PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 5, Fasc. 2 (1985), p. 187-196

STRONG CONVERGENCE OF VECTOR-VALUED PRAMARTS AND SUBPRAMARTS

BY

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Abstract. We prove a subpramart convergence theorem which is a complete solution of a problem raised by L. Egghe, and a pramart convergence theorem which partly solves a problem of L. Sucheston. We obtain also certain decomposition of subpramarts.

1. Introduction. In the present paper we deal with subpramarts and pramarts in Banach spaces. In Section 2 we give necessary definitions and notations, formulate problems of Sucheston and Egghe, and recall known results concerning the problems.

In Section 3 we prove the main result of this paper, it is the subpramart convergence theorem – Theorem 3.5 – which completely solves the problem of Egghe. Theorem 3.6 – the pramart convergence theorem – gives a partial solution of the problem of Sucheston. The clue to the both theorems is Lemma 3.4, which can be regarded independently as a very useful tool in other similar reasonings.

In Section 4 we obtain an extension of a subpramart decomposition theorem due to Millet and Sucheston and we give an example showing that this decomposition does not hold true in the general case.

2. Preliminaries. Let (Ω, F, P) be a probability space, $(F_n)_{n\in N}$ an increasing sequence of sub-σ-algebras of the σ-algebra F, T – the set of all bounded stopping times with respect to $(F_n)_{n\in N}$.

Definition 2.1. For a Banach space E, the adapted sequence $(X_n, F_n)_{n\in N}$ of E-valued random variables is called a *pramart* if $(X_\sigma - E^{F_\sigma} X_\tau)$ converges to zero in probability, uniformly in $\tau \geqslant \sigma$, i.e., for every $\varepsilon > 0$ there exists $\sigma_0 \in T$ such that $\sigma_0 \leqslant \sigma \leqslant \tau$ implies

$$P(\{\|X_\sigma - E^{F_\sigma} X_\tau\| > \varepsilon\}) \leqslant \varepsilon.$$

Sucheston stated in 1979 the question whether Chatterji's result for martingales holds true in the case of pramarts. Let us report it exactly:

Problem 2.2. Let E be a Banach space with the Radon-Nikodym property (RNP), and let (X_n) be an E-valued pramart bounded in $L_1(E)$-norm. Is X_n strongly a.s. convergent?

Let now E be a Banach lattice.

Definition 2.3. The adapted sequence (X_n, F_n) of E-valued random variables is called a *subpramart* if $(X_\sigma - E^{F_\sigma} X_\tau)^+$ converges to zero in probability, uniformly in $\tau \geqslant \sigma$.

In the real-valued case Millet and Sucheston proved

THEOREM 2.4 ([11], p. 96, Th. 3.7). *Let (X_n) be a subpramart which satisfies the following condition*:

$$\liminf_{n\to\infty} \mathrm{E}(X_n^+) + \liminf_{n\to\infty} \mathrm{E}(X_n^-) < \infty .$$

Then X_n converges a.s. to an integrable r.v.

As it is seen from the definitions every submartingale is a subpramart, and for submartingales Heinich proved the following

THEOREM 2.5. ([8]). *Let E be a Banach lattice with the RNP. Then every positive $L_1(E)$-bounded submartingale is a.s. convergent.*

Egghe raised a problem which is connected with the problem of Sucheston.

Problem 2.6. (see [6] or [5]). Let E be a Banach lattice with the RNP, and let (X_n, F_n) be an E-valued positive subpramart with an $L_1(E)$-bounded subsequence. Is X_n strongly a.s. convergent?

Both problems have been affirmatively solved under additional assumptions.

THEOREM 2.7 ([7], p. 360, Th. 2.4, and [10], p. 1043, Th. 3.5). *Let E be a Banach space with the RNP, and let $(X_n, F_n)_{n\in N}$ be an E-valued pramart for which there is a subsequence (X_{n_k}) which is uniformly integrable. Then X_n itself converges strongly a.s.*

THEOREM 2.8. ([13], Th. 3.1). *Let E be a Banach space with an unconditional basis and with the RNP. Let $(X_n, F_n)_{n\in N}$ be a pramart with and $L_1(E)$-bounded subsequence. Then X_n converges strongly a.s.*

In this paper we prove a pramart convergence theorem for weakly sequentially complete Banach spaces (Theorem 3.6).

Let us recall some known results concerning the strong convergence of subpramarts.

THEOREM 2.9. ([6], Cor. 2.8). *Let E be a Banach lattice with the RNP. Then every E-valued positive subpramart $(X_n, F_n)_{n\in N}$ is strongly convergent to an integrable r.v., if there is a subsequence $(n_k) \subset N$ such that $(X_{n_k})_{k\in N}$ is uniformly integrable.*

THEOREM 2.10 ([13], Th. 2.2). *Let E be a Banach lattice with an unconditional basis such that the order is induced by the basis, and let E have the RNP. Then every E-valued positive subpramart* $(X_n, F_n)_{n\in N}$ *is strongly a.s. convergent to an integrable r.v., if there is a subsequence* $(X_{n_k})_{k\in N}$ *which is* $L_1(E)$*-bounded.*

Here we solve affirmatively the problem of Egghe without any additional assumptions.

In the paper we use a method based on a Kadec-Klee lattice renorming theorem due to Davis, Ghoussoub and Lindenstrauss:

THEOREM 2.11 ([2]). *A Banach lattice* $(E, \|\ \|)$ *is order continuous if and only if there is an equivalent lattice norm* $\|\ \|_1$ *on E such that* $\{x_n\}_{n\in N}\subset E$, $x_n \xrightarrow{w} x$ *and* $\|x_n\|_1 \to \|x\|_1$ *imply* $\|x_n - x\|_1 \to 0$.

It is obvious that if E is separable, then the equivalent norm has the Kadec-Klee property with respect to a countable set of functionals.

3. Subpramart and pramart convergence theorems. In order to profit by the Kadec-Klee renorming theorem we have to report two lemmas from [6].

LEMMA 3.1. ([6], L. 2.2). *Let* $(X_n, F_n)_{n\in N}$ *be a positive subpramart with values in a Banach lattice E. Let* $x' \in (E')_+$ *be arbitrary. Then* $(x'(X_n), F_n)_{n\in N}$ *is a positive subpramart.*

Definition 3.2 ([6], Def. 2.3). Let $(X_n^m, F_n)_{n\in N}$ be a sequence of real-valued subpramarts. It is called a *uniform sequence* of subpramarts if $\forall \varepsilon > 0\, \exists \sigma_0 \in T$ such that for every $\sigma, \tau \in T$, $\sigma_0 \leqslant \sigma \leqslant \tau$,

$$P\big(\{\sup_{m\in N}(X_\sigma^m - E^{F_\sigma} X_\tau^m) \leqslant \varepsilon\}\big) \geqslant 1-\varepsilon .$$

LEMMA 3.3. ([6], L. 2.4). *Let* $(X_n^m, F_n)_{n\in N}$ *be a uniform sequence of real-valued positive subpramarts. Suppose that there is a subsequence* $(n_k)_{k\in N}$ *such that:*

$$\sup_{k\in N} \int_\Omega \sup_{m\in N} X_{n_k}^m \, dP < \infty .$$

Then each subpramart $(X_n^m, F_n)_{n\in N}$ *converges a.s. to an integrable r.v.* X_∞^m, *and we have:*

$$\sup_{m\in N} X_n^m \to \sup_{m\in N} X_\infty^m \quad a.s. \qquad for\ n\to\infty .$$

The following lemma reduces our problems to a situation similar to that in Theorem 2.7 and 2.9. The author is very grateful to J. Szulga for a considerable simplification of the original proof of this lemma.

LEMMA 3.4. *Let E be a Banach space and let* $(X_n, F_n)_{n\in N}$ *be an adapted sequence of E-valued random variables such that*

$$\sup_{n\in N} \mathrm{E}\|X_n\| < \infty .$$

Then there exists a subsequence $(n_k)_{k\in N} \subset N$ *such that, for every* k, $X_{n_k} = Y_{n_k} + Z_{n_k}$, *where* Y_{n_k} *and* Z_{n_k} *are* F_{n_k}*-measurable,* $(Y_{n_k})_{k\in N}$ *is uniformly integrable and* $Z_{n_k} \to 0$ *a.s. if* $k \to \infty$.

Proof. Note that it is enough to find such a subsequence (n_k), where Z_{n_k} tends to zero in probability.

For every positive integer m set

$$g_X^m(t) = \sup_{n \geqslant m} \int_{\{\|X_n\| > t\}} \|X_n\| \, dP.$$

Since $g_X^1(t)$ is nonincreasing and nonnegative, then there is a positive real number α such that

$$\lim_{t\to\infty} g_X^1(t) = \alpha.$$

Thus there is an increasing sequence $(t_k)_{k\in N}$ such that $\lim_{k\to\infty} t_k = \infty$, and

$$\alpha \leqslant g_X^1(t_k) < \alpha - 1/k.$$

It is obvious that also $\forall m \in N \lim_{t\to\infty} g_X^m(t) = \alpha$, hence

$$\forall m \forall k \quad g_X^m(t_k) \geqslant \alpha.$$

There is then an increasing sequence of positive integers $(n_k)_{k\in N}$ such that

$$\int_{\{\|X_{n_k}\| > t_k\}} \|X_{n_k}\| \, dP > \alpha - 1/k.$$

Set

$$Y_{n_k} = X_{n_k} \mathrm{I}_{(\|X_{n_k}\| \leqslant t_k)} \quad \text{and} \quad Z_{n_k} = X_{n_k} \mathrm{I}_{(\|X_{n_k}\| > t_k)}.$$

Then Y_{n_k} and Z_{n_k} are F_{n_k}-measurable.
Since

$$P(\|Z_{n_k}\| \geqslant \varepsilon) \leqslant P(\|X_{n_k}\| > t_k) \leqslant 1/t_k \sup_{n\in N} \mathrm{E}\, \|X_n\|,$$

Z_{n_k} tends to zero in probability.

In order to prove that $(Y_{n_k})_{k\in N}$ is uniformly integrable it is enough to show that

$$\lim_{t\to\infty} \sup_{k\in N} \int_{\{\|Y_{n_k}\| > t\}} \|Y_{n_k}\| \, dP = 0.$$

Since

$$g(t) = \sup_{k\in N} \int_{\{\|Y_{n_k}\| > t\}} \|Y_{n_k}\| \, dP$$

is a nonincreasing function, then $\lim_{t\to\infty} g(t) = \lim_{i\to\infty} g(t_i)$. But

$$\begin{aligned} g(t_i) &= \sup_{k\geq i} \int_{\{t_i < \|X_{n_k}\| \leq t_k\}} \|X_{n_k}\|\, dP \\ &= \sup_{k\geq i}\Big(\int_{\{\|X_{n_k}\| > t_i\}} \|X_{n_k}\|\, dP - \int_{\{\|X_{n_k}\| > t_k\}} \|X_{n_k}\|\, dP\Big) \\ &\leq \sup_{k\geq i}\left(\alpha - \frac{1}{i} - \alpha + \frac{1}{k}\right) = \frac{2}{i}. \end{aligned}$$

Therefore $\lim_{i\to\infty} g(t_i) = 0$.

Now we are in a position to solve the problem of Egghe.

THEOREM 3.5. *Let E be a Banach lattice with the RNP, and let $(X_n, F_n)_{n\in N}$ be an E-valued positive subpramart such that*

$$\sup_{k\in N} \mathrm{E}\,\|X_{n_k}\| < \infty$$

for some subsequence $(X_{n_k})_{k\in N}$.

Then X_n is strongly a.s. convergent.

Proof. By Lemma 3.4 we can assume that $X_{n_k} = Y_{n_k} + Z_{n_k}$, where $(Y_{n_k})_{k\in N}$ is uniformly integrable and Z_{n_k} tends to zero a.s.

According to Lemma 3.1, $x'(X_n)$ is for every $x'\in(E')_+$ a real-valued subpramart and, by Theorem 2.4, there is an r.v. $f_{x'}$ such that

$$(*) \qquad x'(X_n) \underset{n\to\infty}{\to} f_{x'} \quad \text{a.s.}$$

Since $x'(Z_{n_k}) \underset{k\to\infty}{\to} 0$ a.s., then $x'(Y_{n_k}) \underset{k\to\infty}{\to} f_{x'}$ a.s. and, by the uniform integrability, in the L_1-norm.

Therefore, for every $A\in F_\infty = \sigma(\bigcup_n F_n)$, $\int_A x'(Y_{n_k})\,dP$ is convergent, and so $(\int_A Y_{n_k}\,dP)_{k\in N}$ is weakly convergent Cauchy sequence.

Since E does not contain any c_0-space, then (by Th. 1.c.4 in [9]) E is weakly sequentially complete; and so $\int_A Y_{n_k}\,dP$ is weakly convergent for every $A\in F_\infty$.

Let

$$\mu(A) = \text{w-}\lim_{k\to\infty} \int_A Y_{n_k}\,dP.$$

Since μ is of bounded variation, by the Caratheodory, Hahn and Kluvanek extension theorem (see [3]) μ is a countable additive measure on F_∞. Moreover, μ is P-continuous. By the RNP there exists an r.v. $X = L_1(\Omega, F_\infty, P; E)$ such that

$$\mu(A) = \int_A X\,dP \quad \text{for every } A\in F_\infty,$$

hence

$$\forall x' \in (E')_+ \ \forall A \in F_\infty \qquad \int_A x'(Y_{n_k})\,dP \underset{k\to\infty}{\to} \int_A x'(X)\,dP.$$

On the other hand,

$$\int_A x'(Y_{n_k})\,dP \underset{k\to\infty}{\to} \int_A f_{x'}\,dP,$$

so we get

$$\int_A x'(X)\,dP = \int_A f_{x'}\,dP \qquad \text{for every } A \in F_\infty$$

and, since $x'(X)$ and $f_{x'}$ are F_∞-measurable, $x'(X) = f_{x'}$ a.s. for every $x' \in (E')_+$. From $(*)$ we have $x'(X_n) \underset{n\in N}{\to} x'(X)$ a.s. for every $x' \in (E')_+$.

Since E has the RNP, E is order continuous and, by Theorem 2.11, there is an equivalent norm $\| \ \|_1$ on E which has the Kadec-Klee property. Let D be a countable subset of $(E')_+$ such that for every $x \in E_+$

$$\|x\|_1 = \sup_{x'\in D} x'(x).$$

Then $\big(x'(X_n), F_n\big)_{n\in N}$ is a uniform sequence of positive subpramarts, and by Lemma 3.3

$$\|X_n\|_1 \underset{n\to\infty}{\to} \|X\|_1 \qquad \text{a.s.}$$

Finally, by the Kadec-Klee property of $(E, \| \ \|_1)$ we get

$$X_n \underset{n\to\infty}{\to} X \qquad \text{strongly a.s.}$$

THEOREM 3.6. *Let E be a weakly sequentially complete Banach space with the RNP, and let $(X_n, F_n)_{n\in N}$ be an E-valued pramart such that*

$$\sup_{k\in N} \mathrm{E}\,\|X_{n_k}\| < \infty$$

for some subsequence (X_{n_k}). Then X_n is strongly a.s. convergent.

Proof. According to Lemma 3.4, we can assume that $X_{n_k} = Y_{n_k} + Z_{n_k}$, where $(Y_{n_k})_{k\in N}$ is uniformly integrable and $Z_{n_k} \to 0$ a.s., $k \to \infty$.

Since we consider a sequence of strongly measurable functions which are separable valued, we can assume that E is separable. By the classical Kadec renorming theorem, there is an equivalent norm $\| \ \|_1$ on E with the Kadec-Klee property. Moreover, there exists a countable set $D \subset \{x' \in E' \mid \|x'\| \leqslant 1\}$ such that for every $x \in E$

$$\|x\|_1 = \sup_{x'\in D} |x'(x)|.$$

For every $x' \in E'$, $x'(X_n)$ is a real-valued pramart. By Theorem 2.4, there is an r.v. $f_{x'}$ such that $x'(X_n) \to f_{x'}$ a.s.

Next, as in the proof of Theorem 3.5, we get that for every $A \in F_\infty$

$$\Big(\int_A Y_{n_k}\, dP\Big)_{k \in N}$$

is a weakly convergent Cauchy sequence. By the weak completeness of E, $\big(\int_A Y_{n_k}\, dP\big)_{k \in N}$ is weakly convergent.

Afterwards, continuing as in the proof of Theorem 3.5, we get an r.v. X such that $\forall x' \in E'$ $x'(X_n) \to x'(X)$ a.s.; hence also $|x'(X_n)| \to |x'(X)|$ a.s., where $|x'(X_n)|$ is a real-valued positive subpramart. By Lemma 3.3

$$\|X_n\|_1 = \sup_{x' \in D} |x'(X_n)| \to \sup_{x' \in D} |x'(X)| = \|X\|_1,$$

and by the classical Kadec renorming theorem $X_n \to X$ a.s.

4. Decomposition of subpramarts. In this section we discuss an extension to Banach lattices of the following subpramart decomposition theorem of Millet-Sucheston:

THEOREM 4.1 ([11], p. 93, Prop. 3.3). *Let $(X_n, F_n)_{n \in N}$ be a real-valued positive integrable adapted sequence. Then (X_n) is a subpramart if and only if there is a positive submartingale (R_n, F_n) such that for every positive integer n, $R_n \leqslant X_n$ a.s., and*

$$\lim_{\tau \in T} (X_\tau - R_\tau) = 0 \textit{ in probability}.$$

We consider now a Banach lattice E with an unconditional basis $\{e_i\}$, such that the lattice order is induced by the basis, i.e.,

$$\sum_{i=1}^{\infty} x^i e_i \leqslant \sum_{i=1}^{\infty} y^i e_i \quad \text{iff} \quad \forall i \in N \quad x^i \leqslant y^i,$$

where

$$\sum_{i=1}^{\infty} x^i e_i \quad \text{and} \quad \sum_{i=1}^{\infty} y^i e_i \in E.$$

Remark 4.2. On every Banach space E with an unconditional basis $\{e_i\}_{i \in N}$ there is a norm $\|\ \|_1$ equivalent to the original one, and such that for this norm the basis $\{e_i\}_{i \in N}$ is strictly hyperorthogonal ([12], Theorem 20.3). $(E, \|\ \|_1)$ is a Banach lattice with the order induced by the basis $\{e_i\}_{i \in N}$.

The following lemma is easy to verify.

LEMMA 4.3. *Let E be a Banach lattice with the order induced by a basis $\{e_i\}_{i \in N}$.*

Let $\{e^i\}_{i\in N}$ *be a biorthogonal sequence of functionals and set*

$$D = \{x' \in E' \mid \|x'\| \leqslant 1,\ x' = \sum_{i\in S} x_i e^i,\ S \subset N,\ x_i \in Q_+\},$$

where Q_+ *is the set of positive rational numbers, and* S *is finite.*

Then D *is countable and*

$$\forall x \in E_+ \qquad \|x\| = \sup_{x'\in D} x'(x).$$

The next lemma may be interesting in itself as it shows a large class of submartingales which are pramarts.

LEMMA 4.4. *Let* E *be a Banach lattice with the RNP, and let* $(R_n, F_n)_{n\in N}$ *be an* E*-valued positive* L_1*-bounded submartingale. Then* $(R_n, F_n)_{n\in N}$ *is a pramart.*

Proof. Let $R_n = M_n - Z_n$ be the Riesz decomposition of R_n, where $(M_n, F_n)_{n\in N}$ is a martingale and $(Z_n, F_n)_{n\in N}$ is a positive potential (see [4]). Since, for every increasing sequence of stopping times $(\tau_k) \subset T$, R_{τ_k} is a positive L_1-bounded submartingale, then (by Theorem 2.5) R_{τ_k} is convergent in probability, and so $(R_\tau)_{\tau\in T}$ also converges in probability. Similarly, by the well known Chatterji theorem, $(M_\tau)_{\tau\in T}$ is convergent in probability. Thus $(z_\tau)_{\tau\in T}$ converges in probability. Moreover, $\lim_{\tau\in T} Z_\tau = 0$, as $\text{w-}\lim_{\tau\in T} Z_\tau = 0$.

Since

$$\forall \sigma, \tau \in T, \qquad \tau \geqslant \sigma \Rightarrow 0 \leqslant E^{F_\sigma} R_\tau - R_\sigma = Z_\sigma - E^{F_\sigma} Z_\tau \leqslant Z_\sigma,$$

then $\sup_{\tau\geqslant\sigma} \|E^{F_\sigma} R_\tau - R_\sigma\| \leqslant \|Z_\sigma\|$, and so R_n is a pramart.

THEOREM 4.5. *Let* E *be a Banach lattice with the order induced by an unconditional basis* $\{e_i\}_{i\in N}$*, and with the RNP. Let* $(X_n, F_n)_{n\in N}$ *be an* E*-valued positive integrable adapted sequence. Then* X_n *is a subpramart iff there exists a positive submartingale* $(R_n, F_n)_{n\in N}$ *such that* $\forall n \in N$ $R_n \leqslant X_n$ *a.s., and* $\lim_{\tau\in T}(X_\tau - R_\tau) = 0$, *in probability.*

Proof. Necessity. Let $\{e^i\}_{i\in N}$ be the biorthogonal sequence of functionals. For $n, i \in N$ set $X_n^i = e^i(X_n)$. Then (X_n^i, F_n) is (for every $i \in N$) a real-valued positive subpramart; and as in the proof of Theorem 4.1 (see [11])

$$r_n^i = \inf_{\tau\geqslant n} \mathrm{E}^{F_n} X_\tau^i$$

is a submartingale such that $\forall n \in N$ $r_n^i \leqslant X_n^i$ a.s., and $\lim_{\tau\in T}(X_\tau^i - r_\tau^i) = 0$, in probability.

Set

$$R_n = \bigwedge_{\tau,n} E^{F_n} X_\tau.$$

Then

$$R_n = \bigwedge_{\tau \geqslant n} E^{F_n} \sum_{i=1}^{\infty} X_\tau^i e_i = \bigwedge_{\tau \geqslant n} \sum_{i=1}^{\infty} (E^{F_n} X_\tau^i) e_i = \sum_{i=1}^{\infty} \inf_{\tau \geqslant n} (E^{F_n} X_\tau^i) e_i = \sum_{i=1}^{\infty} r_n^i e_i.$$

Since, for every i, r_n^i is a submartingale, then R_n is also a submartingale. Moreover, $0 \leqslant R_n \leqslant X_n$ a.s. $\forall n \in N$, and it is easy to verify that if R_n' is a submartingale such that $R_n' \leqslant X_n$, then $R_n' \leqslant R_n$.

Let D be as in Lemma 4.3. Since

$$\lim_{\tau \in T} (X_\tau^i - r_\tau^i) = 0$$

in probability, and $r_n^i = e^i R_n^i$, then

$$\forall x' \in D \quad \lim_{\tau \in T} x'(X_\tau - R_\tau) = 0 \quad \text{in probability}.$$

In order to prove that

$$\lim_{\tau \in T} (X_\tau - R_\tau) = 0 \quad \text{in probability},$$

it is enough to show that for every increasing sequence of stopping times τ_n

$$(X_{\tau_n} - R_{\tau_n}) \underset{n \to \infty}{\to} 0 \text{ a.s.}$$

By Lemma 4.4, R_{τ_n} is a pramart, hence $(X_{\tau_n} - R_{\tau_n})_{n \in N}$ is a positive subpramart, and, for $x' \in D$, $x'(X_{\tau_n} - R_{\tau_n})$ is a uniform sequence of subpramarts. Using Lemma 3.3 we get

$$\|X_{\tau_n} - R_{\tau_n}\| = \sup_{x' \in D} x'(X_{\tau_n} - R_{\tau_n}) \to 0, \quad n \to \infty.$$

Sufficiency. Assume that there exists a submartingale (R_n', F_n) such that $R_n' \leqslant X_n$ and

$$\lim_{\tau \in T} (X_\tau - R_\tau') = 0, \quad \text{in probability}.$$

Since $R_n' \leqslant R_n \leqslant X_n$, we have

$$\lim_{\tau \in T} (X_\tau - R_\tau) = 0, \quad \text{in probability}.$$

But

$$X_\sigma - E^{F_\sigma} X_\tau \leqslant X_\sigma - R_\sigma,$$

thus X_n is a subpramart.

As the following example shows, the above decomposition, i.e., the "necessity" part of the previous theorem, fails even for the Banach lattice $L_2([0, 1])$ and for a constant sequence of trivial σ-algebras.

Example 4.6. We consider $L_2([0, 1])$ as a Banach lattice with the natural pointwise order. Let, for $n \in N$, $F_n = \{\Phi, \Omega\}$.

Set $A_n = [k/2^m, (k+1)/2^m)$ for $n = 2^m + k$, where $m = 0, 1, 2, \ldots, k = 0, 1, \ldots, 2^m - 1$, and set $X_n = \mathbb{1}_{A_n^c}$. Then $(X_n, F_n)_{n \in N}$ is an L_2-valued positive order bounded pramart as it is deterministic norm-convergent sequence. Note that

$$R_n = \bigwedge_{m \geq n} E^{F_n} X_m = \bigwedge_{m \geq n} X_m = \bigwedge_{m \geq n} \mathbb{1}_{A_m^c} = \mathbb{1}_{\bigcap_{m \geq n} A_m^c} = 0.$$

Let (R_n', F_n) be a submartingale such that $R_n' \leqslant X_n$. Then $R_n' \leqslant R_n$, and $X_n - R_n' \geqslant X_n - R_n = X_n$.

However $X_n \underset{n \to \infty}{\to} \mathbb{1}$ strongly. Thus $(X_n - R_n) \underset{n \to \infty}{\not\to} 0$.

The above example has been given to me by J. Szulga (oral comunication).

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Received on 28. 9. 1982;
revised version on 21. 12. 1982