PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 4, Fasc. 2 (1984), p. 167-170

ON THE CONVERGENCE OF SOME DISCRETE PROBABILITY DISTRIBUTIONS

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Abstract. In [4] Zakusilo proved that the random power series $\sum_{n=1}^{\infty} c^n X_n$, where $c \in (0, 1)$ and $X_1, X_2, \ldots$ are i.i.d. random variables, is convergent with probability 1 if and only if $\mathrm{E}\log(|X_1|+1) < \infty$. The purpose of this paper is to prove a discrete analogue of this theorem. Further, we extend the result to multiparameter random series.

Let $P(Z_+^d)$ be the class of all probability distributions on the lattice Z_+^d of all d-vectors with integer components. For $n = (n_1, \ldots, n_d) \in Z_+^d$ we put $|n| = n_1 + \ldots + n_d$. Let e_j $(j = 1, \ldots, d)$ be a vector in Z_+^d whose components are equal to 0 but for the j-th one are equal to 1. Every $\mu \in P(Z_+^d)$ can be represented as

(1) $$\mu = \sum_{n \in Z_+^d} p_n \delta_n,$$

where $p_n \geqslant 0$, $\sum p_n = 1$, and δ_n is the unit mass at the point n. Given a number c in the unit interval (0, 1) and $\mu \in P(Z_+^d)$ with representation (1), we define a distribution $S_c \mu$ on Z_+^d by the formula

(2) $$S_c \mu = \sum_{\substack{n \in Z_+^d \\ n=(n_1,\ldots,n_d)}} p_n \mathop{*}_{j=1}^{d} [(1-c)\delta_0 + c\delta_{e_j}]^{*n_j},$$

where the asterisk $*$ denotes the convolution operation.

It should be noted that S_c is a slight generalization of the Steutel - van Harn transformation on $P(Z_+)$ (cf. [3]). It is not difficult to verify the formulas

$$S_c(\mu_1 * \mu_2) = S_c \mu_1 * S_c \mu_2, \qquad S_{c_1} S_{c_2} \mu = S_{c_1 c_2} \mu,$$
$$S_c(\alpha\mu_1 + \beta\mu_2) = \alpha S_c \mu_1 + \beta S_c \mu_2,$$

where $\alpha, \beta \geqslant 0$, $\alpha+\beta = 1$. Moreover, $S_c \mu$ is jointly continuous in c and μ.

A distribution μ on Z^d_+ is said to be *c-decomposable* if there exists a $\mu_1 \in P(Z^d_+)$ (depending on c and μ) such that

$$\mu = S_c \mu * \mu_1.$$

More generally, μ is said to be *$\langle c_1, \ldots, c_k \rangle$-decomposable,* where $c_1, \ldots, c_k \in (0, 1)$, if there exist $\mu_1, \ldots, \mu_k \in P(Z^d_+)$ such that

$$\mu = S_{c_1} \mu * \mu_1, \tag{3}$$
$$\mu_1 = S_{c_2} \mu_1 * \mu_2, \quad \ldots \quad \mu_{k-1} = S_{c_k} \mu_{k-1} * \mu_k.$$

In this case μ_k is said to *satisfy the convolution equations* (3) for some μ and $c_1, \ldots, c_k$ in (0, 1) (cf. [1]). The aim of this note is to prove the following

THEOREM. *The following statements are equivalent*:

(i) μ_k *satisfies the convolution equations* (3);

(ii) *the infinite convolution*

$$\mathop{*}_{m_1, \ldots, m_k = 0}^{\infty} S_{c_1^{m_1} \ldots c_k^{m_k}} \mu_k \tag{4}$$

is weakly convergent;

(iii) $\sum_{n \in Z^d_+} p_n \log^k(|n|+1) < \infty$, *where* $p_n = \mu_k(\{n\})$.

We prove first the following

LEMMA. *For every* $c \in (0, 1)$ *there exist positive constants A and B such that, for sufficiently large* $q = 1, 2, \ldots$, *the following inequality holds*:

$$B \log^k q \leqslant \sum_{m=0}^{\infty} \left(1-(1-c^m)^q\right) V^m_{m+k-1} \leqslant A \log^k q, \tag{5}$$

where

$$V^i_j = \frac{j!}{i!(j-i)!}.$$

Proof. It is easy to see that

$$\sum_{m=0}^{\alpha} V^m_{m+k-1} = V^\alpha_{\alpha+k} \tag{6}$$

and

$$(7) \qquad \lim_{m\to\infty} \frac{V^m_{m+k}}{m^k} = \frac{1}{k!}.$$

Further, we have the inequalities

$$\sum_{m=0}^{\infty} (1-(1-c^m)^q)\, V^m_{m+k-1} \geqslant \sum_{0\leqslant m\leqslant \log_c q^{-1}} (1-(1-c^m)^q)\, V^m_{m+k-1}$$
$$\geqslant \sum_{0\leqslant m\leqslant \log_c q^{-1}} (1-(1-q^{-1})^q)\, V^m_{m+k-1}$$

which, by (6) and (7), imply that there exists a $B > 0$ such that, for sufficiently large q, the inequality

$$(8) \qquad B \log^k q \leqslant \sum_{m=0}^{\infty} (1-(1-c^m)^q)\, V^m_{m+k-1}$$

holds.

On the other hand, we get

$$\sum_{m=0}^{\infty} (1-(1-c^m)^q)\, V^m_{m+k-1}$$
$$= \sum_{0\leqslant m\leqslant \log_c q^{-1}} (1-(1-c^m)^q)\, V^m_{m+k-1} + \sum_{p=1}^{\infty} \sum_{\substack{p\log_c q^{-1}\leqslant m \\ \leqslant (p+1)\log_c q^{-1}}} (1-(1-c^m)^q)\, V^m_{m+k-1}$$
$$\leqslant \sum_{0\leqslant m\leqslant \log_c q^{-1}} V^m_{m+k-1} + \sum_{p=1}^{\infty} (1-(1-q^{-p})^q) \sum_{\substack{p\log_c q^{-1}\leqslant m \\ \leqslant (p+1)\log_c q^{-1}}} V^m_{m+k-1},$$

which by (6) and by a simple computation implies the existence of a constant A such that

$$(9) \qquad \sum_{m=0}^{\infty} (1-(1-c^m)^q)\, V^m_{m+k-1} \leqslant A \log^k q$$

for sufficiently large q.

Finally, from (8) and (9) we obtain (5), which completes the proof of the Lemma.

Proof of the Theorem. Equivalence (i)$\Leftrightarrow$(ii) can be easily proved. Thus, we prove only the equivalence (ii)$\Leftrightarrow$(iii). Let μ_k be a distribution on Z^d_+ and let $p_n = \mu_k(\{n\})$ $(n \in Z^d_+)$. Further, from the Kolmogorov theorem on three series ([2], p. 323-324) it follows that the series (4) is weakly convergent if and only if

$$(10) \qquad \sum_{m_1,\ldots,m_k=0}^{\infty} (1 - S_{c_1^{m_1}\ldots c_k^{m_k}} \mu_k(\{0\})) < \infty,$$

where 0 is the zero element in Z_+^d. On the other hand, we have

$$S_c\,\mu(\{0\}) = \sum_{n \in Z_+^d} (1-c)^{|n|}\,p_n$$

with $p_n = \mu(\{n\})$. Thus condition (10) can be rewritten as

$$\sum_{n \in Z_+^d} \sum_{m_1,\ldots,m_k=0}^{\infty} p_n\big(1-(1-c_1^{m_1}\ldots c_k^{m_k})^{|n|}\big) < \infty,$$

which implies that for $c = \min(c_1,\ldots,c_k)$

$$\sum_{n \in Z_+^d} \sum_{m=0}^{\infty} p_n\big(1-(1-c^m)^{|n|}\big)\,V_{m+k-1}^m < \infty. \tag{11}$$

By the Lemma the last condition is equivalent to (iii).

Conversely, if (iii) is satisfied, then (11) holds with $c = \max(c_1,\ldots,c_k)$ for any $c_1,\ldots,c_k \in (0, 1)$. Hence (10) is satisfied and, consequently, the series (4) is weakly convergent. Thus the proof is complete.

A simple consequence of the Theorem is the following

COROLLARY. *If* μ_k *on* Z_+^d *has any finite moment, then it satisfies the convolution equation* (3).

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Received on 22. 12. 1981