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RATE OF CONVERGENCE IN THE STRONG LAW OF LARGE NUMBERS

BY

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Abstract. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables such that $EX_n = 0$, $EX_n^2 = \sigma_n^2 < \infty$, $n \geqslant 1$. For each $n \geqslant 1$ let

$$S_n = \sum_{k=1}^{n} X_k, \quad \mathscr{S}_n^2 = \sum_{k=1}^{n} \sigma_k^2;$$

then, under some additional conditions, $S_n/\mathscr{S}_n^{1+\alpha} \to 0$ as $n \to \infty$ with probability 1 for any $\alpha > 0$.

The main purpose of this paper is to give the order of magnitude of

$$\sum_{n=1}^{\infty} P(|S_n| \geqslant t\mathscr{S}_n^{1+2\alpha})$$

as $t \to 0^+$. The rate of convergence in the random strong law of large numbers is established too.

1. Introduction. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with $EX_n = 0$ and $EX_n^2 = \sigma_n^2 < \infty$, $n \geqslant 1$. Define

$$S_n = \sum_{k=1}^{n} X_k, \quad \mathscr{S}_n^2 = \sum_{k=1}^{n} \sigma_k^2, \quad n \geqslant 1.$$

It is well known that if $\mathscr{S}_n^2 \to \infty$ and $\mathscr{S}_{n+1}^2/\mathscr{S}_n^2 \to 1$ as $n \to \infty$, then $S_n/\mathscr{S}_n^{1+2\alpha} \to 0$ (as $n \to \infty$) with probability 1 for any $\alpha > 0$ provided that for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P(|X_n| > \varepsilon\mathscr{S}_n^{1+2\alpha}) < \infty$$

as is necessary (cf. [6], [16]).

Many authors (cf. [13]-[15], [17], [3]) have studied the rate of convergence in the strong law of large numbers (SLLN) under the assumption that

$EX_n^2 = 1$, $n \geq 1$. The most general result in this direction belongs to Chen [3] who proved the following

THEOREM 1. *Suppose that* $\{X_n, n \geq 1\}$ *is a sequence of independent random variables such that* $EX_n = 0$, $EX_n^2 = 1$, $n \geq 1$. *If there exists a function* g *such that*

(1) $g(x)$ *is nondecreasing on the interval* $(0, \infty)$, *is even on* $(-\infty, \infty)$, *and* $g(x) \to \infty$ *as* $x \to \infty$;

(2) *the function* $x/g(x)$ *does not decrease on* $(0, \infty)$;

(3)
$$\limsup_{n\to\infty} n^{-1} \sum_{k=1}^{n} EX_k^2 g(X_k) < \infty;$$

(4) *for some constant* α
$$\sum_{n=1}^{\infty} (\log n)/n^{2\alpha} g(n^{1/2}) < \infty \qquad (0 < \alpha \leq 1/2);$$

then we have

(5)
$$\lim_{t\to 0} t^{1/\alpha} \sum_{n=1}^{\infty} P(|S_n| \geq tn^{1/2+\alpha}) = C_\alpha,$$

where
$$C_\alpha = \pi^{-1/2} 2^{1/2\alpha} \Gamma(1/2 + 1/2\alpha).$$

This result extends to the nonidentically distributed case the theorem of Wu [17] and gives a deeper understanding of the SLLN's.

Recently Ahmad [1] has presented a random version of Theorem 1. Namely he proved that if $\{N_n, n \geq 1\}$ is a sequence of positive integer-valued random variables not necessarily independent of $\{X_n, n \geq 1\}$, then, under some additional assumptions on $\{X_n, n \geq 1\}$, $\{N_n, n \geq 1\}$ and the function g,

(6)
$$\lim_{t\to 0} t^{1/\alpha} \sum_{n=1}^{\infty} P(|S_{N_n}| > tn^{1/2+\alpha}) = \lambda^{1/2\alpha} C_\alpha,$$

where λ is such a positive constant that $N_n/n \to \lambda$ with probability 1 as $n \to \infty$.

It should be mentioned here that the proof of Lemma 4 given in [1], based on "the argument of Landers and Rogge [8]", has a gap. In order to prove Lemma 4 [1], using the ideas of Landers and Rogge [8], one needs extensions of Lemmas 7 and 8, given in [8], for any sequence of independent random variables $\{X_n, n \geq 1\}$ with $EX_n = 0$ and $EX_n^2 = 1$, $n \geq 1$. Lemma 8 [8] can be extended to this case, which in fact is done in the proof of Lemma 4 [1]. But the extension of Lemma 7 [8], to the nonstationary case of $\{X_n, n \geq 1\}$, by the argument of Landers and Rogge, needs the inequality $(*)$ [8, p. 281], which, in general, in this case is incorrect. Taking into account Lemma

1 [10] or Lemma 6.1 [11], one can easily notice that the proof of Lemma 4 [1], based on the argument of Landers and Rogge [8], requires the following assumption: there exist positive constants b_1 and b_2 such that for every n, k, $n > k \geqslant 1$,

$$(**) \qquad b_1 P(S_n - S_k \geqslant 0) \leqslant P(S_n - S_k \leqslant 0) \leqslant b_2 P(S_n - S_k \geqslant 0).$$

Let us observe that if $\{X_n, n \geqslant 1\}$ is a sequence of symmetrical random variables, then $(**)$ holds. On the other hand, if $\{X_n, n \geqslant 1\}$ satisfies the central limit theorem, then

$$\lim_{n\to\infty} P(S_n \leqslant 0) = 1/2 = \lim_{n\to\infty} P(S_n \geqslant 0),$$

which proves that if, in addition, $\{X_n, n \geqslant 1\}$ is stationary, then $(**)$ holds too, and this is just the case considered in [8] and, therefore, in [1].

Unfortunately, in Lemma 1 [10] the assumption $(**)$ is omitted too but in the proof we used it (cf. [10], p. 233, lines 5 and 6).

We would also like to mention that Lemma 3 in [1] does not follow from Petrov's Theorem 10 [9], because, under the assumed assumptions, the variance of $S_{N_n}/(n\lambda)^{1/2}$ need not be equal to one (as is required in Theorem 10 [9]) even if N_n, $n \geqslant 1$, are assumed to be independent of $\{X_n, n \geqslant 1\}$.

Let h be a finite and positive function defined on $[0, \infty)$. Assume that h has a continuous derivative $h'(x)$ for all $x \geqslant 0$. Furthermore, let, for every $t > 0$, f_t be an increasing and positive function which has a continuous derivative $f_t'(x)$ for all $x \geqslant 0$. Let us put

$$A_n(h, f_t) = \sum_{k=1}^{n} h(k) P(|S_k| > \mathscr{S}_k f_t(k)),$$

$$A_\infty(h, f_t) = \lim_{n\to\infty} A_n(h, f_t),$$

$$F(h, f_t) = 2 \sum_{n=1}^{\infty} h(n) \Phi(-f_t(n)),$$

where Φ denotes the standard normal distribution function.

The main purpose of this paper is to study the order of magnitude of $A_\infty(h, f_t)$ and $F(h, f_t)$ as $t \to 0^+$. The results obtained generalize the theorems given by Chen [3], Wu [17], Severo and Slivka [13], Szynal [15], Ahmad [1], Sirazdinov Gafurov and Komekov [14].

2. The rate of convergence in the SLLN's. Let G be the class of functions satisfying (1) and (2). For a given function g of the set G let

$$b_n(g) = \sum_{k=1}^{n} \mathrm{E} X_k^2 g(X_k) / \mathscr{S}_n^2 g(\mathscr{S}_n), \qquad n \geqslant 1.$$

THEOREM 2. *Suppose that*

$$\sum_{n=1}^{\infty} h(n)\, n^{-r} < \infty \quad \textit{for some number } r \geqslant 2.$$

If there exists a function g in G such that $b_n(g) \to 0$ as $n \to \infty$ and

$$T(h, f_t, g) = \sum_{n=1}^{\infty} b_n(g)\, h(n) \log n/(1+f_t^2(n)) < \infty, \tag{7}$$

then

$$|A_\infty(h, f_t) - F(h, f_t)| \leqslant C(1 + T(h, f_t, g))$$

provided $F(h, f_t) < \infty$, where C is some positive constant independent of the function f_t.

Proof. At first let us observe that, by our assumptions and the central limit theorem, $F(h, f_t) < \infty$ implies $A_\infty(h, f_t) < \infty$. Thus, taking into account that $F(h, f_t) < \infty$, we get

$$|A_\infty(h, f_t) - F(h, f_t)| \leqslant \sum_{n=1}^{\infty} h(n) \left| P(|S_n| > \mathscr{S}_n f_t(n)) - 2\Phi(-f_t(n)) \right|. \tag{8}$$

Let us put

$$\Delta_n(x) = |P(S_n < x\mathscr{S}_n) - \Phi(x)|;$$

by Theorem 5 [9] we have

$$\Delta_n = \sup_x \Delta_n(x) \leqslant C b_n(g).$$

Hereafter C denotes a positive constant (independent of the function f_t), and the same symbol may be used for different constants. Choose an integer n_0 such that, for every $n \geqslant n_0$, $\Delta_n \leqslant e^{-1/2}$. This can always be done because of $b_n(g) \to 0$ as $n \to \infty$. Now we get

$$\sum_{n=1}^{n_0} h(n) \left| (|S_n| > \mathscr{S}_n f_t(n)) - 2\Phi(-f_t(n)) \right| \leqslant 2 \sum_{n=1}^{n_0} h(n) \leqslant C \tag{9}$$

and

$$\left| P(|S_n| > \mathscr{S}_n f_t(n)) - 2\Phi(-f_t(n)) \right| \leqslant \Delta_n(f_t(n)) + \Delta_n(-f_t(n)). \tag{10}$$

On the other hand, by Theorem 11 [9], for every $n \geqslant n_0$

$$\Delta_n(x) \leqslant C\Delta_n \log \Delta_n^{-1}/(1+x^2). \tag{11}$$

Hence, putting $A_0 = \{n\colon \Delta_n \leqslant n^{-r}\}$, $A_1 = \{n \geqslant n_0\colon \Delta_n > n^{-r}\}$, where the number $r \geqslant 2$ is given in the assumptions of Theorem 2, we obtain

$$\sum_{n \in A_0} h(n)\, \Delta_n(\pm f_t(n)) \leqslant \sum_{n \in A_0} h(n)\, n^{-r} \leqslant C \tag{12}$$

and, by (11),

$$\sum_{n\in A_1} h(n)\Delta_n(\pm f_t(n)) \leqslant C \sum_{n\in A_1} b_n(g)h(n)\log n/(1+f_t^2(n)). \tag{13}$$

Thus Theorem 2 follows from (8)-(13).

Suppose $f_t(n)\to\infty$ as $n\to\infty$. Let $G_t(x) = h(x)\Phi(-f_t(x))$. Then, by the Euler-Maclaurin sum formula [4], p. 122, we have

$$2\sum_{n=0}^{m} G_t(n) = G_t(0)+G_t(m)+2\int_0^m G_t(x)\,dx - 2\int_0^m P(x)\,dG_t(x),$$

where $P(x) = [x]-x+1/2$ and $[x]$ denotes the integral part of x. Thus, by the monotone convergence theorem,

$$2\sum_{n=0}^{\infty} G_t(n) = G_t(0)+2\int_0^\infty G_t(x)\,dx - 2\int_0^\infty P(x)\,dG_t(x)$$

provided $G_t(m) = h(m)\Phi(-f_t(m))\to 0$ as $m\to\infty$. But $-1/2 \leqslant P(x) \leqslant 1/2$. Hence

$$2\int_0^\infty G_t(x)\,dx - \int_0^\infty |dG_t(x)| \leqslant 2\sum_{n=0}^{\infty} G_t(n) - G_t(0)$$
$$\leqslant 2\int_0^\infty G_t(x)\,dx + \int_0^\infty |dG_t(x)|.$$

Define $H(x) = \int_0^x h(u)\,du$, $x\geqslant 0$, $H(\infty) = \lim_{x\to\infty} H(x)$. Then

$$2\int_0^\infty G_t(x)\,dx = \int_0^{H(\infty)} P\big(H(f_t^{-1}(|N|)) > u\big)\,du,$$

where N is a standard normal random variable and f_t^{-1} is the inverse of f_t. Furthermore

$$\int_0^\infty |dG_t(x)| \leqslant \int_0^\infty |h'(x)|\,P\big(f_t^{-1}(|N|) > x\big)\,dx +$$
$$+\int_0^\infty h(x)f_t'(x)\exp\{-f_t^2(x)/2\}\,dx/(2\pi)^{1/2}.$$

Thus, taking into account the relations given above, we obtain

$$\Big|F(h, f_t) - \int_0^{H(\infty)} P\big(H(f_t^{-1}(|N|)) > u\big)\,du + h(0)\Phi(-f_t(0))\Big|$$
$$\leqslant \int_0^\infty |h'(x)|\,P\big(f_t^{-1}(|N|) > x\big)\,dx + \int_0^\infty h(x)f_t'(x)\exp\{-f_t^2(x)/2\}\,dx/(2\pi)^{1/2}.$$

Let us observe that if, e.g., $f_t(x) = tx^\alpha$, $h(x) \equiv 1$, $x \geqslant 0$, α, $t > 0$, then $H(x) = x$ and, in this case, we get

$$\mathrm{E}\,|N|^{1/\alpha}/t^{1/\alpha} - 1 \leqslant F(h, f_t) \leqslant \mathrm{E}\,|N|^{1/\alpha}/t^{1/\alpha}.$$

Thus, by Theorem 2,

$$\lim_{t \to 0+} t^{1/\alpha} \sum_{n=1}^{\infty} P(|S_n| > t\mathscr{S}_n n^\alpha) = C_\alpha$$

for every $0 < \alpha \leqslant 1/2$ provided

$$\sum_{n=1}^{\infty} b_n(g) \log n/n^{2\alpha} < \infty.$$

This, in the special case, $\mathscr{S}_n^2 = n$, $b_n(g) = 1/g(n^{1/2})$, $n \geqslant 1$, gives the main result of Chen [3]. On the other hand, by Theorem 2 and the relations given above, one can obtain much more general results. For example, putting $h(x) = (\log x)^\beta/x^\gamma$, $f_t(x) = t(\log x)^\alpha$, $x \geqslant 1$, $h(x) = f_t(x) = 0$, $0 \leqslant x < 1$, by Theorem 2 we get

$$\lim_{t \to 0+} t^{(1+\beta)/\alpha} \sum_{n=1}^{\infty} (\log n)^\beta P\big(|S_n| > t\mathscr{S}_n(\log n)^\alpha\big)/n^\gamma = I(\gamma, \alpha, \beta)$$

for every $\alpha > 0$, $\beta, \gamma \geqslant 0$ such that $1+\beta \geqslant 2\alpha$, and

$$\sum_{n=1}^{\infty} b_n(g)(\log n)^{1+\beta-2\alpha} n^{-\gamma} < \infty,$$

where

$$I(\gamma, \alpha, \beta) = 0, \ +\infty, \text{ or } C_{\alpha/(\beta+1)}/(\beta+1) \text{ for } \gamma > 1, \ \gamma < 1 \text{ or } \gamma = 1,$$

respectively. This assertion seems to be unknown even in the case where X_n, $n \geqslant 1$, are independent and identically distributed. As another consequence of Theorem 2 or the statement given above, we get

$$\lim_{t \to 0+} t^{(1+\beta)/\alpha} \sum_{n=3}^{\infty} (\log\log n)^\beta P\big(|S_n| \geqslant t\mathscr{S}_n(\log\log n)^\alpha\big)/n^\gamma (\log n)^\delta\big)$$
$$= I(\delta, \gamma, \alpha, \beta)$$

for every $\alpha > 0$, δ, β, $\gamma \geqslant 0$ such that $(1+\beta) \geqslant 2\alpha$, and

$$\sum_{n=3}^{\infty} b_n(g)(\log\log n)^{\beta-2\alpha}(\log n)^{1-\delta} n^{-\gamma} < \infty,$$

where $I(\delta, \gamma, \alpha, \beta) = 0$ or $+\infty$ for $\gamma > 1$ or $\gamma < 1$, respectively, and $I(\delta, 1, \alpha, \beta) = 0$, $+\infty$ or $C_{\alpha/(1+\beta)}/(\beta+1)$ for $\delta > 1$, $\delta < 1$ or $\delta = 1$, respectively.

Let us observe that the consequences of Theorem 2 given above can also be considered as the study of asymptotic behaviour (as $t \to 0^+$ and $n \to \infty$) of the probabilities $P(|S_n| > t\mathscr{S}_n^{1+2\alpha})$. In fact, the assertions obtained have covered the following cases: $\mathscr{S}_n^2 = n$, $\mathscr{S}_n^2 = \log n$ and $\mathscr{S}_n^2 = \log\log n$, $n \geqslant 3$. On the other hand, in order to consider probabilities of the type $P(|S_n| > t\mathscr{S}_n(\log\log \mathscr{S}_n^2)^\alpha)$, $P(|S_n| > t\mathscr{S}_n(\log \mathscr{S}_n^2)^\alpha)$ or other ones, one can find an appropriate increasing and positive function $f_t(x)$ which is continuous and has a continuous derivative $f_t'(x)$ for all $x \geqslant 0$, and $f_t(0) = 0$, $f_t(n) = t\log\log \mathscr{S}_n^2$ or $f_t(n) = t\log \mathscr{S}_n^2$, respectively, and then use Theorem 2 with such a function. At the same time we must, as we have seen, find an appropriate function $h(n)$, which will also depend on $\mathscr{S}_n^2$.

One can also note that Theorem 2 may be useful in the study of "strong limit laws" (for example, such as the law of the iterated logarithm) for nonidentically distributed random variables. Namely, let us consider a positive function $f(x)$ which has a positive and continuous derivative $f'(x)$. Let us put $f_t(x) = (a+t)f(x)$. Then

$$P\big(\limsup_{n\to\infty} |S_n|/\mathscr{S}_n f(n) = a\big) = 1$$

iff, for every $t > 0$, $P\big(N_\infty(t) < \infty\big) = 1$, and, for every $t < 0$, $P\big(N_\infty(t) = \infty\big) = 1$, where

$$N_\infty(t) = \sum_{n=1}^{\infty} I\big(|S_n| \geqslant \mathscr{S}_n f_t(n)\big).$$

It is obvious that if h is a positive and nondecreasing function such that $h(x) \to \infty$ as $x \to \infty$ and $\mathrm{E}h\big(N_\infty(t)\big) < \infty$, then $\limsup_{n\to\infty} |S_n|/\mathscr{S}_n f(n) \leqslant a$. By the monotone convergence theorem, we get

$$\mathrm{E}h\big(N_\infty(t)\big) = \lim_{n\to\infty} \mathrm{E}h\big(N_n(t)\big), \quad \text{where} \quad N_n(t) = \sum_{k=1}^{n} I\big(|S_k| \geqslant \mathscr{S}_k f_t(k)\big).$$

Furthermore, if by definition $h(0) = N_0(t) = 0$, then

$$\lim_{m\to\infty} \mathrm{E}h\big(N_m(t)\big) = \lim_{m\to\infty} \sum_{n=1}^{m} \mathrm{E}\big(h(N_n(t)) - h(N_{n-1}(t))I\big(|S_n| \geqslant \mathscr{S}_n f_t(n)\big)\big)$$

$$\geqslant \sum_{n=1}^{\infty} \min_{1\leqslant k\leqslant n} \big(h(k)-h(k-1)\big)P\big(|S_n| \geqslant \mathscr{S}_n f_t(n)\big).$$

On the other hand, by the same way we get

$$\mathrm{E}h\big(N_\infty(t)\big) \leqslant \sum_{n=1}^{\infty} \max_{1\leqslant k\leqslant n} \{h(k)-h(k-1)\}\, P\big(|S_n| \geqslant \mathscr{S}_n f_t(n)\big).$$

Thus we have proved the following

THEOREM 3. *Assume that there exist functions h_1 and h_2 such that for every $n \geqslant 1$*

$$h_1(n) \leqslant \min_{1\leqslant k\leqslant n} [h(k)-h(k-1)] \leqslant \max_{1\leqslant k\leqslant n} [h(k)-h(k-1)] \leqslant h_2(n),$$

where h is a given nondecreasing and positive function. Then

$$\sum_{n=1}^{\infty} h_2(n) P(|S_n| \geqslant \mathscr{S}_n f_t(n)) < \infty \quad \textit{implies} \quad \mathrm{E}h(N_\infty(t)) < \infty,$$

and

$$\sum_{n=1}^{\infty} h_1(n) P(|S_n| \geqslant \mathscr{S}_n f_t(n)) = \infty \quad \textit{implies} \quad \mathrm{E}h(N_\infty(t)) = \infty.$$

Thus, for example, putting $h(n) = (\log\log n)^\alpha$, or $h(n) = n\gamma(\log n)^\beta$, for $n \geqslant 3$ and some $\alpha, \beta > 0$, $\gamma \geqslant 1$ $h(n) = 1$, $n = 1, 2$, and using Theorem 3 we get

$$\sum_{n=3}^{\infty} (\log\log n)^{\alpha-1} P(|S_n| \geqslant \mathscr{S}_n f_t(n))/n \log n = \infty$$

implies

$$\mathrm{E}(\log\log N_\infty(t))^\alpha = \infty,$$

and

$$\sum_{n=3}^{\infty} n^{\gamma-1} (\log n)^\beta P(|S_n| \geqslant \mathscr{S}_n f_t(n)) < \infty$$

implies

$$\mathrm{E}\{N_\infty^\gamma(t)(\log N_\infty(t))^\beta\} < \infty.$$

Of course, sufficient conditions for the convergence of these series are given in Theorem 2 and in Theorem 4 below.

Let us assume that $\mathrm{E}|X_n|^{2+s} = \beta_n^{2+s} < \infty$, $n \geqslant 1$, for some fixed $s > 0$. Let

$$B_n^{2+s} = \sum_{k=1}^{n} \beta_k^{2+s}, \quad L_n^s = B_n^{2+s}/\mathscr{S}_n^{2+s}, \quad L_n^{s*} = B_n^{2+s*}/\mathscr{S}_n^{2+s*},$$

where $s^* = \min(1, s)$, and let

$$L(n, s, \varepsilon) = \sum_{k=1}^{n} \mathrm{E}|X_k|^{2+s} I(|X_k| > \varepsilon \mathscr{S}_n)/B_n^{2+s}, \quad \log_+ x = \max(0, \log x).$$

The following theorem is a consequence of the results presented in [12], combined with Theorem 2.

THEOREM 4. *Assume that*

$$S_1(s, f_t, h) = \sum_{n=1}^{\infty} L_n^s L(n, s, f_t(n)) h(n) f_t^{-2-s}(n) < \infty.$$

(i) *If* $f_t^2(n) \geqslant 2s^{-1}(1+s)\log_+(1/L_n^s)$ *and*

$$S_2(s, f_t, h) = \sum_{n=1}^{\infty} L_n^s f_t^{-2(2+s)}(n)\, h(n) < \infty,$$

then

$$|A_\infty(h, f_t) - F(h, f_t)| \leqslant C\big(S_1(s, f_t, h) + S_2(s, f_t, h)\big).$$

(ii) *If* $f_t^2(n) \leqslant 2s^{-1}(1+s)\log_+(1/L_n^s)$ *and*

$$S_3(s, f_t, h) = \sum_{n=1}^{\infty} L_n^{s^*} h(n)\exp\{-(2+2s-s^*) f_t^2(n)/4(1+s)\} < \infty,$$

then

$$|A_\infty(h, f_t) - F(h, f_t)| \leqslant C\big(S_1(s, f_t, h) + S_3(s, f_t, h)\big).$$

Let us note that from Theorem 4 we immediately obtain the following

COROLLARY. *Suppose that* X_n, $n \geqslant 1$, *are independent random variables with* $EX_n = 0$, $L_n^s \leqslant Cn^{-s/2}$, $L_n^{s*} \leqslant Cn^{-s*/2}$, $n \geqslant 1$, *for some positive constants* $s > 0$ *and* $C > 0$. *Then*

$$\lim_{t \to 0^+} t^{(r+1)/\alpha} \sum_{n=1}^{\infty} n^r P(|S_n| > t\mathcal{S}_n n^\alpha) = C_{\alpha/(r+1)}/(r+1)$$

for every α, $s > 0$, $r \geqslant 0$, *such that* $(r+1)/(2+s) - s/2(2+s) < \alpha < (r+1)/(2+s)$.

3. The rate of convergence in the random SLLN's. Let us put

$$S_{N_n} = \sum_{k=1}^{N_n} X_k, \qquad M_n^2 = \sum_{k=1}^{N_n} \sigma_k^2,$$

where $\{N_n, n \geqslant 1\}$ is a sequence of positive integer-valued random variables not necessarily independent of $\{X_n, n \geqslant 1\}$.

Define

$$Z_\infty(t, \alpha) = \sum_{n=1}^{\infty} P|S_{N_n}| > tM_n^{1+2\alpha}), \qquad t > 0, \alpha > 0,$$

$$H(t) = \sum_{n=1}^{\infty} P(|M_n^2 - \lambda \mathcal{S}_n^2| \geqslant t\mathcal{S}_n^2),$$

where λ is a positive random variable such that, for some $0 < a \leqslant b < \infty$, $P(a \leqslant \lambda \leqslant b) = 1$.

PROPOSITION. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent random variables with* $EX_k = 0$, $EX_k^2 = \sigma_k^2 < \infty$, $k \geqslant 1$, *and for some* $0 < \alpha \leqslant 1/2$

$$\lim_{K \to \infty} \limsup_{t \to 0} t^{1/\alpha} \sum_{n > Kt^{-1/\alpha}} (1 + t^2 \mathcal{S}_n^{4\alpha})^{-1/\alpha} = 0, \tag{14}$$

(15) $$\lim_{K\to\infty} \limsup_{t\to 0} t^{1/\alpha} \sum_{n>Kt^{-1/\alpha}} \Phi(-t\mathscr{S}_n^{2\alpha}) = 0,$$

(16) $$\lim_{K\to\infty} \limsup_{t\to 0} t^{1/\alpha} \sum_{n>Kt^{-1/\alpha}} \sum_{k=1}^{\beta_n} P(|X_k| \geqslant t\mathscr{S}_n^{1+2\alpha}) = 0,$$

where $\beta_n = \max\{k\colon \mathscr{S}_k^2 \leqslant (b+t)\mathscr{S}_n^2\}$.

If $S_n/\mathscr{S}_n \xrightarrow{\mathscr{D}} N(0, 1)$, *then*

$$\liminf_{t\to 0} t^{1/\alpha}[Z_\infty(t, \alpha)+H(t)] \geqslant \liminf_{t\to 0} F(\alpha, t, b)\, t^{1/\alpha},$$

and

$$\limsup_{t\to 0} t^{1/\alpha}[Z_\infty(t, \alpha)-H(t)] \leqslant \limsup_{t\to 0} t^{1/\alpha} F(\alpha, t, a),$$

where $0 < a \leqslant b < \infty$ *are given constants such that* $P(a \leqslant \lambda \leqslant b) = 1$ *and*

$$F(\alpha, t, x) = 2\sum_{n=1}^{\infty} \Phi(-tx^\alpha \mathscr{S}_n^{2\alpha}).$$

From our Proposition we easily get the following

THEOREM 5. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent and identically distributed random variables with* $\mathrm{E}X_n = 0$ *and* $\mathrm{E}X_n^2 = 1$, *and let* $\{N_n, n \geqslant 1\}$ *be a sequence of positive integer-valued random variables such that for every* $t > 0$

$$H(t) = \sum_{n=1}^{\infty} P(|N_n - n\lambda| \geqslant tn) < \infty,$$

where λ *is a random variable. If for some* $0 < a \leqslant b < \infty$, $P(a \leqslant \lambda \leqslant b) = 1$, *and, for some* $\alpha > 0$,

$$\lim_{K\to\infty} \limsup_{t\to 0} t^{(1-2\alpha)/\alpha(1+2\alpha)} \mathrm{E}|X_1|^{4/(1+2\alpha)} I(|X_1| \geqslant Kt^{-1/2\alpha}) = 0,$$

then

$$\limsup_{t\to 0} t^{1/\alpha}\Big[\sum_{n=1}^{\infty} P(|S_{N_n}| \geqslant tN_n^{1/2+\alpha}) - H(t)\Big] \leqslant C_\alpha/a,$$

$$\liminf_{t\to 0} t^{1/\alpha}\Big[\sum_{n=1}^{\infty} P(|S_{N_n}| \geqslant tN_n^{1/2+\alpha}) + H(t)\Big] \geqslant C_\alpha/b,$$

$$\lim_{t\to 0} t^{1/\alpha} \sum_{n=1}^{\infty} P(|S_n| \geqslant tn^{1/2+\alpha}) = C_\alpha.$$

Note that our Proposition, even in the case $P(N_n = n) = 1$, $n \geqslant 1$, gives a generalization of the main result of Chen [3]. Furthermore, Theorem 5 presents an extension of the main results of Szynal [15], Sirazdinov, Gafurov and Komekov [14].

Proof of Proposition. Let us put $a_n(t) = (a-t)\mathcal{S}_n^2$, $b_n(t) = (b+t)\mathcal{S}_n^2$, $I_n(t) = [|M_n^2 - \lambda\mathcal{S}_n^2| \leqslant t\mathcal{S}_n^2]$. Then we have

$$(17)\qquad \sum_{n=1}^{\infty} P(|S_{N_n}| > tM_n b_n^{\alpha}(t),\, I_n(t)) \leqslant Z_{\infty}(t, \alpha) \leqslant \sum_{n=1}^{\infty} P(|S_{N_n}| > tM_n a_n^{\alpha}(t),\, I_n(t)) + H(t).$$

On the other hand, by the random central limit theorem given in [5], we have

$$\Delta_n = \sup_x |P(S_{N_n} < xM_n) - \Phi(x)| \to 0 \quad \text{as } n \to \infty.$$

Let, for every $t > 0$, $n_0(t)$ be such a positive integer that $n_0(t) \to \infty$ and $t^{1/\alpha} n_0(t) \to 0$ as $t \to 0$. Then, for every positive number K, we get

$$(18)\qquad t^{1/\alpha} \sum_{n \leqslant Kt^{-1/\alpha}} \left|P\left(|S_{N^n}| > tM_n a_n^{\alpha}(t)\right) - 2\Phi\left(-ta_n^{\alpha}(t)\right)\right| \leqslant 2t^{1/\alpha}\Big[\sum_{n=1}^{n_0(t)-1} \Delta_n + \sum_{n=n_0(t)}^{Kt^{-1/\alpha}} \Delta_n\Big] \leqslant 4t^{1/\alpha} n_0(t) + 2t^{1/\alpha}\left(Kt^{-1/\alpha} - n_0(t)\right) \max_{n_0(t) \leqslant k \leqslant Kt^{-1/\alpha}} \Delta_k \to 0 \quad \text{as } t \to 0.$$

Furthermore, for every $0 < \varepsilon < 1$, we obtain

$$(19)\qquad I(K, t, \alpha) = t^{1/\alpha} \sum_{n > Kt^{-1/\alpha}} P\left(|S_{N_n}| > tM_n a_n^{\alpha}(t),\, I_n(t)\right) \leqslant t^{1/\alpha} \sum_{n > Kt^{-1/\alpha}} P\left(|S_{\alpha_n}| > t\varepsilon a_n^{1/2+\alpha}(t)\right) + t^{1/\alpha} \sum_{n > Kt^{-1/\alpha}} P\left(\max_{\alpha_n \leqslant k \leqslant \beta_n} |S_k - S_{\alpha_n}| > (1-\varepsilon)\, t a_n^{1/2+\alpha}(t)\right),$$

where $\alpha_n = \min\{k\colon \mathcal{S}_k^2 \geqslant a_n(t)\}$ and $\beta_n = \max\{k\colon \mathcal{S}_k^2 \leqslant b_n(t)\}$.

By the results of Fuk [7] (Corollary 3 with $\beta = \alpha = 1/2$, $x = t\varepsilon\left(a_n(t)\right)^{1/2+\alpha}$, $y_1 = y_2 = \ldots = y = \alpha x/2$, $B_n^2 = C_{2,Y} = \mathcal{S}_n^2$) we get

$$P\left(|S_{\alpha_n}| > t\varepsilon\left(a_n(t)\right)^{1/2+\alpha}\right) \leqslant \sum_{k=1}^{\alpha_n} P\left(|X_k| \geqslant \alpha t\varepsilon\left(a_n(t)\right)^{1/2+\alpha}/2\right) + 2/\left[1 + \alpha t^2\varepsilon^2\left(a_n(t)\right)^{1+2\alpha}/4\mathcal{S}_{\alpha_n}^2\right]^{1/\alpha} + 2\exp\left\{-t^2\varepsilon^2\left(a_n(t)\right)^{1+2\alpha}/8e^2\mathcal{S}_{\alpha_n}^2\right\}.$$

On the other hand, from the results of Fuk [7] (Corollary 3 with β, α, $y_1 = \ldots = y$ in it as in above and $x = (1-\varepsilon)\, t\left(a_n(t)\right)^{1/2+\alpha}$) or by the result of

Borovkov [2], we obtain

$$P\big(\max_{\alpha_n\leqslant k\leqslant \beta_n} |S_k - S_{\alpha_n}| \geqslant (1-\varepsilon)\big(a_n(t)\big)^{1/2+\alpha}\big)$$
$$\leqslant \sum_{k=\alpha_n}^{\beta_n} P\big(|X_k| \geqslant \alpha(1-\varepsilon)\, t a_n^{1/2+\alpha}(t)/2\big) +$$
$$+2/[1+\alpha t^2(1-\varepsilon^2)\, a_n^{1+2\alpha}(t)/4(b-a+2t)\, \mathcal{S}_n^2]^{1/\alpha} +$$
$$+2\exp\{-t^2(1-\varepsilon^2)\, a_n^{1+2\alpha}(t)/8e^2(b-a+2t)\,\mathcal{S}_n^2\}.$$

Thus, by (19) and the definition of $a_n(t)$ and $b_n(t)$, for $\varepsilon = 1/2$, we have

$$(20)\qquad I(K, t, \alpha) \leqslant Ct^{1/\alpha} \sum_{n>Kt^{-1/\alpha}} \sum_{k=1}^{\beta_n} P\big(|X_k| \geqslant \alpha t(a-t)^{1/2+\alpha}\, \mathcal{S}_n^{1+2\alpha}/4\big) +$$
$$+Ct^{1/\alpha} \sum_{n>Kt^{-1/\alpha}} [1+t^2(a-t)^{1+2\alpha}\, \mathcal{S}_n^4/(b-a+2t)]^{-1/\alpha} +$$
$$+Ct^{1/\alpha} \sum_{n>K^{-1/\alpha}} [1+t^2(a-t)^{1+2\alpha}\, \mathcal{S}_n^{4\alpha}]^{-1/\alpha} +$$
$$+Ct^{1/\alpha} \sum_{n>Kt^{-1/\alpha}} \exp\{-t^2(a-t)^{1+2\alpha}\, \mathcal{S}_n^{4\alpha}/32e^2\} +$$
$$+Ct^{1/\alpha} \sum_{n>Kt^{-1/\alpha}} \exp\{-t^2\, \mathcal{S}_n^{4\alpha}(a-t)^{1+2\alpha}/32e^2(b-a+2t)\}.$$

But, by our assumptions, a and b are given positive constants and $b-a \geqslant 0$. Hence from (14)-(20) it is not difficult to obtain the first part of our Proposition. The second one follows in a similar way.

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REFERENCES

[1] I. A. Ahmad, *A remark on the SLLN for random partial sums*, Z. Wahrsch. Verw. Gebiete 54 (1980), p. 119-124.

[2] A. A. Borovkov, *Notes on inequalities for sums of independent variables*, Teor. Verojatnost. i Primenen. 17 (1972), p. 588-590.

[3] R. Chen, *A remark on the strong law of large numbers*, Proc. Amer. Math. Soc. 61 (1976), p. 112-116.

[4] H. Cramér, *Mathematical methods of statistics*, Princeton 1946.

[5] M. Csörgő and Z. Rychlik, *Weak convergence of sequences of random elements with random indices*, Math. Proc. Camb. Phil. Soc. 88 (1980), p. 171-174.

[6] V. A. Egorov, *On the strong law of large numbers and the law of the iterated logarithm for sequences of independent random variables*. Teor. Verojatnost. i Primenen. 15 (1970), p. 520-527.

[7] D. H. Fuk, *Some probability inequalities for martingales*, Sibirskij Mat. Ž., 14 (1973), p. 185-193.

[8] D. Landers and L. Rogge, *The exact approximation order in the central limit theorem for random summation*, Z. Wahrsch. Verw. Gebiete 36 (1976), p. 269-283.

[9] V. V. Petrov, *Sums of independent random variables*, New York, Springer 1972.

[10] Z. Rychlik, *The order of approximation in the random central limit theorem*, Lect. Notes in Math. 656 (1978), p. 225-236.

[11] Z. Rychlik, *Rozkłady graniczne losowo indeksowanych ciągów zmiennych losowych*, Rozprawa habilitacyjna, Lublin 1980.

[12] Z. Rychlik, *Nonuniform central limit bounds with applications to probabilities of deviations*, Teor. Verojatnost. i Primenen 28 (1983), p. 646-652.

[13] N. C. Severo and J. Slivka, *On the strong law of large numbers*, Proc. Amer. Math. Soc. 24 (1970), p. 729-734.

[14] S. H. Siraždinov, M. U. Gafurov and B. Komekov, *Some remarks on the strong law of large numbers for sums of a random number of random variables*, Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk 22 (1978), p. 28-34.

[15] D. Szynal, *On almost complete convergence for the sum of a random number of independent random variables*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys. 20 (1972), p. 571-574.

[16] H. Teicher, *Generalized exponential bounds, iterated logarithm and strong laws*, Z. Wahrsch. Verw. Gebiete 48 (1979), p. 293-307.

[17] C. F. Wu, *A note on the convergence rate of the strong law large numbers*, Bull. Inst. Math. Acad. Sinica 1 (1973), p. 121-124.

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