[
\text{Cov}(T_{1s}^{(\ln^6 m)}, T_{1r}^{(\ln^6 m)}) = \sum_{j_1} \sum_{j_2} \sum_{k_1} \sum_{k_2} a_{1,j_1} a_{1,k_1} \text{Cov}(\varepsilon_{j_1,j_2}, \varepsilon_{k_1,k_2})
\]

\[
\leq \sum_{j_1} \sum_{j_2} \sum_{k_1} \sum_{k_2} a_{1,j_1} a_{1,k_1} \sqrt{\text{Var}(\varepsilon_{j_1,j_2}) \sqrt{\text{Var}(\varepsilon_{k_1,k_2})}},
\]

which for \(s, r = 1, 2, 3, 4 (s < r)\) can be calculated by using the Appendix A. Now we can conclude that

\[
\text{Var}(d_1) = \frac{\pi^2}{24m^2} + o \left( \frac{1}{m^2} \right) + O \left( \frac{\ln^{14} m}{m^2} \right) + O \left( \frac{\ln^{26} m^2}{m^4} \right)
\]

\[
+ O \left( \frac{\ln^{20} m}{m^3} \right) + O \left( \frac{\ln^{26} m^4}{m^4} \right) + O \left( \frac{\ln^{14} m^2}{m^2} \right)
\]

\[
= \frac{\pi^2}{24m^2} + o \left( \frac{1}{m^2} \right) + O \left( \frac{\ln^{14} m}{m^2} \right),
\]
which completes the proof of part (c). The proof of part (d) is the same as that of part (c).

Now, since from (2.14), (2.5) and by the notation in (2.11) we have

\[
\text{Cov}(\hat{d}_1, \hat{d}_2) = \frac{1}{4m^2S_{x_1}S_{x_2}} \text{Cov} \left( \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} a_{1,j_1} \varepsilon_{j_1,j_2}, \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} a_{2,j_2} \varepsilon_{j_1,j_2} \right)
\]

\[
= \frac{1}{4m^4(1+o(1))} \text{Cov} \left( \sum_{s=1}^{4} T_{1s}^{l_{m}}, \sum_{s=1}^{4} T_{2s}^{l_{m}} \right)
\]

\[
= \frac{1}{4m^4(1+o(1))} \sum_{s=1}^{4} \sum_{r=1}^{4} \text{Cov}(T_{1s}^{l_{m}}, T_{2r}^{l_{m}}),
\]

in which for \(s = r = 4\), by Lemma 2.4,

\[
\text{Cov}(T_{14}^{l_{m}}, T_{24}^{l_{m}}) = \sum_{j_1=(l_{m})+1}^{\sum_{j_1=1}^{m}} \sum_{j_2=(l_{m})+1}^{\sum_{j_2=1}^{m}} \sum_{k_1=(l_{m})+1}^{\sum_{k_1=1}^{m}} \sum_{k_2=(l_{m})+1}^{\sum_{k_2=1}^{m}} a_{1,j_1} a_{2,k_2} \text{Cov}(\varepsilon_{j_1,j_2}, \varepsilon_{k_1,k_2})
\]

\[
= \sum_{j_1=(l_{m})+1}^{\sum_{j_1=1}^{m}} \sum_{j_2=(l_{m})+1}^{\sum_{j_2=1}^{m}} \sum_{k_1=(l_{m})+1}^{\sum_{k_1=1}^{m}} \sum_{k_2=(l_{m})+1}^{\sum_{k_2=1}^{m}} a_{1,j_1} a_{2,k_2} \text{O}(\alpha_{j_1,j_2,k_1,k_2})
\]

\[
= O \left( \frac{m^2}{\ln^{6} m} \right) = o(m^2),
\]

for \(s \neq r \neq 4\) we can write

\[
\text{Cov}(T_{1s}^{l_{m}}, T_{2r}^{l_{m}}) = \sum_{j_1} \sum_{j_2} \sum_{k_1} \sum_{k_2} a_{1,j_1} a_{2,k_2} \text{Cov}(\varepsilon_{j_1,j_2}, \varepsilon_{k_1,k_2})
\]

\[
\leq \sum_{j_1} \sum_{j_2} \sum_{k_1} \sum_{k_2} a_{1,j_1} a_{2,k_2} \sqrt{\text{Var}(\varepsilon_{j_1,j_2})} \sqrt{\text{Var}(\varepsilon_{k_1,k_2})}.
\]

Using the Appendix A, it can be easily shown that

\[
\text{Cov}(T_{12}^{l_{m}}, T_{22}^{l_{m}}) = \text{Cov}(T_{12}^{l_{m}}, T_{23}^{l_{m}}) = \text{Cov}(T_{14}^{l_{m}}, T_{24}^{l_{m}})
\]

\[
= \text{Cov}(T_{14}^{l_{m}}, T_{23}^{l_{m}}) = O(m^2 \ln^{14} m) + o(m^2),
\]

and \(\text{Cov}(T_{1s}^{l_{m}}, T_{2r}^{l_{m}}) = o(m^2)\) for the remaining values of \(s\) and \(r\). Thus

\[
\text{Cov}(\hat{d}_1, \hat{d}_2) = o \left( \frac{1}{m^2} \right) + O \left( \frac{\ln^{14} m}{m^2} \right).
\]
COROLLARY 2.3. Since the mean squared errors of $\hat{d}_1$ and $\hat{d}_2$,

$$\text{MSE}(\hat{d}_1) = \text{Var}(\hat{d}_1) + \mathbb{E}^2(\hat{d}_1 - d_1)$$

$$= -\frac{4\pi^4}{81} \left( \frac{f_i''(0)}{f_i'(0)} \right)^2 \frac{m^4}{n^4} + \frac{\pi^2}{24m^2}$$

$$+ O\left( \frac{m \ln^3 m}{n^2} \right) + O\left( \frac{\ln^{14} m}{m^2} \right) + o\left( \frac{m^4}{n^4} \right) + o\left( \frac{1}{m^2} \right)$$

and

$$\text{MSE}(\hat{d}_2) = \text{Var}(\hat{d}_2) + \mathbb{E}^2(\hat{d}_2 - d_2)$$

$$= -\frac{4\pi^4}{81} \left( \frac{f_2''(0)}{f_2'(0)} \right)^2 \frac{m^4}{n^4} + \frac{\pi^2}{24m^2}$$

$$+ O\left( \frac{m \ln^3 m}{n^2} \right) + O\left( \frac{\ln^{14} m}{m^2} \right) + o\left( \frac{m^4}{n^4} \right) + o\left( \frac{1}{m^2} \right),$$

tend to zero under Condition A; $\hat{d}_1$ and $\hat{d}_2$ are asymptotically consistent.

By omitting the negligible terms in the mean squared errors of $\hat{d}_1$ and $\hat{d}_2$ and minimizing with respect to $m$, we obtain the theoretical (THR) asymptotically optimal choice for $m$ as follows:

$$m_{\text{THR}} = \left( \frac{27}{128\pi^2} \right)^{1/5} \left( \frac{f_i'(0)}{f_i''(0)} \right)^{2/5} n^{4/5} \quad \text{for } i = 1, 2.$$  

3. NUMERICAL RESULTS

In this section we report the numerical results of our study. We considered the FISSAR(1, 1) model of the form

$$(1 - \phi_{10}B_1)(1 - \phi_{01}B_2)(1 - B_1)^{d_1}(1 - B_2)^{d_2}X_{ij} = Z_{ij},$$

where $|\phi_{10}| < 1, |\phi_{01}| < 1, -0.5 < d_1, d_2 < 0.5$, and $\{Z_{ij}\}$ is a two-dimensional Gaussian white noise process with mean zero and variance $\sigma_z^2 = 1$.

Table 1 shows the theoretical values of the bias, standard deviation (SD), MSE and covariance (Cov) of the GPH estimators of the memory parameters and estimators based on the optimal choice of $m_1$ and $m_2$ mentioned in (2.15) (termed as THR) using Theorem 2.1 and Corollary 2.3 by omitting the negligible terms. We considered $m_1 = m_2 = \sqrt{n}$ as Geweke and Porter-Hudak [5] proposed in the one-dimensional case. The values for $(\phi_{10}, \phi_{01}, d_1, d_2)$ were $(i) = (0.1, 0.7, 0.2, 0.2)$ and $(ii) = (0.3, 0.3, 0.1, 0.4)$. For each of these two processes we calculated the characteristics mentioned above for four sample sizes: $n = 50, 100, 200, 300$. Table 2 shows the simulated values. For simulation study we generated 1000 realizations of FISSAR(1, 1) model using the method mentioned in [6].
Table 1. Theoretical results: the bias, SD, MSE and covariance of \( \hat{d}_1 \) and \( \hat{d}_2 \) by the THR and GPH methods for the FISSAR(1, 1) model for two sets of parameters
(i): \((\phi_{10}, \phi_{01}, d_1, d_2) = (0.1, 0.7, 0.2, 0.2)\) and (ii): \((\phi_{10}, \phi_{01}, d_1, d_2) = (0.3, 0.3, 0.1, 0.4)\)

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<tr>
<td></td>
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<td>THR</td>
<td>10</td>
<td>10</td>
<td>0.107</td>
<td>0.107</td>
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<tr>
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<td>17</td>
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<td>0.008</td>
<td>0.037</td>
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Note that \( f^*_1(\omega_1) \) and \( f^*_2(\omega_2) \) for the FISSAR(1, 1) model used in Theorem 2.1, Corollary 2.3 and equation (2.15) are given as:

\[
 f^*_1(\omega_1) = \frac{\sigma^2}{2\pi} \frac{1}{1 + \phi_{10}^2 - 2\phi_{10}\cos(\omega_1)}, \\
 f^*_2(\omega_2) = \frac{\sigma^2}{2\pi} \frac{1}{1 + \phi_{01}^2 - 2\phi_{01}\cos(\omega_2)}.
\]

According to Theorem 2.1, Corollaries 2.1 and 2.3 and equations (2.15), (3.1) and (3.2), the value of each of \( m_1 \), Bias(\( \hat{d}_1 \)), SE(\( \hat{d}_1 \)) and MSE(\( \hat{d}_1 \)), depends on the value of \( \phi_{10} \), and the value of each of \( m_2 \), Bias(\( \hat{d}_2 \)), SE(\( \hat{d}_2 \)) and MSE(\( \hat{d}_2 \)), depends on the value of \( \phi_{01} \). This can be seen in Tables 1 and 2. Although the values of \( d_1 \) and \( d_2 \) are equal in (\( \phi_{10}, \phi_{01}, d_1, d_2 \) = (0.1, 0.7, 0.2, 0.2), the values of bias, SD and MSE of \( \hat{d}_2 \) are greater than those of \( \hat{d}_1 \), due to the value of \( \phi_{01} \) which is greater than the value of \( \phi_{10} \). It can also be seen that the simulated values of MSEs are less than the theoretical values for small \( m_1 \) and \( m_2 \). These values are approximately equal for large
Table 2. Simulation results: the bias, SD, MSE and covariance of \( \hat{d}_1 \) and \( \hat{d}_2 \) by the THR and GPH methods for the FISSAR(1, 1) model for two sets of parameters

(i): \((\phi_{10}, \phi_{01}, d_1, d_2) = (0.1, 0.7, 0.2, 0.2)\) and (ii): \((\phi_{10}, \phi_{01}, d_1, d_2) = (0.3, 0.3, 0.1, 0.4)\)

<table>
<thead>
<tr>
<th>Set</th>
<th>n</th>
<th>Method</th>
<th>Bias</th>
<th>SD</th>
<th>MSE</th>
<th>Cov</th>
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<tr>
<td></td>
<td></td>
<td></td>
<td>( d_1 )</td>
<td>( d_2 )</td>
<td>( d_1 )</td>
<td>( d_2 )</td>
</tr>
<tr>
<td>(i)</td>
<td>50</td>
<td>THR</td>
<td>0.023</td>
<td>0.197</td>
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<tr>
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<td>0.244</td>
<td>0.107</td>
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<td></td>
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<td>THR</td>
<td>0.042</td>
<td>0.170</td>
<td>0.041</td>
<td>0.052</td>
</tr>
<tr>
<td></td>
<td></td>
<td>GPH</td>
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<td>0.244</td>
<td>0.068</td>
<td>0.041</td>
</tr>
<tr>
<td></td>
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<td>THR</td>
<td>0.039</td>
<td>0.134</td>
<td>0.021</td>
<td>0.025</td>
</tr>
<tr>
<td></td>
<td></td>
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<td>0.174</td>
<td>0.043</td>
<td>0.042</td>
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<tr>
<td></td>
<td>300</td>
<td>THR</td>
<td>0.034</td>
<td>0.105</td>
<td>0.014</td>
<td>0.018</td>
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<td></td>
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<td>GPH</td>
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<td>GPH</td>
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<td>0.012</td>
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<td>0.052</td>
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<tr>
<td></td>
<td>200</td>
<td>THR</td>
<td>0.065</td>
<td>0.069</td>
<td>0.018</td>
<td>0.015</td>
</tr>
<tr>
<td></td>
<td></td>
<td>GPH</td>
<td>0.021</td>
<td>0.024</td>
<td>0.043</td>
<td>0.038</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>THR</td>
<td>0.052</td>
<td>0.059</td>
<td>0.013</td>
<td>0.012</td>
</tr>
<tr>
<td></td>
<td></td>
<td>GPH</td>
<td>0.012</td>
<td>0.021</td>
<td>0.034</td>
<td>0.034</td>
</tr>
</tbody>
</table>

From Tables 1 and 2 we can also see that the MSE decreases when the grid size increases for both theoretical and simulated values.

In both theoretical and simulation studies, the biases and the MSEs of \( \hat{d}_2 \) obtained by the THR method are less than those obtained by the GPH method when \((\phi_{10}, \phi_{01}, d_1, d_2) = (0.1, 0.7, 0.2, 0.2)\), these differences decrease when \( n \) increases. Note that in this case the \( \phi_{01} \) value is large. The MSEs of \( \hat{d}_1 \) when \((\phi_{10}, \phi_{01}, d_1, d_2) = (0.1, 0.7, 0.2, 0.2)\) and the MSEs of \( \hat{d}_1 \) and \( \hat{d}_2 \) when \((\phi_{10}, \phi_{01}, d_1, d_2) = (0.3, 0.3, 0.1, 0.4)\) in both THR and GPH methods are almost equal, but the biases in the THR method are greater and the SDs are smaller.

In the theoretical case, the bias, SD and MSE of \( \hat{d}_1 \) and \( \hat{d}_2 \) when \( \phi_{10} = \phi_{01} \) are equal and do not depend on the values of \( \hat{d}_1 \) and \( \hat{d}_2 \). In simulation, this happens when \( n \) is large.

Finally, from Tables 1 and 2 it is easy to see that there is an agreement between the theoretical and simulated covariance of \( \hat{d}_1 \) and \( \hat{d}_2 \).
n = 300. From Figure 1 it can be seen that the biases of the THR estimators are greater than the biases of the GPH estimators. However, the standard deviations of the THR estimators are smaller than the standard deviations of the GPH estimators.

Figure 2 shows Q–Q plots of the bias of \( \hat{d}_1 \) and \( \hat{d}_2 \) by (a) the THR method and (b) the GPH method when \((\phi_{10}, \phi_{01}, d_1, d_2) = (0.3, 0.3, 0.1, 0.4)\) and \(n = 300\). All tables and figures underscore the suboptimality of GPH estimators.

By Figures 1 and 2 and Tables 1 and 2, we suggest that \( m_{1/2}(\hat{d}_1 - d_1) \) and \( m_{1/2}(\hat{d}_2 - d_2) \) are independent and identically distributed as \( N(0, \pi^2/24) \) when \( m = o(n^{4/5}) \) and \( \ln^2 n = o(m) \).

**Figure 1.** Boxplots of the bias of \( \hat{d}_1 \) and \( \hat{d}_2 \) by (a) the THR method and (b) the GPH method when \((\phi_{10}, \phi_{01}, d_1, d_2) = (0.3, 0.3, 0.1, 0.4)\) and \(n = 300\)

**Figure 2.** Normal Q–Q plots of the bias of \( \hat{d}_1 \) and \( \hat{d}_2 \) by (a) the THR method and (b) the GPH method when \((\phi_{10}, \phi_{01}, d_1, d_2) = (0.3, 0.3, 0.1, 0.4)\) and \(n = 300\)
4. CONCLUSION

In this article, we studied the properties of the regression estimators of the FISSARMA models, in particular we established the asymptotic bias, variance, covariance and MSE of the memory parameters of the model. We also derived the spatial version of Theorem 2 of \cite{11}. Some numerical results have also been provided to verify theoretical results that we obtained. By the numerical results it is found that \( m^{1/2}(d_1 - d_1) \) and \( m^{1/2}(d_2 - d_2) \) are independent and identically distributed as \( N(0, \pi^2/24) \) when \( m = o(n^{4/5}) \) and \( \ln^2 n = o(m) \). Our results contribute to the theory of spatial models, in particular the FISSARMA models.

5. APPENDIX A

To prove Theorem 2.1 we need the following:

\begin{align}
(5.1) \sum_{j_1=1}^{\ln^6 m} \sum_{j_2=1}^{\ln^6 m} a_{1,j_1} \sqrt{\Var(\varepsilon_{j_1,j_2})} &= \sum_{j_1=1}^{\ln^6 m} \sum_{j_2=1}^{\ln^6 m} a_{2,j_2} \sqrt{\Var(\varepsilon_{j_1,j_2})} = O(\ln^{13} m), \\
(5.2) \sum_{j_1=(\ln^6 m)+1}^{\ln^6 m} \sum_{j_2=(\ln^6 m)+1}^{\ln^6 m} a_{1,j_1} \sqrt{\Var(\varepsilon_{j_1,j_2})} &= \sum_{j_1=(\ln^6 m)+1}^{\ln^6 m} \sum_{j_2=(\ln^6 m)+1}^{\ln^6 m} a_{2,j_2} \sqrt{\Var(\varepsilon_{j_1,j_2})} = O\left((m - \ln^6 m) \ln^7 m\right), \\
(5.3) \sum_{j_1=(\ln^6 m)+1}^{\ln^6 m} \sum_{j_2=(\ln^6 m)+1}^{\ln^6 m} a_{1,j_1} \sqrt{\Var(\varepsilon_{j_1,j_2})} &= O\left((\ln^6 m)\left(\sum_{j_1=1}^{\ln^6 m} a_{1,j_1} - \sum_{j_1=1}^{\ln^6 m} a_{1,j_1}\right)\right) = O(\ln^{13} m), \\
(5.4) \sum_{j_1=(\ln^6 m)+1}^{\ln^6 m} \sum_{j_2=(\ln^6 m)+1}^{\ln^6 m} a_{2,j_2} \sqrt{\Var(\varepsilon_{j_1,j_2})} &= O(\ln^{13} m), \\
(5.5) \sum_{j_1=(\ln^6 m)+1}^{\ln^6 m} \sum_{j_2=(\ln^6 m)+1}^{\ln^6 m} a_{1,j_1} \sqrt{\Var(\varepsilon_{j_1,j_2})} &= O\left(\sum_{j_1=(\ln^6 m)+1}^{\ln^6 m} \sum_{j_2=(\ln^6 m)+1}^{\ln^6 m} a_{1,j_1} \sqrt{\pi^2/6 + O(\gamma_{j_1,j_2})}\right) \\
&= O\left(\sum_{j_1=(\ln^6 m)+1}^{\ln^6 m} \sum_{j_2=(\ln^6 m)+1}^{\ln^6 m} a_{1,j_1} (1 + O(\gamma_{j_1,j_2}))\right) \\
&= O\left(\sum_{j_1=(\ln^6 m)+1}^{\ln^6 m} \sum_{j_2=(\ln^6 m)+1}^{\ln^6 m} a_{1,j_1} + (\ln m) \sum_{j_1=(\ln^6 m)+1}^{\ln^6 m} \sum_{j_2=(\ln^6 m)+1}^{\ln^6 m} O(\gamma_{j_1,j_2})\right) \\
&= O((m - \ln^6 m)(\ln^7 m)) + O((m - \ln^6 m)(\ln^3 m)) = O((m - \ln^6 m)(\ln^7 m)),
\end{align}
GPH estimators of the FISSARMA models (5.6) \[
\sum_{j_1=(\ln^6 m)+1}^{m} \sum_{j_2=(\ln^6 m)+1}^{m} a_{2,j_2} \sqrt{\text{Var}(\varepsilon_{j_1,j_2})} = O\left((m - \ln^6 m)(\ln^7 m)\right).
\]

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REFERENCES


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