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LIMITING PROPERTIES OF DIFFERENCE BETWEEN THE SUCCESSIVE k-TH RECORD VALUES

BY

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Abstract. Let $\{Y_n^{(k)}\}$ denote the sequence of the k-th record statistics corresponding to the sequence $\{X_i\}$ of i.i.d. random variables. In this paper it is shown that $k(Y_{n+1}^{(k)} - Y_n^{(k)})$ tends weakly (for $k \to \infty$) to the exponentially distributed random variable for a wide class of absolutely continuous random variables X_i.

1. Introduction. Suppose that $\{X_n\}$, $n = 1, 2, \ldots,$ is a sequence of independent random variables with common distribution function (d.f.). Let $X_1^{(n)} \leqslant \ldots \leqslant X_n^{(n)}$ denote order statistic in the sequence $X_1, X_2, \ldots, X_n$. By

$$Y_n^{(k)} = X_{L_k(n)}^{(L_k(n)+k-1)}, \quad n = 0, 1, 2, \ldots, k \geqslant 1,$$

where

$$L_k(0) = 1,$$
$$L_k(n+1) = \min\{j: X_{L_k(n)}^{(L_k(n)+k-1)} < X_j^{(j+k-1)}\}, \quad n = 0, 1, 2, \ldots,$$

we define a sequence of the k-th record statistics.

Properties of the k-th record statistics were discussed extensively in a lot of papers. Limiting distributions of the k-th record values for $n \to \infty$ were obtained by Resnick [5], and Dziubdziela and Kopociński [1]. Some characterizations of the geometric and exponential distributions by k-th record values one can find in papers due to Srivastava [6], [7], Grudzień [2], Grudzień and Szynal [3], and Nagaraja [4].

Write $Z_n^{(k)} = Y_{n+1}^{(k)} - Y_n^{(k)}$. Grudzień [2] discussed extensively the characterization of the exponential and geometric distribution by random variables $Z_n^{(k)}$.

In this paper limiting distribution of random variables

(1) $$U_n^{(k)} = kZ_n^{(k)}, \quad n = 1, 2, \ldots,$$

is obtained for $k \to \infty$.

2. Limiting distributions of random variables $U_n^{(k)}$. Grudzień [2] showed that if random variable X has the exponential distribution with probability density function (p.d.f), with respect to the Lebesgue measure, of the form

(2) $$f(x; \lambda, \mu) = \begin{cases} \lambda^{-1} \exp[-(x-\mu)\lambda^{-1}] & \text{if } x > \mu, \\ 0 & \text{if } x \leqslant \mu, \end{cases}$$

then $Z_n^{(k)}$ has (p.d.f.) $f(z; \lambda/k, 0)$. Thus, from definition (1), it follows that $U_n^{(k)}$ has p.d.f. $f(u; \lambda, 0)$. In this section we prove that, for a wide class of absolutely continuous random variables X, there exists a $\lambda > 0$ such that $U_n^{(k)} \xrightarrow{w} U_n$, where U_n has p.d.f. $f(u; \lambda, 0)$.

The following convention is used: a.e. means almost everywhere with respect to the Lebesgue measure.

Firstly we prove the following

THEOREM 2.1 *Suppose that $F(x)$ is a d.f. with p.d.f. $f(x)$ and the interval S as the support.*

If a sequence $\{F_k\}_{k=1,2,\ldots}$ of d.f. is of the form

(3) $$F_k(u) = \begin{cases} 1 - \int\limits_S \left[\dfrac{1-F(x+u/k)}{1-F(x)}\right]^k dG_k(x) & \text{for } u \geqslant 0, \\ 0 & \text{for } u < 0, \end{cases}$$

where $\{G_k\}_{k=1,2,\ldots}$ is a sequence of d.f., and

(i) *$f(x)/[1-F(x)]$ is a differentiable function with the first derivative bounded a.e. on S,*

(ii) *$G_k \xrightarrow{w} G$, where $G(x)$ is 0 for $x \leqslant x_0$ and 1 otherwise, with $x_0 \in \partial S$, then $F_k \to F_\lambda^*$ such that $F_\lambda^*(u) = 1 - \exp(-u/\lambda)$ and $\lambda^{-1} = f(x_0)/[1-F(x_0)]$, where $f(x_0)/[1-F(x_0)]$ means respectively the right or the left limit in the case $x_0 \in \partial S$ (F_0^*, F_∞^* mean distributions concentrated at infinity and 0).*

Proof. Let us notice that from assumption (i) it follows that, for arbitrary fixed $u > 0$,

$$\log[(1-F(x+u/k))/(1-F(x))] = -r(x)u/k - r'(x+\theta u/k)u^2/k^2,$$

where $0 < \theta < 1$ and $r(x) = f(x)/[1-F(x)] \geqslant 0$. Thus

(4) $$1 - F_k(u) = \int\limits_S \exp[-r'(x+\theta u/k)u^2/k] \exp[-r(x)u]\, dG_k(x).$$

Let's define the new function $H_k(u)$ as follows:

$$H_k(u) = \int_S \exp[-r(x)u]\, dG_k(x). \tag{5}$$

From assumption (i) it follows that there exists a number $0 < M < \infty$ such that $|r'(x)| \leqslant M$ a.e. on S. Hence it follows by (4) that

$$H_k(u)\exp[-Mu^2/k] \leqslant 1 - F_k(u) \leqslant H_k(u)\exp[Mu^2/k]. \tag{6}$$

From (5) and assumption (ii) it follows that

$$H_k(u) \to \exp[-r(x_0)u]. \tag{7}$$

Applying (6) and (7) we can obtain that $F_k(u) \to 1 - \exp(-u/\lambda)$, where $\lambda^{-1} = r(x_0)$. From (3) it follows that $F_k(u) \to 0$ for $u \leqslant 0$ and this completes the proof.

COROLLARY 2.1. *Since* $\exp(-u/\lambda)$ *is a continuous function,*

$$F_k(u) \rightrightarrows 1 - \exp(-u/\lambda).$$

The next theorem concerns the limiting distribution of $U_n^{(k)}$.

THEOREM 2.2. *Suppose that* X *has d.f.* $F(x)$, *p.d.f.* $f(x)$, *the interval* S *as the support and* $f(x)/(1-F(x))$ *is a differentiable function with the first derivative bounded a.e. on* S.

Then random variables $U_n^{(k)} \xrightarrow{w} U_n$, *where* U_n *has d.f.* $F_\lambda^*(u) = 1 - \exp(-u/\lambda)$ *with* $\lambda^{-1} = f(x_0)$, *where* $x_0 = \inf\{x \in S\}$.

Proof. The distribution function of $Z_n^{(k)}$ is (see Grudzień [2])

$$F_n^k(z) = \begin{cases} \dfrac{1}{(n-1)!}\displaystyle\int_S [-k\log(1-F(x))]^{n-1} \dfrac{kf(x)}{1-F(x)} \times \\ \quad \times [1-F(x)]^k \left\{1 - \left[\dfrac{1-F(x+z)}{1-F(x)}\right]^k\right\} dx & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let's notice that the function $G_k(x)$ defined as

$$G_k(x) = \int_{-\infty}^{x} \frac{1}{(n-1)!}[-k\log(1-F(y))]^{n-1}[1-F(y)]^k \frac{kf(y)}{1-F(y)}\, dy$$

is a distribution function. Thus the random variable $U_n^{(k)} = kZ_n^{(k)}$ has the following d.f.:

$$F_{U_n^{(k)}}(u) = 1 - \int_S \left[\frac{1-F(x+u/k)}{1-F(x)}\right]^k dG_k(x).$$

Since

$$G_k(x) = \frac{1}{(n-1)!} \int_0^{-k\log(1-F(x))} u^{n-1} e^{-u} du,$$

it is easy to see that $G_k(x) \to G(x)$, where $G(x)$ is 0 for $x \leqslant x_0$ and 1 otherwise. Thus assumptions of Theorem 2.1 are fulfilled and this completes the proof.

Remark. Let's notice that if $f(x_0) = 0$ then, for arbitrary $0 < u < \infty$, $P\{k(Y_{n+1}^{(k)} - Y_n^{(k)}) < u\} \to 0$, $k \to \infty$. It seems to be reasonable that there exists a sequence $\{a_k\}$ such that $a_k/k \to 0$ and

$$P\{a_k(Y_{n+1}^{(k)} - Y_n^{(k)}) < u\} \underset{k\to\infty}{\longrightarrow} F^*(u) > 0$$

for arbitrary $0 < u < \infty$.

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