PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. IV, Fasc. 1 (1984), p. 79-89

# WEAK CONVERGENCE OF AN EMPIRICAL MONOTONIC DEPENDENCE FUNCTION UNDER DEPENDENCE

BY

T. BEDNARSKI AND T. LEDWINA (WROCŁAW)

*Abstract.* The weak convergence of a consistent estimator of a monotonic dependence function of two random variables $X$ and Y is studied. The estimator is treated as a random element of $D[0, 1]$ and of $L_2([0, 1], \lambda)$, where $\lambda$ stands for the Lebesgue measure. Its asymptotic distribution is derived for the two spaces in the following cases: independence of $X$ and $Y$, distributions contiguous to independence, and dependence of $X$ and $Y$. Except for the case of independence the asymptotic distributions depend strongly on the marginals of $X$ and $Y$. Therefore, the asymptotic distribution of rank counterpart of the estimator is also considered. The obtained results extend the possibility of practical applications of the measure of monotonic dependence and its consistent estimator.

**1. Introduction.** Consider a two-dimensional random vector $(X, Y)$ with continuous marginal distributions and finite expectations. A functional measure of monotonic dependence $\mu_{X,Y}(p)$, $p \in (0, 1)$, was introduced and discussed in detail in [11] and [9]. Under the above assumptions the monotonic dependence function $\mu_{X,Y}(p)$ is defined as

$$\mu_{X,Y}(p) = \begin{cases} \mu^{+}_{X,Y}(p) & \text{if } \mu^{+}_{X,Y}(p) \geqslant 0, \\ \mu^{-}_{X,Y}(p) & \text{if } \mu^{+}_{X,Y}(p) \leqslant 0, \end{cases}$$

where

$$\mu^{+}_{X,Y}(p) = [\mathrm{E}(X \mid Y > y_p) - \mathrm{E}X]/[\mathrm{E}(X \mid X > x_p) - \mathrm{E}X]$$

and

$$\mu^{-}_{X,Y}(p) = -\mu^{+}_{-X,Y}(p),$$

while $x_p$ and $y_p$ denote the $p$-th quantiles of $X$ and $Y$, respectively. The function $\mu_{X,Y}(p)$ can be estimated via $\mu_n(p)$, the analogue of $\mu_{X,Y}(p)$ for the sample distribution. Kowalczyk [9] proved that, for every fixed $p \in (0, 1)$, $\mu_n(p) \to \mu_{X,Y}(p)$ a.e. The practical usefulness of the measure of monotonic dependence is conditioned by the possibility of calculation of asymptotic distributions of $\mu_n$ under various circumstances. The most important of them are: independence of $X$ and $Y$, alternatives contiguous to independence, and dependence of $X$ and $Y$. The availability of the asymptotic distributions and their properties usually give the basis for construction of optimal tests and confidence regions. In this paper we deal with the first of the mentioned problems and study the asymptotic distributions of $\mu_n^+$, a consistent estimator of $\mu_{X,Y}^+$.

If $(X_1, Y_1), \ldots, (X_n, Y_n)$ is a sample, then $\mu_n^+(p) = L_n(p)/M_n(p)$, for

$$L_n(p) = \sum_{i=1}^{n} X_i(I[Y_i \leqslant y_{p,n}] - p) + (np - [np] - 1) \sum_{i=1}^{n} X_i I[Y_i = y_{p,n}],$$

$$M_n(p) = \sum_{i=1}^{n} X_i(I[X_i < x_{p,n}] - p) + x_{p,n}(np - [np]),$$

and $x_{p,n}$, $y_{p,n}$ are the $p$-th quantiles of the empirical marginal distributions chosen as the $k$-th order statistics for $k = [np]+1$. The function $I[A]$ stands for the indicator of the set $A$. We put additionally $L_n(1) = L_n(1-)$ and $M_n(1) = M_n(1-)$. The estimator $\mu_n^-$ of $\mu_{X,Y}^-$ can be defined analogously.

The asymptotic behaviour of $\mu_n^+$, where $\mu_n^+$ is treated as a random element of $D[0, 1]$, is (under independence of $X$ and $Y$) given by the following three results (see [1]).

LEMMA 1.1. *If $X$ and $Y$ are independent and* $\mathrm{E}X^2 < +\infty$, *then*

$$L_n/(n^{1/2}\sigma_n) \xrightarrow{D} W^0,$$

*where $\sigma_n$ stands for a consistent estimator of the standard deviation of $X$ while $W^0$ denotes the Brownian bridge on $D[0, 1]$.*

The convergence of $M_n$ is established by

LEMMA 1.2. *If the quantiles of $X$ are uniquely determined, then*

$$n^{-1} M_n \xrightarrow{P} f \quad \text{in } D[0, 1],$$

*where* $f(p) = \mathrm{E}X(I[X < x_p] - p)$.

THEOREM 1.1. *Assume $X$ and $Y$ are independent,* $\mathrm{E}X^2 < +\infty$, *and the quantiles of $X$ are uniquely determined. Then*

$$n^{1/2}\mu_n^+/\sigma_n \xrightarrow{D} \mu^+ \quad \text{on } D[\varepsilon, 1-\varepsilon],$$

*where $\varepsilon \in (0, 1/2)$ and $\mu^+$ is a Gaussian process such that*

$$\mathrm{E}\mu_p^+ = 0, \quad \mathrm{E}\mu_p^+ \mu_q^+ = p(1-q)/f(p)f(q) \quad \text{for } p \leqslant q,\ p, q \in [\varepsilon, 1-\varepsilon].$$

In Section 2 of this paper we prove, using a central limit theorem in $D[0, 1]$, that a suitably normalized $L_n$ converges to a Gaussian process under dependence of $X$ and $Y$.

In Section 3 we study the asymptotic behaviour of $L_n$ in $D[0, 1]$ under sequence of distributions alternative to the hypothesis of independence of $X$ and $Y$. As in [2], [3] and [13], [14] the sequence of alternatives is

$$P_1^n = \mathop{\mathsf{X}}_{i=1}^{n} P_n,$$

where $dP_n/dP_0 = 1+n^{-1/2}a_n$ while $P_0$ is a fixed product distribution of $X$ and $Y$, and $\{a_n\}_{n\geqslant 1}$ is a convergent sequence of measurable functions having the property $\int a_n dP_0 = 0$ for every $n \geqslant 1$. We prove that if $\{P_1^n\}$ is contiguous to $\{P_0^n\}$, then $L_n/(n^{1/2}\sigma_n)$ converges under $P_1^n$ to the Brownian bridge shifted by a deterministic function $a^*(p)$ $(a^*(1) = a^*(0) = 0)$.

Sections 4 and 5 contain results analogous to those obtained in Sections 2 and 3 but concerning the convergence in $L_2([0, 1], \lambda)$. We obtain a more convenient limiting distribution of $L_n$ under the hypothesis of dependence of $X$ and $Y$.

Practically, in all the already-mentioned cases the asymptotic distribution has parameters strongly dependent on the distribution of the vector $(X, Y)$. Therefore, and also partly for the sake of completeness, in Section 6 we study the asymptotic behaviour of $\mu_n^+$ where instead of $(X_i, Y_i)$ we put their respective ranks. This ensures independence of the limiting distributions on the marginals of $X$ and $Y$.

In the sequel, without additional reference, we shall deal with the random element

$$R_n(p) = n^{-1/2} \sum_{i=1}^{n} X_i(I[Y_i \leqslant y_{p,n}]-p) \tag{1.1}$$

which, practically in all of the considered situations, fulfills the condition $L_n/n - R_n \xrightarrow{P} 0$. The asymptotic results will be formulated for $L_n$, the numerator of $\mu_n^+$. If the distribution of $(X, Y)$ is fixed, then by Lemma 1.2 the limit of $\mu_n^+$ can be established as in Theorem 1.1. In the case of contiguous alternatives one can prove an analogue of Lemma 1.2 and obtain the limit of $\mu_n^+$ in a similar way.

**2. The weak convergence of $L_n$ in $D[0, 1]$ under dependence of $X$ and $Y$.** Without loss of generality we assume $\mathrm{E}X = 0$, for if $\mathrm{E}X = m$, we have

$$R_n(p) = n^{-1/2} \sum_{i=1}^{n} (X_i - m)(I[Y_i \leqslant y_{p,n}]-p) + mn^{-1/2} \sum_{i=1}^{n} (I[Y_i \leqslant y_{p,n}]-p),$$

where the last term equals $mn^{-1/2}([np]\to 1-np)$ and converges to 0 in $D[0, 1]$.

Let $\varphi_n$ be a random element in $D[0, 1]$ defined by $\varphi_n(p) = F(y_{p,n})$ for $p \in [0, 1)$ and put $\varphi_n(1) = 1$. Here $F$ stands for the distribution function of $Y$. We also define a function $K$ on $[0, 1]$ by

$$K(p) = \mathrm{E}X\big(I[Y \leqslant F^{-1}(p)] - p\big)$$

and we denote by $K \circ \varphi_n$ the superposition of $K$ and $\varphi_n$. The main result of this section is the following

THEOREM 2.1. *Assume* $\mathrm{E}X^4 < +\infty$ *and suppose that* $Y$ *has uniquely determined quantiles. Then*

$$n^{-1/2} L_n - n^{1/2} K \circ \varphi_n \xrightarrow{D} T \quad \text{in } D[0, 1],$$

*where* $T$ *is a Gaussian random element of* $D[0, 1]$ *such that* $P(T \in C[0, 1]) = 1$, $\mathrm{E}T_p = 0$ *for all* $p \in [0, 1]$, *and* $\mathrm{E}T_p T_q$ *is given by*

$$\mathrm{E}\{X(I[Y \leqslant y_p] - p) - K(p)\}\{X(I[Y \leqslant y_q] - q) - K(q)\}. \tag{2.1}$$

The assertion of the theorem will follow from lemmas given below. Observe that for $p \in [0, 1]$ we have

$$R_n(p) = V_n(p) + n^{-1/2}[\varphi_n(p) - p] \sum_{i=1}^{n} X_i,$$

where

$$V_n(p) = n^{-1/2} \sum_{i=1}^{n} X_i\big(I[Y_i \leqslant F^{-1}(\varphi_n(p))] - \varphi_n(p)\big). \tag{2.2}$$

The monotonicity of $\varphi_n$ and the convergence of $\varphi_n(p) - p$ to 0 in probability for every $p \in [0, 1]$ imply

LEMMA 2.1. *If the quantiles of* $Y$ *are uniquely determined, then*

$$\varphi_n \xrightarrow{P} \varphi \quad \text{on } D[0, 1], \quad \text{where } \varphi(p) \equiv p.$$

*Therefore we have*

$$R_n - V_n \xrightarrow{P} 0 \quad \text{on } D[0, 1]. \tag{2.3}$$

To find the asymptotic distribution of $V_n$ we shall apply the standard random change of time argument. Let us first notice that $V_n(p) = T_n[\varphi_n(p)]$ for

$$T_n(p) = n^{-1/2} \sum_{i=1}^{n} X_i\{I[Y_i \leqslant y_p] - p\}. \tag{2.4}$$

Therefore, from Lemma 2.1 and [4], p. 145, we infer that if $T_n$ converges to a limit on $D[0, 1]$, so does $V_n$. Next, let us notice that $T_n$ is a normalized sum of independent identically distributed random elements $Z_i(p)$

$= X_i(I[Y_i \leqslant y_p]-p)$ defined on $D[0, 1]$. To derive the asymptotic distribution of $T_n$ we shall use an appropriate central limit theorem (CLT).

Recall that a $D$-valued random element $Z$ distributed as $Z_1$ is said to satisfy the CLT if there exists a $D$-valued random element $T$ such that $T$ is the limit in distribution of the sequence $n^{-1/2} \sum_{i=1}^{n} (Z_i - \mathrm{E}Z_i)$. Let $U$ stand for $Z-\mathrm{E}Z$ and let $\mathscr{L}(T)$ denote the law of $T$. Sufficient conditions for $U$ to satisfy the CLT are given by

THEOREM 2.2 (see [6]). *Let* $\mathrm{E}U^2(t) < \infty$ *for all* $t \in [0, 1]$. *Assume that there exist nondecreasing continuous functions* $G^*$ *and* $F^*$ *on* $[0, 1]$ *and numbers* $\alpha > 1/2$ *and* $\beta > 1$ *such that for all* $s, t, u$ $(0 \leqslant s \leqslant t \leqslant u \leqslant 1)$ *the following two conditions hold*:

(i) $\mathrm{E}[U(u)-U(t)]^2 \leqslant [G^*(u)-G^*(t)]^\alpha$,

(ii) $\mathrm{E}[U(u)-U(t)]^2[U(t)-U(s)]^2 \leqslant [F^*(u)-F^*(s)]^\beta$.

*Then* $U$ *satisfies the* CLT *in* $D[0, 1]$ *and* $\mathscr{L}(T)\{C[0, 1]\} = 1$.

The random element $Z(p)-\mathrm{E}Z(p)$ fulfills conditions (i) and (ii) with $F^*$ and $G^*$ proportional to the identity function and with $\alpha = 3/4$ and $\beta = 3/2$, respectively. To verify this, one multiplies all terms under the expectations and applies the Hölder inequality to the random variable $XI[y_t < Y \leqslant y_u]$ for $t < u$, $t, u \in [0, 1]$. Indeed, for $\alpha \in (1/2, 1)$ we have

$$|\mathrm{E}XI[y_t < Y \leqslant y_u]| \leqslant [\mathrm{E}|X|^{1/(1-\alpha)}]^{1-\alpha}(u-t)^\alpha.$$

The above gives the following result which together with Lemma 2.1 implies the assertion of Theorem 2.1.

LEMMA 2.2. *If* $\mathrm{E}X^4 < +\infty$, *then* $T_n - \mathrm{E}T_n \xrightarrow{\mathscr{D}} T$, *where* $T$ *is a Gaussian process such that* $P(T \in C[0, 1]) = 1$, $\mathrm{E}T_p = 0$ *for* $p \in [0, 1]$, *and* $\mathrm{E}T_pT_q$ *is given by* (2.1).

Remark 2.1. Observe that parameters of the Gaussian distribution given in the theorem depend on the distribution of $(X, Y)$. Under the independence of $X$ and $Y$ the asymptotic distribution of $L_n/(n^{1/2}\sigma_n)$ is free of this drawback.

**3. The weak convergence of $L_n$ in $D[0, 1]$ under alternatives contiguous to independence.** Assume $P_0$ is a product distribution on $R^2$ and let the sequence of distributions $\{P_1^n\}$ be defined as $n$-products of $P_n$, where

$$dP_n/dP_0 = 1 + n^{-1/2} a_n.$$

It is assumed that the measurable functions $a_n$ are uniformly bounded and converge pointwisely to some function $a(x, y)$ and $\int a_n dP_0 = 0$ for all $n \geqslant 1$. By Theorem 2.1 of [13], the sequence of product measures $P_1^n = \mathop{\mathsf{X}}_{i=1}^{n} P_n$ is contiguous to $P_0^n = \mathop{\mathsf{X}}_{i=1}^{n} P_0$. This kind of alternatives ($\{P_1^n\}$) was considered, among others, by Chibisov [5],Behnen [2], [3], and Neuhaus [14]. For the sake of completeness we recall some facts about contiguity (see [7], [8], and [12]).

Let $\mathscr{X}$ be an arbitrary space and $\mathscr{A}_n$ a sequence of $\sigma$-fields of subsets of $\mathscr{X}$. For each $n$ let $Q_n$ and $Q_n'$ be two probability measures defined on $\mathscr{A}_n$. The sequence $\{Q_n'\}$ is said to be *contiguous* to $\{Q_n\}$ if, for every sequence $A_n$ of $\mathscr{A}_n$-measurable sets, $Q_n(A_n) \to 0$ implies $Q_n'(A_n) \to 0$. The condition of contiguity is equivalent to each of the following statements:

(a) A sequence $\{T_n\}$ of $\mathscr{A}_n$-measurable random variables converges in $Q_n'$-probability to 0 if it converges to 0 in $Q_n$.

(b) For every $\varepsilon > 0$ there are an $n(\varepsilon) < +\infty$ and $\delta(\varepsilon) > 0$ such that if $n \geqslant n(\varepsilon)$ and $A_n \in \mathscr{A}_n$, then the inequality $Q_n(A_n) \leqslant \delta(\varepsilon)$ implies $Q_n'(A_n) < \varepsilon$.

LEMMA 3.1. *Let $N_n$ be a sequence of measurable mappings from $(\mathscr{X}, \mathscr{A}_n)$ to a measurable space $(\mathscr{Y}, \mathscr{B})$. If a sequence $\{Q_n'\}$ is contiguous to $\{Q_n\}$, then the respective sequence of probability measures induced by distributions of $N_n$ on $(\mathscr{Y}, \mathscr{B})$ are also contiguous.*

The lemma is a direct consequence of (a).

Let $(\mathscr{X}, \mathscr{A}_n) = (\mathscr{X}, \mathscr{A})$ and let $\mathscr{X}$ be a metric space with Borel $\sigma$-field $\mathscr{A}$ such that every probability measure on $\mathscr{A}$ is tight. Suppose $\{Q_n'\}$ is contiguous to $\{Q_n\}$. The following lemma is well known and can be easily proved by the application of (b).

LEMMA 3.2. *Under the above assumptions the tightness of $\{Q_n\}$ implies the tightness of $\{Q_n'\}$.*

The remaining part of this section is the following generalization of Lemma 1.1:

THEOREM 3.1. *If all the quantiles of $Y$ are uniquely determined and* $\mathrm{E}X^4 < +\infty$, *then under $P_1^n$ we have*

$$L_n/(n^{1/2}\sigma_n) \xrightarrow{D} T^* \quad \text{on } D[0, 1],$$

*where $T^*$ is a Gaussian process with continuous sample paths and with moments*

$$\mathrm{E}T_p^* = \mathrm{E}_{P_0}\{X(I[Y \leqslant y_p]-p)a(X, Y)\}, \qquad \mathrm{E}T_p^* T_q^* = p(1-q) \quad \text{for } p \leqslant q.$$

Proof. Recall that under $P_1^n$ the element $L_n/(n^{1/2}\sigma_n)$ has the same limiting distribution as $R_n/(n^{1/2}\sigma_n)$, where $R_n$ is given by (1.1). Moreover, similarly as in (2.3), we have $R_n - V_n \to 0$ in $P_1^n$-probability. This is due to the contiguity of $\{P_1^n\}$ to $\{P_0^n\}$ and to Lemma 2.1. Since $V_n(p) = T_n[\varphi_n(p)]$ for $T_n$ given by (2.4), the random change of time argument implies a further reduction. Therefore, we shall now study the limiting distribution of $T_n$ under the contiguous alternatives $\{P_1^n\}$.

By Lemma 2.2 the sequence $T_n$ converges in distribution under $P_0^n$. It is then tight. Lemma 3.2 yields tightness under $P_1^n$. To consider the convergence of finite-dimensional distributions of $T_n$ under $P_1^n$ let us fix $p_1 < p_2 < \ldots < p_k$, $p_i \in [0, 1]$, and define random vectors

$$W_i = [X_i(I[Y_i < y_{p_1}] - p_1), \ldots, X_i(I[Y_i < y_{p_k}] - p_k)].$$

Then $(T_n(p_1) - \mathrm{E}T_n(p_1), \ldots, T_n(p_k) - \mathrm{E}T_n(p_k))$ coincides with

$$Z_n = n^{-1/2} \sum_{i=1}^{n} (W_i - \mathrm{E}W_i).$$

If $\varphi_{Z_n}(t)$ stands for the characteristic function of $Z_n$, then

$$\begin{aligned}\varphi_{Z_n}(t) &= \{\mathrm{E} \exp [n^{-1/2} i(t, W_1 - \mathrm{E}W_1)]\}^n \\ &= \{1 - (2n)^{-1} \mathrm{E}(t, W_1 - \mathrm{E}W_1)^2 + o(n^{-1})\}^n \\ &= \{1 - (2n)^{-1} \textstyle\int (W_1 - \mathrm{E}W_1, t)^2 (1 + n^{-1/2} a_n) dP_0 + o(n^{-1})\}^n \\ &= \{1 - n^{-1} [2^{-1} \textstyle\int (W_1 - \mathrm{E}W_1, t)^2 dP_0 - \\ &\quad - (2n^{1/2})^{-1} \textstyle\int (W_1 - \mathrm{E}W_1, t)^2 a_n dP_0] + o(n^{-1})\}^n.\end{aligned}$$

The last expression tends to $\exp \{(-1/2)\mathrm{E}(W_1 - \mathrm{E}W_1, t)^2\}$, where the expectation is taken under $P_0$; and this is the characteristic function of the $k$-dimensional normal distribution with mean zero and covariance matrix with the $i\,j$-th element given by

$$\mathrm{E}[X(I[Y \leqslant y_{p_i}] - p_i)][X(I[Y \leqslant y_{p_j}] - p_j)] = \mathrm{E}X^2 p_i(1 - p_j) \quad \text{for } i \leqslant j.$$

**4. The weak convergence of $n^{-1/2} L_n$ in $L_2([0, 1], \lambda)$ under dependence.** The results proved in this section are analogous to those obtained in Section 2. We study the convergence of the random element $L_n(p)$, $p \in [0, 1]$, in the Hilbert space $L_2([0, 1], \lambda)$, where $\lambda$ stands for the Lebesgue measure. It turns out that in the present case one obtains a more convenient formula, from the practical point of view, for the limiting distribution under dependence of $X$ and $Y$. As before we apply a CLT to the random element

$$T_n(p) = n^{-1/2} \sum_{i=1}^{n} X_i(I[Y_i \leqslant y_p] - p).$$

Let $Z_i(p) = X_i(I[Y_i \leqslant y_p] - p)$. Then $Z_i(p) - \mathrm{E}Z_i(p)$ is a random element of $L_2([0, 1], \lambda)$ and if $\mathrm{E}X_1^2 < +\infty$, then by [15], Chap. IV, there exists a weak limit of the expression

$$n^{-1/2} \sum_{i=1}^{n} [Z_i(p) - \mathrm{E}Z_i(p)] \quad \text{in } L_2([0, 1], \lambda)$$

and it is a Gaussian process $T$ with expectation zero and covariance kernel (2.1). Hence, as an analogue of Theorem 2.1, we shall prove

THEOREM 4.1. *Assume* $EX_1^2 < +\infty$ *and suppose that the quantiles of* $Y$ *are uniquely determined. Then*

$$n^{-1/2}L_n - n^{1/2}EZ_1 \xrightarrow{D} T \quad in\ L_2([0, 1], \lambda).$$

Proof. Let $\|\cdot\|$ denote the norm in $L_2([0, 1], \lambda)$. The theorem will follow if we prove

$$\|n^{-1/2}L_n - R_n\| + \|R_n - T_n\| \xrightarrow{P} 0. \tag{4.1}$$

We have

$$n^{-1/2}L_n - R_n = n^{-1/2}\sum_{i=1}^{n} X_i I[Y_i = y_{p,n}](np-[np]-1).$$

Therefore

$$\|n^{-1/2}L_n - R_n\|^2 \stackrel{\text{a.e.}}{=} n^{-1}\int_0^1 [X_{i(p,n)}(np-[np]-1)]^2 dp$$
$$\leqslant n^{-1}\int_0^1 X_{i(p,n)}^2 dp = n^{-2}\sum_{i=1}^{n} X_i^2,$$

where $i(p, n)$ is the index of the $([np]+1)$-st order statistics of $Y_1, \ldots, Y_n$. Clearly, the last expression converges to 0 with probability 1. The second term of (4.1) converges to 0 by the following argument:

$$\|R_n - T_n\|^2 = \int_0^1 \{n^{-1/2}\sum_{i=1}^{n} X_i(I[Y_i \leqslant y_p] - I[Y_i \leqslant y_{p,n}])\}^2 dp$$
$$\leqslant (2/n)\sum_{i=1}^{n} X_i^2 |F(Y_i) - F_n(Y_i)|,$$

where $F$ is the distribution function of $Y$ and $F_n$ is the empirical distribution for the sample $Y_1, \ldots, Y_n$. Since $\sup_x |F(x) - F_n(x)|$ converges to 0 in probability, we obtain the convergence of $\|R_n - T_n\|$. This completes the proof.

COROLLARY 4.1. *Under the assumptions of Theorem* 4.1, *if in addition* $X$ *and* $Y$ *are independent, then the process* $T$ *has the covariance kernel of the Brownian bridge multiplied by* $EX^2$.

**5. The weak convergence of $n^{-1/2}L_n$ in $L_2([0, 1], \lambda)$ under alternatives contiguous to independence.** As in Section 3 we consider sequences of probability measures $\{P_1^n\}$ given by $n$-fold products of the probabilities $P_n$, where $dP_n/dP_0 = 1 + n^{-1/2}a_n$ and $P_0$ is a product distribution on $R^2$. Since arguments used to prove our basic result do not differ much from those presented in Section 3, we shall simply state the result.

THEOREM 5.1. *Let the quantiles of* $Y$ *be uniquely determined and* $EX^2 < +\infty$ *under* $P_0$. *Suppose* $\sigma_n$ *is a positive consistent estimator of the standard* *deviation of $X$. Then under $P_1^n$ the sequence $n^{-1/2}\sigma_n^{-1} L_n$ converges weakly in $L_2([0, 1], \lambda)$ to a Gaussian random element $T^*$, where*

$$ET^*(p) = E_{P_0}\{X(I[Y \leqslant y_p]-p)\,a(X, Y)\},$$

$$ET^*(p)\,T^*(q) = p(1-q) \quad \textit{for } p \leqslant q.$$

**6. The weak convergence of $\mu_{rn}$ in $D[0, 1]$ under independence and alternatives contiguous to independence.** To avoid influence of marginal distributions on $\mu_{X,Y}(p)$ and on the distribution of $\mu_n$ a grade monotone dependence function and its consistent estimator $\mu_{rn}$ were introduced in [10]. This estimator, being a function of ranks $R_1, \ldots, R_n$ of $X_1, \ldots, X_n$ and $S_1, \ldots, S_n$ of $Y_1, \ldots, Y_n$, is defined as follows:

$$\mu_{rn}(p) = L_{rn}(p)/M_{rn}(p), \qquad p \in (0, 1),$$

where

$$L_{rn}(p) = \sum_{i=1}^{n} (R_i/(n+1))\,I[S_i < [np]+1] - np/2 + R_{k(p)}(np-[np])/(n+1),$$

$$M_{rn}(p) = ([np]+1)(2np-[np])/2(n+1) - np/2,$$

while $k(p)$ is that element of $1, \ldots, n$ for which $S_{k(p)} = [np]+1$. Since $M_{rn}(p)/n \to (p-1)p/2$ as $n \to \infty$, we shall concentrate on the asymptotic distribution of $L_{rn}$, where for $p = 1$ we put $L_{rn}(p) = 0$.

THEOREM 6.1. *Under the independence of $X$ and $Y$ we have*

$$(12/n)^{1/2} L_{rn} \xrightarrow{D} W^0 \quad \textit{in } D[0, 1],$$

*where $W^0$ stands for the Brownian bridge on $D[0, 1]$.*

Proof. Let us first note that $R_{k(p)}(np-[np])/(n+1)n^{1/2} \xrightarrow{P} 0$ in $D[0, 1]$. Then observe that under the independence of $X$ and $Y$ the finite-dimensional distributions of the process

$$R_{rn}(p) = n^{-1/2} \sum_{i=1}^{n} R_i(I[S_i < [np]+1]-p)/(n+1) \tag{6.1}$$

coincide with the corresponding finite-dimensional distributions of the simpler process

$$n^{-1/2} \sum_{i=1}^{[np]} (R_i/(n+1) - 1/2). \tag{6.2}$$

Therefore, we can consider the asymptotic distribution of (6.2) instead of (6.1) and our theorem follows from Theorem 24.1 of [4].

Let $W^*$ be a Gaussian process on $D[0, 1]$ such that

$$EW_p^* = E_{P_0}\{U(I[V \leqslant p]-p)\,a(U, V)\},$$

$$EW_p^* W_q^* = p(1-q) \quad \text{for } p \leqslant q,$$

while $U = G(X)$, $V = F(Y)$, and let $G$ and $F$ stand for the marginals of $X$ and $Y$, respectively. The asymptotic behaviour of $(12/n)^{1/2} L_{rn}$ under $P_1^n$ is given by

THEOREM 6.2. *Under* $\{P_1^n\}$ *we have*

$$(12/n)^{1/2} L_{rn} \xrightarrow{D} W^* \quad in\ D[0, 1].$$

Proof. By Theorem 6.1 the distributions of $(12/n)^{1/2} L_{rn}$ are tight under $\{P_0^n\}$, and hence by Lemma 3.2 they are tight under $\{P_1^n\}$. To prove that finite-dimensional distributions of $(12/n)^{1/2} L_{rn}$ (or, equivalently, of $12^{1/2} R_{rn}$) converge, we use the result of Ruymgaart [16] on asymptotic normality of some rank test statistics.

Recall that

$$\mathrm{R}_{rn}(p) = n^{-1/2} \sum_{i=1}^{n} R_i(I[S_i \leqslant np] - p)/(n+1)$$

and introduce

$$Z_n(p) = n^{-1/2} \sum_{i=1}^{n} U_i(I[V_i \leqslant p] - p),$$

where $U_i = G(X_i)$, $V_i = F(Y_i)$, $i = 1, \ldots, n$.

In particular, by Ruymgaart [16], for every fixed $p \in [0, 1]$, $R_{rn}(p) - Z_n(p)$ tends to 0 in $P_0^n$ and, consequently, in $P_1^n$-probability. Since by Theorem 3.1 we have the convergence of $Z_n - \mathrm{E}_{P_1^n} Z_n$ in $D[0, 1]$ under $P_1^n$, the above implies the convergence of the finite-dimensional distributions of $R_{rn} - \mathrm{E}_{P_1^n} Z_n$. This completes the proof.

## REFERENCES

[1] T. Bednarski and T. Ledwina, *A note on the weak convergence of an estimator of monotonic dependence function of two random variables*, Math. Operationsforsch. Statist., Ser. Statistics, Vol. 13 (1982), No. 4, p. 555-560.

[2] K. Behnen, *Asymptotic optimality and ARE of certain rank-order tests under contiguity*, Ann. Math. Statist. 42 (1971), p. 325-329.

[3] — *A characterization of certain rank-order tests with bounds for the asymptotic relative efficiency*, ibidem 43 (1972), p. 1839-1851.

[4] P. Billingsley, *Convergence of probability measures*, John Wiley, New York 1968.

[5] D. M. Chibisov, *An investigation of the asymptotic power of the test of fit*, Theor. Probability Appl. 10 (1965), p. 421-437.

[6] M. G. Hahn, *Central limit theorems in* $D[0, 1]$, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 44 (1978), p. 89-101.

[7] J. Hájek, *Asymptotically most powerful rank-order tests*, Ann. Math. Statist. 33 (1962), p. 1124-1147.

[8] — and Z. Šidák, *Theory of rank tests*, Academic Press, New York 1967.

[9] T. Kowalczyk, *General definition and sample counterparts of monotonic dependence functions of bivariate distributions*, Math. Operationsforsch. Statist. 8 (1977), p. 351-369.

[10] — and T. Ledwina, *Some properties of chosen grade parameters and their rank counterparts* Math. Operationsforsch. Statist., Ser. Statistics, Vol. 13 (1982), No. 4, p. 547-553.

[11] T. Kowalczyk and E. Pleszczyńska, *Monotonic dependence functions of bivariate distributions*, Ann. Statist. 5 (1977), p. 1221-1227.

[12] L. LeCam, *Locally asymptotically normal families of distributions*, Univ. of California Publ. Statist. 3 (1960), p. 37-98.

[13] G. Neuhaus, *Asymptotic properties of the Cramér-von Mises statistic when parameters are estimated*, p. 257-297 in: Proc. Prague Symp. on Asymptotic Statistics, J. Hájek ed., Universita Karlova Praha, Prague 1974.

[14] — *Asymptotic power properties of the Cramér-von Mises test under contiguous alternatives*, J. Multivariate Anal. 6 (1976), p. 95-110.

[15] K. R. Parthasarathy, *Probability measures on metric spaces*, Academic Press, New York – London 1967.

[16] F. M. Ruymgaart, *Rank tests for independence with best strong exact Bahadur slope*, Z Wahrscheinlichkeitstheorie und Verw. Gebiete 36 (1976), No. 2, p. 119-127.

T. Bednarski
Institute of Mathematics
Polish Academy of Sciences
ul. Kopernika 18
51-617 Wrocław, Poland

T. Ledwina
Institute of Mathematics
Technical University of Wrocław
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland

*Received on 27. 5. 1981*