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RENEWAL FUNCTION ASYMPTOTICS REFINED À LA FELLER

BY

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Dedicated to Tomasz Rolski, a constant colleague over forty years, distance notwithstanding.

Abstract. Feller's volume 2 shows how to use the Key Renewal Theorem to prove that in the limit $x \to \infty$, the renewal function $U(x)$ of a renewal process with nonarithmetic generic lifetime X with finite mean $\mathrm{E}(X) = 1/\lambda$ and second moment differs from its linear asymptote λx by the quantity $\frac{1}{2}\lambda^2 \mathrm{E}(X^2)$. His first edition (1966) (but not the second in 1971) asserted that a similar approach would refine this asymptotic result when X has finite higher order moments. The paper shows how higher order moments may justify drawing conclusions from a recurrence relation that exploits a general renewal equation and further appeal to the Key Renewal Theorem.

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1. INTRODUCTION

This paper concerns two remarks in the first edition of Volume 2 of Feller's (1966) treatise on probability theory that are missing in the second edition (1971), see [6]. The context is that of the renewal function U of a renewal process whose generic lifetime random variable X has a non-lattice distribution function F whose first two moments are finite (here, $U(x) = \sum_{n=0}^{\infty} F^{n*}(x)$ for $x \geqslant 0$ with F^{n*} the n-fold convolution power of F). Write $\mathrm{E}(X) = 1/\lambda$. After using the Key Renewal Theorem (Feller, 1966, Theorem XI.1.2) to show that

$$U(x) - \lambda x \to \tfrac{1}{2}\lambda^2 \mathrm{E}(X^2) \quad \text{as } x \to \infty, \tag{1.1}$$

Feller wrote (p. 357) "[this] asymptotic expansion of U may be further refined if F has moments of higher order", and (p. 372) "this method [of using the general renewal equation to establish (1.1)] can be used for better estimates when higher moments exist."

Associated with any given renewal process as described above is the *general renewal equation*

(1.2) $$Z(x) = z(x) + \int_0^x Z(x-u)\, F(\mathrm{d}u), \text{ so } Z(x) = \int_0^x z(x-u)\, U(\mathrm{d}u),$$

in which z is the *generator* for the *solution function* Z. By linearity, if the pairs (Z_1, z_1) and (Z_2, z_2) satisfy the same general renewal equation with given lifetime d.f. F, then so does also the pair $(a_1 Z_1 - a_2 Z_2, a_1 z_1 - a_2 z_2)$ for any real finite a_1, a_2, as Feller exploited. In this paper I establish and build on a result asserted in Daley and Mohan [3]. That result enables us to construct a sequence of functions which involve iterated integrals of the renewal function, its asymptote and, where applicable, finiteness of these iterated differences for which the Key Renewal Theorem holds. A necessary and sufficient condition for the n-th step in this reasoning is that the lifetime r.v. X should have a finite $(n+1)$-st order moment, as in Feller's first assertion.

It remains unclear to us as to how to exploit the Key Renewal Theorem to obtain "better estimates when higher moments exist" (second quotation from Feller), because detailed asymptotic behaviour of the difference function $Z_1(x) := U(x) - \lambda x$ is needed. Stronger properties (limits of analytic functions and the Riemann–Lebesgue theorem) are exploited in Stone [8], based also on Feller and Orey's work [7]. To be fair, this second quotation was made in the (two-sided) random walk setting.

2. BASIC RESULT AND RECURRENCE RELATIONS

Our basic result appeared without proof in Daley and Mohan [3] in the setting of a general random walk; we give it here with proof in the simpler case of a renewal process, i.e. on $\mathbb{R}_+$.

THEOREM A. *Let the real-valued function Z on $\mathbb{R}_+$ satisfy the general renewal equation* (1.2), *with generator z, in which the lifetime d.f. F of the nonnegative r.v. X has first moment $1/\lambda$.*

(a) *When z is directly Riemann integrable and $C = \int_0^\infty z(u)\,\mathrm{d}u$, $Z(x) \to \lambda C$ $(x \to \infty)$.*

(b) *Then $\widetilde{Z}(x) := \int_0^x [Z(y) - \lambda C]\,\mathrm{d}y$ satisfies* (1.2) *with generator*

$$\tilde{z}(x) := \int_x^\infty [C\lambda \overline{F}(v) - z(v)]\,\mathrm{d}v,$$

where $\overline{F}(x) = 1 - F(x)$.

(c) *When $\int_x^\infty z(v)\,\mathrm{d}v$ is directly Riemann integrable and X has finite second moment,*

$$\widetilde{Z}(x) \to \lambda \widetilde{C} := \lambda \int_0^\infty \tilde{z}(y)\,\mathrm{d}y \quad (x \to \infty),$$

where $\widetilde{C} = \lambda C\, \frac{1}{2}\mathrm{E}(X^2) - \int_0^\infty v\, z(v)\,\mathrm{d}v$.

P r o o f. The assertion (a) is simply the Key Renewal Theorem (e.g., Theorem XI.1.2 of Feller, 1966).

For (b), the integral representation in (1.2) implies that

$$\int_0^x Z(y)\,\mathrm{d}y = \int_0^x \mathrm{d}y \int_0^y z(y-u)\,U(\mathrm{d}u) = \int_0^x U(\mathrm{d}u) \int_u^x z(y-u)\,\mathrm{d}y$$
$$= \int_0^x U(\mathrm{d}u) \int_0^{x-u} z(v)\,\mathrm{d}v = \int_0^x \big[C - \int_{x-u}^{\infty} z(v)\,\mathrm{d}v\big] U(\mathrm{d}u).$$

Thus, $\int_0^x Z(u)\,\mathrm{d}u$ as a solution function of (1.2) has as its generator $\int_0^x z(u)\,\mathrm{d}u$, $= C - \int_x^\infty z(u)\,\mathrm{d}u$ when z is directly Riemann integrable. $C\lambda x$ as a solution to (1.2) has as its generator the function equal to

$$\text{(2.1)}\qquad C\lambda x - C\int_0^x \lambda(x-y)\,F(\mathrm{d}y) = C\int_0^x \lambda[1-F(v)]\,\mathrm{d}v$$
$$= C - C\int_x^\infty \lambda\overline{F}(v)\,\mathrm{d}v.$$

Then the generator for the solution function $\widetilde{Z}(x) = \int_0^x Z(u)\,\mathrm{d}u - \lambda C x$ is the difference function

$$\text{(2.2)}\quad \tilde{z}(x) = C - \int_x^\infty z(v)\,\mathrm{d}v - C + C\int_x^\infty \lambda\overline{F}(v)\,\mathrm{d}v = \int_x^\infty [C\lambda\overline{F}(v) - z(v)]\,\mathrm{d}v.$$

For (c), $\mathrm{E}(X^2) < \infty$ implies

$$\lim_{x\to\infty} \int_0^x \mathrm{d}y \int_y^\infty \lambda\overline{F}(u)\,\mathrm{d}u = \lim_{x\to\infty} \int_0^\infty \min(x,u)\,\overline{F}(u)\,\mathrm{d}u < \infty,$$

and with direct Riemann integrability of the function $\int_x^\infty z(v)\,\mathrm{d}v$ this implies (c) via the Key Renewal Theorem. ∎

In the notation of Theorem A, set

$$\text{(2.3)}\qquad \begin{aligned} &Z_1(x) := Z(x) = U(x) - \lambda x,\\ &z_1(x) := z(x) = \int_x^\infty \lambda\overline{F}(u)\,\mathrm{d}u, \quad C_1 := C = \int_0^\infty z_1(u)\,\mathrm{d}u, \end{aligned}$$

where C is finite. We have

$$Z_1(x) - \lambda C_1 = \int_0^x z_1(x-u)\,U(\mathrm{d}u) - \lambda\int_0^\infty z_1(u)\,\mathrm{d}u;$$

hence

$$\begin{aligned}\lambda\int_x^\infty z_1(u)\,\mathrm{d}u - [\lim_{y\to\infty} Z_1(y) - Z_1(x)] &= \int_0^x z_1(x-u)\,[U(\mathrm{d}u) - \lambda\,\mathrm{d}u] \\ &= \lambda\int_0^x [U(\mathrm{d}u) - \lambda\,\mathrm{d}u] \int_{x-u}^\infty \overline{F}(v)\,\mathrm{d}v = \lambda\int_0^\infty \overline{F}(v)\,\mathrm{d}v \int_{x-v}^x [U(\mathrm{d}u) - \lambda\,\mathrm{d}u] \\ &= \lambda\int_0^\infty \big[U(x) - U\big((x-v)_+\big) - \lambda\min\{x,v\}\big]\,\overline{F}(v)\,\mathrm{d}v.\end{aligned}$$

In the last integrand, $U(x) - U\big((x-v)_+\big) \leqslant U(v)$ and $0 \leqslant U(v) - \lambda v \leqslant \lambda C_1$ for all v, so the integrand is uniformly bounded above. Split the range of integration to $(0, V)$ and (V, ∞). Take V sufficiently large that $\int_V^\infty \overline{F}(v)\,\mathrm{d}v$ is arbitrarily small positive; then for fixed V, take x sufficiently large that $|U(x) - U(x-v) - \lambda v|$ is arbitrarily small positive for all $0 < v < V$, by Blackwell's form of the renewal theorem. Then the right-hand side is $o(1)$ as $x \to \infty$, so

(2.4)
$$\lambda C_1 - Z_1(x) = \lambda\int_x^\infty z_1(u)\,\mathrm{d}u + o(1) = \lambda\int_x^\infty (u-x)\lambda\overline{F}(u)\,\mathrm{d}u + o(1) \quad (x \to \infty).$$

Now, when $\int_1^\infty y^\gamma \overline{F}(y)\,\mathrm{d}y < \infty$ for any $\gamma > 1$, $\int_x^\infty z_1(u)\,\mathrm{d}u$ is of smaller order than $o(1)$ for $x \to \infty$. In other words, the argument of Theorem A has not (yet) yielded any finer estimate of the asymptotic behaviour of $U(x) - \lambda x$ than the constant λC_1.

In the setting of Theorem A, define $(Z_2, z_2, C_2) = (\widetilde{Z}, \tilde{z}, \widetilde{C})$ ($\widetilde{C}$ need not be finite). More generally, for integers $n = 2, 3, \ldots$, let $(Z, z, C) = (Z_n, z_n, C_n)$ and, provided C_n is finite, define $(Z_{n+1}, z_{n+1}, C_{n+1}) = (\widetilde{Z}, \tilde{z}, \widetilde{C})$ as in Theorem A (here $C_{n+1} = \widetilde{C}$ need not be finite). It now follows that when $\{(Z_j, z_j) : j = 1, \ldots, n\}$ are well defined with $\{C_1, \ldots, C_n\}$ finite, the following recurrence relations hold for $j = 1, \ldots, n$:

(2.5) $$z_{j+1}(x) = C_j \int_x^\infty \lambda\overline{F}(u)\,\mathrm{d}u - \int_x^\infty z_j(u)\,\mathrm{d}u,$$

(2.6) $$Z_{j+1}(x) = \int_0^x [Z_j(v) - \lambda C_j]\,\mathrm{d}v = \int_0^x \tilde{z}_{j+1}(x-u)\,U(\mathrm{d}u).$$

In (2.5), when $\mathrm{E}(X^2)$ is finite, $\widetilde{C}_{j+1}$ is finite if and only if $\int_0^\infty \mathrm{d}x \int_x^\infty z_j(u)\,\mathrm{d}u = \int_0^\infty u z_j(u)\mathrm{d}u$ is well defined and finite. For $j = 1$, using (2.3), it follows that $\int_0^\infty u z_1(u)\,\mathrm{d}u = \int_0^\infty \frac{1}{2} v^2 \lambda\overline{F}(v)\,\mathrm{d}v$, which is finite if and only if $\mathrm{E}(X^3) < \infty$. Applying the Key Renewal Theorem to the second equality in (2.6) yields the following.

COROLLARY 2.1. *Let* $m_j = \mathrm{E}(X^j)$, $j = 1, 2, \ldots$ *For* $m_2 < \infty$, *whether* $\mathrm{E}(X^3)$ *is finite or infinite*, *we have*

$$\lambda C_2 := \lim_{x\to\infty} \int_0^x \left[U(y) - \lambda y - \tfrac{1}{2}\lambda^2 \mathrm{E}(X^2)\right] \mathrm{d}y \tag{2.7}$$

$$= \left(\tfrac{1}{2}\lambda \mathrm{E}(X^2)\right)^2 - \tfrac{1}{6}\lambda^2 \mathrm{E}(X^3) = \lambda C_1 \frac{m_2}{2!} - \frac{\lambda m_3}{3!}.$$

Return to (2.5) but now take $j = 2$. Let us assume $\mathrm{E}(X^3) < \infty$. Then C_2 is finite and $\int_0^\infty z_3(x)\,\mathrm{d}x$ is well defined because the integral over $\mathbb{R}_+$ of the first term on the right-hand side is necessarily finite. Then C_3 is finite if and only if $\int_0^\infty u z_2(u)\,\mathrm{d}u$ is finite; substitution for z_2 from (2.5) shows this to occur if and only if $\int_0^\infty \mathrm{d}x \int_x^\infty u\,\mathrm{d}u \int_u^\infty \lambda \overline{F}(v)\,\mathrm{d}v$ is finite, i.e. because the integrand is non-negative, if and only if $\int_0^\infty v^3 \overline{F}(v)\,\mathrm{d}v < \infty$, i.e. $\mathrm{E}(X^4) < \infty$.

This argument can be continued for as long as $\mathrm{E}(X^j)$ is finite, justifying the next result.

COROLLARY 2.2. *When* C_{n-1} *is finite*, C_n *is well defined*; C_n *is finite if and only if* $\mathrm{E}(X^{n+1})$ *is finite.*

A more transparent proof of Corollary 2.2 proceeds via a chain of substitutions exploiting (2.5) as follows. Assume C_n is finite. Then from (2.5) with, successively, $j = n, n-1, \ldots, 1$, we see that $z_{n+1}(x)$ equals

$$C_n \int_x^\infty \lambda \overline{F}(u)\,\mathrm{d}u - C_{n-1} \int_x^\infty \mathrm{d}u \int_u^\infty \lambda \overline{F}(v)\,\mathrm{d}v + \int_x^\infty \mathrm{d}u \int_u^\infty z_{n-1}(v)\,\mathrm{d}v$$

$$= C_n \int_x^\infty \lambda \overline{F}(u)\,\mathrm{d}u - C_{n-1} \int_x^\infty (u-x)\lambda \overline{F}(u)\,\mathrm{d}u + \int_x^\infty (u-x) z_{n-1}(u)\,\mathrm{d}u$$

$$= C_n \int_x^\infty \lambda \overline{F}(u)\,\mathrm{d}u - C_{n-1} \int_x^\infty (u-x)\lambda \overline{F}(u)\,\mathrm{d}u$$

$$+ C_{n-2} \int_x^\infty \frac{(u-x)^2}{2!} \lambda \overline{F}(u)\,\mathrm{d}u - \int_x^\infty \frac{(u-x)^2}{2!} z_{n-2}(u)\,\mathrm{d}u$$

$$= \int_x^\infty \left[\sum_{i=0}^{r} \frac{C_{n-i}(-1)^i (u-x)^i}{i!}\right] \lambda \overline{F}(u)\,\mathrm{d}u - (-)^r \int_x^\infty \frac{(u-x)^r}{r!} z_{n-r}(u)\,\mathrm{d}u,$$

which for $r = 1, \ldots, n-1$ is equal to

$$\int_x^\infty \left[\sum_{i=0}^{n-1} \frac{C_{n-i}(-1)^i (u-x)^i}{i!}\right] \lambda \overline{F}(u)\,\mathrm{d}u + (-)^n \int_x^\infty \frac{(u-x)^{n-1}}{(n-1)!} z_1(u)\,\mathrm{d}u \tag{2.8}$$

$$= \int_x^\infty \left[\sum_{i=0}^{n-1} \frac{C_{n-i}(-1)^i (u-x)^i}{i!}\right] \lambda \overline{F}(u)\,\mathrm{d}u + (-)^n \int_x^\infty \frac{(v-x)^n}{n!} \lambda \overline{F}(v)\,\mathrm{d}v.$$

All the integrals in (2.8) are finite if and only if $n!\, m_{n+1} = \int_0^\infty v^n \overline{F}(v)\,\mathrm{d}v < \infty$ as in Corollary 2.2. When this condition is met, we can set $x = 0$ in (2.8) and obtain the next result; (2.8) is the simplest non-trivial case of (2.9) below.

THEOREM 2.1. *When* m_{n+1} *is finite*, $\lim_{x\to\infty} Z_n(x) = \lambda C_n$ *finite, and* $\{C_j : j = 1, \dots, n\}$ *satisfy*

$$(2.9)\quad C_r = \sum_{i=1}^{r-1} (-1)^{i+1} C_{r-i} \frac{m_{i+1}}{(i+1)!} + (-1)^{r+1} \frac{m_{r+1}}{(r+1)!} \qquad (r = 1, 2, \dots, n).$$

Corollary 2.2 shows that the sequence of functions $\{Z_n\}$ is finite only for as many moments of the generic r.v. X are finite. This is some amplification of Feller's first remark.

For Feller's other remark, suppose $m_{r+1} < \infty$ for some integer $r > n$ in (2.8); defining $C_0 = 1$, rewrite (2.8) as

$$(2.10)\qquad z_{n+1}(x) = \sum_{i=0}^{n} \int_x^\infty \frac{C_{n-i}(-1)^i}{i!} \big(u - x\big)_+^i \lambda \overline{F}(u)\,\mathrm{d}u.$$

The finiteness of m_{r+1} implies that the magnitude of the i-th integral here is bounded above by $\lambda |C_{n-i}|/i!$ times

$$(2.11)\qquad \int_x^\infty \frac{1}{u^{r-i}} \left(1 - \frac{x}{u}\right)^i u^r \overline{F}(u)\,\mathrm{d}u < \frac{1}{x^{r-i}} \int_x^\infty u^r \overline{F}(u)\,\mathrm{d}u.$$

Since the last integral is $o(1)$ for $x \to \infty$, uniformly in $i = 0, 1, \dots, n$, we can conclude that $x^{r-n} z_{n+1}(x) = o(1)$ for $x \to \infty$.

In terms of explaining Feller's second remark about "better estimates of the renewal function when higher moments exist", this property is the best we have been able to find. Manipulations similar to (2.11) in (2.8) did not lead to any recognizable series in powers of $1/x$ analogous to standard Taylor series expansion. Further, because the key relations (2.5) and (2.6) are recursive in nature, only integer powers are involved and results as powerful as Stone [8] obtained using Fourier methods do not appear to be accessible by Daley and Mohan's approach [3].

3. ANOTHER RECURRENCE RELATION

Using (2.6) as a basis of recurrence relations yields the following expansion:

$$\begin{aligned}(3.1)\qquad Z_{n+1}(x) &= \int_0^x [Z_n(u) - \lambda C_n]\,\mathrm{d}u \\ &= -\lambda C_n x + \int_0^x \mathrm{d}u \int_0^u [Z_{n-1}(v) - \lambda C_{n-1}]\,\mathrm{d}v \\ &= -\lambda C_n x - \lambda C_{n-1} \frac{x^2}{2} + \int_0^x (x - v) Z_{n-1}(v)\,\mathrm{d}v \\ &= -\sum_{i=0}^{n-1} \lambda C_{n-i} \frac{x^{i+1}}{(i+1)!} + \int_0^x \frac{(x - v)^{n-1}}{(n-1)!} Z_1(v)\,\mathrm{d}v,\end{aligned}$$

and

$$(3.2)\qquad \int_0^x \frac{(x-u)^{n-1}}{(n-1)!} Z_1(u)\,\mathrm{d}u = x^n \int_0^x \frac{1}{(n-1)!}\left(1-\frac{u}{x}\right)^{n-1} Z_1(u)\,\frac{\mathrm{d}u}{x} = x^n\,\frac{\lambda C_1}{n!}[1+o(1)].$$

The term $i = n-1$ in the sum occurring in the last equality of (3.1) equals $-\lambda C_1 x^n/n!$, which is the negative of the dominant term in (3.2), so

$$Z_{n+1}(x) + \sum_{i=0}^{n-2} \lambda C_{n-i}\frac{x^{i+1}}{(i+1)!} = o(x^n).$$

However, for large x, the right-hand side dominates all terms on the left-hand side (when C_{n+1} is finite), so the expansion in (3.1) does not add information in the sense of Feller's comments.

Equivalently, we have

(3.3)

$$Z_{n+1}(x) + \sum_{i=0}^{n-1} \lambda C_{n-i}\frac{x^{i+1}}{(i+1)!} = \int_0^x \frac{(x-v)^{n-1}}{(n-1)!}\,\mathrm{d}v \int_0^v z_1(v-u)\,U(\mathrm{d}u)$$
$$= \int_0^x \frac{w^{n-1}}{(n-1)!}\,\mathrm{d}w \int_0^{x-w} z_1(x-w-u)\,U(\mathrm{d}u)$$
$$= \int_0^x U(\mathrm{d}u) \int_0^{x-u} \frac{w^{n-1}}{(n-1)!}\, z_1(x-u-w)\,\mathrm{d}w.$$

4. FURTHER RESULTS

The referee has drawn my attention to older work by Carlsson [2] and more recent papers by Blanchet and Glynn [1], and Dombry and Rabehasaina [5] all concerned with asymptotic expansions of the renewal function U. Carlsson's analysis exploits properties of the Fourier transform $f(t) = \mathrm{E}(e^{itX})$, notably of $1 - f(t)$ and expansions (t is real). Blanchet and Glynn apply Carlsson's results to the particular problem of random geometric sums. I have not seen Dombry and Rabehasaina's work. Carlsson's study includes examples of lifetime distributions F that are weakly nonlattice, i.e. of d.f.s that are nonarithmetic but differ from a lattice distribution only by a "very small amount".

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