PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 5, Fasc. 2 (1985), p. 251-263

MULTIPLY c-DECOMPOSABLE PROBABILITY MEASURES ON BANACH SPACES

BY

NGUYEN VAN THU (HANOI)

Abstract. In the present paper we define α-times c-decomposable $(0 < c < 1,\ \alpha > 0)$ probability measures on a Banach space X in such a way that they form a continuous subclassification of infinitely divisible measures into decreasing classes $L_{c,\alpha}(X)$ each of which is closed under convolution, shifts, changes of scales and passages to weak limits. Moreover, every $L_{c,\alpha}(X)$ admits a universal element (in a generalized Doeblin's sense).

1. Introduction and notation. Throughout the paper we shall denote by X a real separable Banach space with the norm $\|\cdot\|$. We shall consider only Borel σ-additive measures on X. Given a bounded linear operator A and a measure μ on X let $A\mu$ denote the image of μ under A. In particular, if $Ax = ax$ for some $a \in R^1$ and for all $x \in X$, then $A\mu$ will be denoted by the usual symbol $T_a \mu$. Let δ_x denote the unit mass at x $(x \in X)$. For $r > 0$ let B_r denote the ball $[x \in X\colon \|x\| \leqslant r]$, and B_r' its complement.

The concept of c-decomposable probability measures (p.m.'s) was first introduced by Loéve ([6], Exercise 16, page 334) and studied further by Miszejkis [8], Rajba [10], Urbanik [18], Zakusilo [20], among others. A generalization of such a concept to the multiple case is given in [12], [13]. Namely, for a given sequence $c_1, \ldots, c_d$ of numbers from the interval $(0,1)$ and a p.m. μ on X we say that μ is $\langle c_1, \ldots, c_d\rangle$-decomposable if there exist p.m.'s $\mu_1, \ldots, \mu_d$ on the space such that

$$(1.1)\qquad \mu = T_{c_1}\mu * \mu_1, \quad \mu_1 = T_{c_2}\mu_1 * \mu_2, \quad \ldots, \quad \mu_{d-1} = T_{c_d}\mu_{d-1} * \mu_d,$$

where the asterisk $*$ denotes the convolution of measures. In particular, for $c_1 = \ldots = c_d = c$, $\langle c_1, \ldots, c_d\rangle$-decomposable p.m.'s will be called d-times c-decomposable.

By (1.1) it follows that μ is d-times c-decomposable ($d = 1, 2, \ldots$) if and only if there exists a p.m. V on X such that

$$\mu = \mathop{*}_{k=0}^{\infty} T_{c^k} V^{r_{k,d}}, \tag{1.2}$$

where the power is taken in the convolution sense and $r_{k,d}$ is the number of solutions of the equation $x_1 + \ldots + x_d = k$ in nonnegative integers. It is easy to check that

$$r_{k,d} = \binom{d+k-1}{k} = d(d+1) \ldots (d+k-1)/k!. \tag{1.3}$$

Furthermore, in (1.2) and in the sequel the convergence of p.m.'s will be understood in the weak sense.

The formulas (1.2) and (1.3) suggest us to generalize the concept of d-times c-decomposable p.m.'s to the non-integer case. Namely, for every $\alpha > 0$ we put

$$\binom{\alpha}{k} = \begin{cases} 1, & k = 0, \\ \alpha(\alpha-1) \ldots (\alpha-k+1)/k!, & k = 1, 2, \ldots, \end{cases} \tag{1.4}$$

and

$$r_{k,\alpha} = \binom{\alpha+k-1}{k} = \Gamma(\alpha+k)/\Gamma(\alpha)\Gamma(k+1). \tag{1.5}$$

Let $L_0(X)$ denote the class of all infinitely divisible (i.d.) p.m.'s on X. A p.m. μ on X is said to be α-times c-decomposable ($0 < c < 1$, $\alpha > 0$) if there exists a p.m. V in $L_0(X)$ such that

$$\mu = \mathop{*}_{k=0}^{\infty} T_{c^k} V^{r_{k,\alpha}}. \tag{1.6}$$

Let $L_{c,\alpha}(X)$ denote the subclass of $L_0(X)$ consisting of p.m.'s μ such that the equation (1.6) holds for some $V \in L_0(X)$.

In the sequel we shall fix numbers $0 < c < 1$ and $\alpha > 0$. Further, we shall identify a p.m. μ in $L_0(X)$ with the triple $[x_0, R, M]$ in the Tortrat-Levy-Chinczyn representation of μ, where x_0 is a vector in X, R a covariance operator corresponding to the Gaussian component of μ and M a Levy's measure i.e. a generalized Poisson exponent (cf. [19]). In particular, we shall write $[0, 0, M]$ simply by $[M]$.

The paper is organized as follows. In Section 1 we introduce a new concept of α-times c-decomposable p.m.'s. In Section 2 we give a generalized logarithmic criterion which guarantees the existence of multiply c-decomposable p.m.'s on X. In Section 3 an equivalent definition of α-times c-decomposable p.m.'s is given. Moreover, we show that the classes $L_{c,\alpha}(X)$

constitute a continuous monotone system of subsemigroups of $L_0(X)$. Further, in §4 we prove that for every symmetric p.m. μ in $L_{c,\alpha}(X)$ its support denoted by S_μ is a closed subspace of X. Finally, in § 5 we give a further example of A-universal p.m.'s for a subclass K of $L_0(X)$, namely for $K = L_{c,\alpha}(X)$. This stands for an analogue of our results in [16].

Remark. It is the same as in [12] and [17] p.m.'s in $L_\alpha(X): = \bigcap_{c\in(0,1)} L_{c,\alpha}(X)$ are called α-times selfdecomposable. The study on such measures will be communicated elsewhere.

2. A generalized logarithmic criterion. In [20] Zakusilo proved that for $d = 1$ and $X = R^1$ the infinite convolution (1.2) is convergent if and only if

$$\int_X \log(1+\|x\|)\, V(dx) < \infty. \tag{2.1}$$

Such a result was generalized to the multiple case in [13]. Namely, we proved that (1.2) is convergent if and only if

$$\int_X \log^d(1+\|x\|)\, V(dx) < \infty. \tag{2.2}$$

The same is true for every $d > 0$. Namely, we get the following

2.1. THEOREM. *Let $V = [x_0, R, M]$ be an i.d.p.m. on X. Then the following conditions are equivalent*:

(i) *the infinite convolution* (1.6) *is convergent*,

(ii) *the following infinite convolution is convergent*:

$$\mathop{*}_{k=0}^{\infty} T_{c^k} V^{n^{\alpha-1}}, \tag{2.1}$$

(iii) *V has a finite* $\log^\alpha$-*moment, i.e.*

$$\int_X \log^\alpha(1+\|x\|)\, V(dx) < \infty. \tag{2.2}$$

We precede the proof of the Theorem by proving the following

2.2. LEMMA. *For every $\alpha > 0$ there exist positive constants, say $A_1(\alpha)$ and $A_2(\alpha)$, such that for $n = 1, 2, \ldots$*

$$A_1(\alpha)\, n^\alpha \leqslant \sum_{k=0}^{n} r_{k,\alpha} \leqslant A_2(\alpha)\, n^\alpha. \tag{2.3}$$

Proof. Recall [4] that for $0 \leqslant \alpha \leqslant 1$, $x \geqslant y > 1$ and $k = 1, 2, \ldots$, the following inequalities hold:

$$k^{1-\alpha} \leqslant \Gamma(k+1)/\Gamma(k+\alpha) \leqslant (k+1)^{1-\alpha} \tag{2.4}$$

and

$$x^{x-1} e^y / y^{y-1} e^x \leqslant \Gamma(x)/\Gamma(y) \leqslant x^{x-1/2} e^y / y^{y-1/2} e^x. \tag{2.5}$$

On the other hand, for every $\alpha > 0$ there exist positive constants, say $B_1(\alpha)$ and $B_2(\alpha)$, such that for $n = 1, 2, \ldots$

$$B_1(\alpha) n^\alpha \leqslant \sum_{k=1}^{n} k^{\alpha-1} \leqslant B_2(\alpha) n^\alpha \tag{2.6}$$

which together with (1.5) and (2.4) implies (2.3) for the case $0 < \alpha < 1$.

Next suppose that $\alpha > 1$. Putting $x = k+\alpha$ and $y = k+1$ in (2.5) we infer that

$$C_1(\alpha) k^{\alpha-1} \leqslant \Gamma(k+\alpha)/\Gamma(k+1) \leqslant C_2(\alpha) k^{\alpha-1} \tag{2.7}$$

for some positive constants $C_1(\alpha)$ and $C_2(\alpha)$. Finally, combining (2.6) and (2.7) we get (2.3) for $\alpha > 1$ which completes the proof of the Lemma.

Proof of Theorem 2.1. Recall ([17], Lemma 2.5) that for every i.d.p.m. $V = [x_0, R, M]$ the condition (2.2) is equivalent to the following:

$$\int_{B_1'} \log^\alpha \|x\| \, M(dx) < \infty. \tag{2.8}$$

Hence to prove the Theorem it suffices to show that (i) and (ii) are equivalent to (2.8), respectively.

Suppose first that (1.6) is convergent. Then

$$\sum_{k=0}^{\infty} r_{k,\alpha} T_{c^k} M$$

is a Levy's measure. Therefore,

$$\sum_{k=0}^{\infty} r_{k,\alpha} T_{c^k} M(B_1') = \int_{B_1'} \sum_{k=0}^{[\log\|x\|/\log c^{-1}]} r_{k,\alpha} M(dx), \tag{2.9}$$

where $[a]$ denotes the integer part of a, which by Lemma 2.2 implies that (2.9) holds if and only if the condition (2.8) is satisfied.

Conversely, suppose that the condition (2.8) is satisfied. Define $V_1 = [x_0, R, M|_{B_1'}]$ and $V_2 = [M|_{B_1}]$, where $M|_E$ is the restriction of M to a subset E of X. By Lemma 2.2 it follows that the measure

$$\sum_{k=0}^{\infty} r_{k,\alpha} T_{c^k} M|_{B_1}$$

is finite and, consequently, the following infinite convolution is convergent:

$$\mathop{*}_{k=0}^{\infty} T_{c^k} V_1^{r_{k,\alpha}}. \tag{2.10}$$

Furthermore, since the Levy's measure corresponding to V_2 is concentrated on B_1, it follows by [5] that all positive moments of V_2 exist. Let $z, z_0, z_1, \ldots$ be a sequence of independent X-valued random variables with distributions V_2, $V_2^{r_{0,\alpha}}$, $V_2^{r_{1,\alpha}}$, ..., respectively. Then it is easy to check that, for every $k = 0, 1, 2, \ldots$,

$$\text{(2.11)} \qquad \mathrm{E}\,\|z_k\| \leqslant ([r_{k,\alpha}]+1)\,\mathrm{E}\,\|z\| \leqslant (r_{k,\alpha}+1)\,\mathrm{E}\,\|z\|$$

which, together with the fact that

$$\text{(2.12)} \qquad \sum_{k=0}^{\infty} (r_{k,\alpha}+1)\,c^k < \infty,$$

implies

$$\text{(2.13)} \qquad \sum_{k=0}^{\infty} c^k\,\mathrm{E}\,\|z_k\| < \infty.$$

Consequently, the power random series

$$\sum_{k=0}^{\infty} c^k z_k$$

is convergent in L_1-norm and hence the convolution

$$\mathop{*}_{k=0}^{\infty} T_{c^k} V_2^{r_{k,\alpha}}$$

is convergent. Finally, since $V = V_1 * V_2$, we conclude that the convolution (1.6) is convergent. Thus the equivalence (i)$\leftrightarrow$(2.8) is proved. The proof of (ii)$\leftrightarrow$(2.8) is similar and will be omitted. The Theorem is thus fully proved.

3. An equivalent definition of multiply c-decomposable p.m.'s on X. Let $G_\alpha(X)$ $(\alpha > 0)$ denote the subclass of $L_0(X)$ consisting of all p.m.'s V for which the condition (2.2) is satisfied. By virtue of Theorem 2.1, one can define an operator $I_{c,\alpha}$ from $G_\alpha(X)$ onto $L_{c,\alpha}(X)$ as follows:

$$\text{(3.1)} \qquad I_{c,\alpha} V = \mathop{*}_{k=0}^{\infty} T_{c^k} V^{r_{k,\alpha}} \qquad (V \in G_\alpha(X)).$$

Further, for every $0 < \alpha \leqslant 1$ we define an operator $T_{c,\alpha}$ on the whole of $L_0(X)$ by

$$\text{(3.2)} \qquad T_{c,\alpha}\,\mu = \mathop{*}_{k=1}^{\infty} T_{c^k}\,\mu^{\left|\alpha \atop k\right|} \qquad (\mu \in L_0(X)),$$

where $\left|\alpha \atop k\right| = \left|\binom{\alpha}{k}\right|$ $(k = 0, 1, 2, \ldots)$. It should be noted that

$$\sum_{k=1}^{\infty} \left|\alpha \atop k\right| = 1$$

and hence the infinite convolution (3.2) is convergent for every $\mu \in L_0(X)$. The operator $T_{c,\alpha}$ can be regarded as an analogue of T_c in the study of multiply c-decomposable p.m.'s. Namely, we get the following

3.1. THEOREM. *A p.m. μ on X is α-times c-decomposable, where $0 < \alpha < 1$, if and only if there exists an i.d.p.m. V on X such that*

$$\mu = T_{c,\alpha}\,\mu * V. \tag{3.3}$$

Proof. Suppose first that $\mu \in L_{c,\alpha}(X)$ i.e. $\mu = I_{c,\alpha} V$ for some $V \in G_\alpha(X)$. By (3.2) and by the fact that

$$\sum_{k=1}^{m} \left|{\alpha \atop k}\right| r_{m-k,\alpha} = r_{m,\alpha} \quad (m = 1, 2, \ldots), \tag{3.4}$$

we get the equation

$$T_{c,\alpha}\,\mu = \mathop{*}_{m=1}^{\infty} T_{c^m} V^{r_{\alpha,m}} \tag{3.5}$$

which, by (3.1), implies (3.3).

Converselly, suppose that (3.3) holds with $\mu = [x_0, R, M]$ and

$$V = [(1-c)^\alpha)\,x_0,\ (1-c^2)^\alpha)\,R,\ \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} T_{c^k} M]. \tag{3.6}$$

Further, since

$$\sum_{k=0}^{\infty} r_{k,\alpha}\, x^k = (1-x)^{-\alpha} \quad (0 < x < 1), \tag{3.7}$$

and

$$\sum_{n=0}^{m} (-1)^{m-n} r_{n,\alpha} \binom{\alpha}{m-n} = \begin{cases} 1, & m = 0, \\ 0, & m = 1, 2, \ldots, \end{cases} \tag{3.8}$$

it follows, by (3.6), that

$$\mu = [x_0, R, M] = I_{c,\alpha} V, \tag{3.9}$$

which shows that $\mu \in L_{c,\alpha}(X)$. Thus the Theorem is fully proved.

From the above Theorem we get the following Corollaries:

3.2. COROLLARY. *The operator $I_{c,\alpha}$ is one-to-one. Moreover, for any α_1, $\alpha_2 > 0$ and $V \in G_{\alpha_1+\alpha_2}(X)$*

$$I_{c,\alpha_1+\alpha_2} V = I_{c,\alpha_1} I_{c,\alpha_2} V. \tag{3.10}$$

Proof. Let $\mu = I_{c,\alpha} V$. By Theorem 3.1 the p.m. V is uniquely determined by μ, which shows that $I_{c,\alpha}$ is one-to-one. Further, the equation (3.10) follows immediately from the definition of $I_{c,\alpha}$. The Corollary is thus proved.

3.3. COROLLARY. *Suppose that $\alpha_1, \alpha_2, \ldots$ is a sequence of numbers from the interval* (0, 1) *such that $\alpha = \sum \alpha_k < \infty$. Then, $\mu \in L_{c,\alpha}(X)$ if and only if there* *exists a sequence* $\mu_1, \mu_2, \ldots$ *of p.m.'s in* $L_0(X)$ *such that*

$$\mu = T_{c,\alpha_1}\mu * \mu_1, \quad \mu_1 = T_{c,\alpha_2}\mu_1 * \mu_2, \ldots \tag{3.11}$$

Proof. Suppose first that $\mu \in L_{c,\alpha}(X)$, where $\alpha = \sum \alpha_k$ and $\alpha_k \in (0, 1]$ ($k = 1, 2, \ldots$). Then $\mu = I_{c,\alpha} V$ for some p.m. $V \in G_\alpha(X)$. Putting

$$S_n = \sum_{k=1}^{n} \alpha_k, \quad \mu_1 = I_{c,\alpha_1} V, \quad \mu_n = I_{c,\alpha_n} \mu_{n-1} \quad (n = 2, 3, \ldots)$$

and taking into account (3.10) and Theorem 3.1, we get a sequence $\mu_1, \mu_2, \ldots$ of p.m.'s satisfying (3.11).

To prove the "if" part of the Corollary one may assume, without loss of generality, that $0 < \alpha < 1$. Then, it is easy to check that if $\mu_1, \mu_2, \ldots$ satisfy (3.11), then for every $n = 1, 2, \ldots$

$$\mu = T_{c,S_n}\mu * \mu_n. \tag{3.12}$$

Letting $n \to \infty$ we infer, by the above equation, that μ_n converges to some $\mu_{c,\alpha}$ and $T_{c,S_n}\mu$ converges to $T_{c,\alpha}\mu$. Thus $\mu \in L_{c,\alpha}(X)$, which completes the proof of the Corollary.

The following theorem is concerned with the continuity and the monotonicity of the classes $L_{c,\alpha}(X)$.

3.4. THEOREM. *If* $0 \leqslant \alpha < \beta$, *then*

$$L_{c,\beta}(X) \subset L_{c,\alpha}(X). \tag{3.13}$$

Moreover, we get the formulas

$$L_{c,\beta}(X) = \bigcap_{\alpha<\beta} L_{c,\alpha}(X) \tag{3.14}$$

and

$$L_{c,\alpha}(X) = \operatorname{closure}\Big(\bigcap_{\beta>\alpha} L_{c,\beta}(X)\Big), \tag{3.15}$$

where the closure is taken in the weak topology.

Proof. The formulas (3.13) and (3.14) can be easily deduced from Corollary 3.3. We shall prove (3.15).

Accordingly, let μ be a p.m. from $L_{c,\alpha}(X)$ and $\{\beta_n\}$ a sequence of numbers such that $\beta_n \downarrow \alpha$. Our aim is to show that there exists a sequence $\{\mu_n\}$ of p.m.'s such that $\mu_n \in L_{c,\beta_n}(X)$ ($n = 1, 2, \ldots$) and μ_n converges to μ.

We first consider the case $\alpha > 0$. Let V be a p.m. in $G_\alpha(X)$ such that $\mu = I_{c,\alpha} V$. Without loss of generality one may assume that

$$\int_X \|x\| \, V(dx) < \infty. \tag{3.16}$$

Hence V belongs to $G_{\beta_n}(X)$ ($n = 1, 2, \ldots$).

Putting $\mu_n = I_{c,\beta_n} V$ $(n = 1, 2, \ldots)$ and taking into account the fact that $r_{k,\beta_1} > r_{k,\beta_2} > \ldots$ for every $k = 1, 2, \ldots$, we get the decomposition

$$\mu_1 = \mu_n * \mathop{*}_{k=0}^{\infty} T_{c^k} V^{(r_{k,\beta_1} - r_{k,\beta_n})} \tag{3.17}$$

which implies, by Theorem 2.2 [9], that the sequence $\{\mu_n\}$ is convergent. Moreover, since $\beta_n \downarrow \alpha$, it follows that μ_n converges to μ. Thus the case $\alpha > 0$ is proved.

Next we consider the case $\alpha = 0$. Let μ be a p.m. in $L_0(X)$. Without loss of generality one may assume that the first moment of μ exists. Thus one may define $\mu_n = I_{c,\beta_n} \mu$ $(n = 1, 2, \ldots)$. By a similar argument as above, we infer that $\mu_n \in L_{c,\beta_n}(X)$ and μ_n converges to μ, which proves the case $\alpha = 0$ and completes the proof of the Theorem.

4. The support of measures in $L_{c,\alpha}(X)$. In [14] we proved that the support of a symmetric c-decomposable i.d.p.m. on a Hausdorff LCTVS is a closed subspace. In particular, it follows that the support of symmetric stable and semistable p.m.'s on X are closed subspace of X (cf. [7] [11]). The same is true for symmetric p.m.'s in $L_{c,\alpha}(X)$. Namely, we get the following

4.1. Theorem. *For every symmetric p.m. μ in $L_{c,\alpha}(X)$ its support S_μ is a closed subspace of X.*

Proof. By Theorem 3.4 it suffices to prove the Theorem for $0 < \alpha < 1$. Let μ be a symmetric measure in $L_{c,\alpha}(X)$. Then, by Theorem 3.1 it follows that there exists a symmetric p.m. V in $L_0(X)$ such that the equation (3.3) holds. Hence we get the equation

$$S_\mu = \text{closure}(S_{T_{c,\alpha}\mu} + S_V). \tag{4.1}$$

Since, by [11], S_V is a group, we get the inclusion

$$S_\mu \supset S_{T_{c,\alpha}\mu} \tag{4.2}$$

and consequently, by definition of $T_{c,\alpha}$, we have

$$S_\mu \supset c^k S_\mu \supset nc^k S_\mu \tag{4.3}$$

for any n, $k = 1, 2, \ldots$, which implies that for every $a \geqslant 0$

$$S_\mu \supset aS_\mu. \tag{4.4}$$

Hence and by the fact that S_μ is a group, we conclude that S_μ is a subspace of X, which completes the proof of the Theorem.

5. Generalized Doeblin's universal p.m.'s for $L_{c,\alpha}(X)$. Let A be a bounded linear operator on X, and K a subclass of $L_0(X)$. Recall [16] that a p.m. P on X is A-universal for K if $P \in K$ and for every $\mu \in K$ there exist sequences $\{n_k\}$ and $\{m_k\}$ of natural numbers such that the sequence $\{A^{n_k} P^{m_k}\}$ is shiftconvergent to μ. In other words, every element of K is a shift-cluster point of the double sequence $\{A^n P^m\}$. It should be noted that such a concept is a generalization of the concept of universal p.m.'s for i.d.p.m.'s introduced by Doeblin [1]. The existence of A-universal p.m.'s for $L_0(X)$ and

$$L_\alpha(X) := \bigcap_{c\in(0,1)} L_{c,\alpha}(X)$$

was discussed in [15] and [16], respectively. Our present aim is to give a further example of A-universal p.m.'s, namely for $K = L_{c,\alpha}(X)$. The general problem what subclass K of $L_0(X)$ admits an A-universal element remains to be unsolved. We start the study with the following lemmas:

5.1. LEMMA. *Suppose that X is finite-dimensional and P is A-universal for $L_{c,\alpha}(X)$. Then P is a full measure on X, A is invertible and*

$$\|A^n\| \to 0. \tag{5.1}$$

Proof. It is evident that A is invertible because in the opposite case all cluster points of the sequence $\{A^n \bar{P}^m\}$, where $\bar{P}$ denotes the symmetrization of P, should be concentrated on the proper hyperplane $A(X)$ in X. Further, if P is not full, then so are $\bar{P}$ and all cluster points of $\{A^n \bar{P}^m\}$ which is impossible since $\bar{P}$ is A-universal for symmetric p.m.'s in $L_{c,\alpha}(X)$ and among them there are full ones. Thus P must be full.

On the other hand, since δ_0 is a cluster point of $\{A^n \bar{P}^m\}$ it follows that the sequence $\{A^n\}$ is bounded. Let B be a cluster point of $\{A^{n_k}\}$, where $A^{n_k} \bar{P}^{m_k}$ converges to δ_0 for an appropriate sequence $\{m_k\}$. Then we get the equation $B\bar{P} = \delta_0$ which, by the fact that P is full, implies that $B = 0$. Thus 0 is a cluster point of $\{A^n\}$ which is equivalent to (5.1). The Lemma is thus proved.

5.2. LEMMA. *For every $m = 0, 1, 2, \ldots$ we have the inequality*

$$\sum_{k=m+1}^{\infty} r_{k,\alpha} c^k \leqslant (m+1) r_{m+1,\alpha} (1-c)^{-\alpha-1} c^{m+1}. \tag{5.2}$$

Proof. Let us denote the left-hand side of (5.2) by $R_m(c)$ and note that it is the m-rest in the Maclaurin expansion of the function $f(c) := (1-c)^{-\alpha}$. By the well-known integral formula

$$R_m(c) = \frac{1}{m!} \int_0^c f^{(m+1)}(t)(c-t)^m \, dt \tag{5.3}$$

it follows that

$$R_m(c) = \alpha(\alpha+1) \ldots (\alpha+m)/m! \int_0^c (1-t)^{-\alpha-m-1} (c-t)^m \, dt. \tag{5.4}$$

Hence, and by the fact that $(c-t)/(1-t) \leqslant c$ with $0 \leqslant t \leqslant c < 1$, we get the inequality (5.2), which completes the proof of the Lemma.

5.3. LEMMA. *Suppose that N and H are measures on X such that*

$$H = \sum_{k=0}^{\infty} r_{k,\alpha} T_{c^k} N. \tag{5.5}$$

Then there exists a positive constant $A(c, \alpha)$ depending only upon c and α such that

$$\int_{B_1} \|x\| H(dx) \leqslant A(c, \alpha)\{N(X) + \int_{B_1'} \log^{\alpha} \|x\| N(dx)\}. \tag{5.6}$$

Proof. From the equation (5.5) it follows that

$$\begin{aligned} \int_{B_1} \|x\| H(dx) &= \int_X \|x\| \mathbf{1}_{B_1}(x) \sum_{k=0}^{\infty} r_{k,\alpha} T_{c^k} N(dx) \\ &= \int_X \|x\| \{\sum_{k=0}^{\infty} r_{k,\alpha} c^k - \sum_{k=0}^{\infty} r_{k,\alpha} c^k \mathbf{1}_{B_1'}(c^k x)\} N(dx) \\ &= \int_{B_1} \|x\| (1-c)^{-\alpha} N(dx) + \int_{B_1'} R_m(c) \|x\| N(dx), \end{aligned} \tag{5.7}$$

where $m = [\log_d \|x\|]$, $d = c^{-1}$, and $R_m(c)$ is the same as in the proof of Lemma 5.2. Further, by Lemma 5.2 we get

$$R_m(c) \leqslant (m+1) r_{m+1,\alpha} (1-c)^{-\alpha-1} \|x\|^{-1}. \tag{5.8}$$

Hence and by (2.7) it follows that

$$\begin{aligned} R_m(c) &\leqslant K(c, \alpha)(m+1)^{\alpha} \|x\|^{-1} \leqslant 2^{\alpha} K(c, \alpha)(m^{\alpha}+1) \|x\|^{-1} \\ &\leqslant 2^{\alpha} K(c, \alpha)(\log_d^{\alpha} \|x\| + 1) \|x\|^{-1}, \end{aligned} \tag{5.9}$$

where $K(c, \alpha)$ is a positive constant depending upon c and α only. Finally, combining (5.7) and (5.9) we get the inequality (5.6) with $A(c, \alpha) = \max((1-c)^{-\alpha}, 2^{\alpha} K(c, \alpha))$. Thus the Lemma is fully proved.

The following theorems stand for a discrete analogue of Theorem 3.3 and 3.4 in [16].

5.4. THEOREM. *Suppose that X is finite-dimensional and A is a linear operator on it. Then there exists an A-universal p.m. for $L_{c,\alpha}(X)$ if and only if A is invertible and the condition* (5.1) *is satisfied.*

Proof. The necessity follows from Lemma 5.1; the sufficiency follows from Theorem 5.5 below.

5.5. THEOREM. *Let A be an invertible bounded linear operator on an arbitrary separable Banach space X such that the condition* (5.1) *is satisfied. Then for any $0 < c < 1$ and $\alpha > 0$ there exists an A-universal p.m. for $L_{c,\alpha}(X)$.*

Proof. It is easy to check that condition (5.1) is equivalent to the existence of constants $b > 0$ and $a > 1$ such that, for every $k = 1, 2, \ldots$,

$$\|A^k\| \leqslant b a^{-k}. \tag{5.10}$$

Let $\{P_k\}$ be a countable dense subset of $L_{c,\alpha}(X)$ with the property that $P_k = [x_k, 0, M_k]$,

$$M_k = \sum_{n=0}^{\infty} r_{n,\alpha} T_{c^n} G_k, \tag{5.11}$$

where G_k is a finite measure concentrated on B_k, $G_k(\{0\}) = 0$ and $G_k(X) \leqslant k$ $(k = 1, 2, \ldots)$.

Put

$$G = [a^{k^2}]^{-1} A^{-k^3} G_k, \tag{5.12}$$

where a is the same as in (5.10). Then G is a finite measure on X vanishing at 0. Moreover, since for $k = 1, 2, \ldots,$

$$\int_{B'_1} \log^\alpha \|x\| \, A^{-k^2} G_k(dx) \leqslant 2^\alpha k^{3\alpha+1} \log^\alpha \beta, \tag{5.13}$$

where $\beta = \max(e, \|A^{-1}\|)$, it follows that

$$\int_{B'_1} \log^\alpha \|x\| \, G(dx) < \infty, \tag{5.14}$$

which, together with Theorem 2.1, implies that the measure M, defined by the formula

$$M = \sum_{n=0}^{\infty} r_{n,\alpha} T_{c^n} G, \tag{5.15}$$

is a Levy's measure. Put $P = [M]$. We shall prove that P is A-universal for $L_{c,\alpha}(X)$.

Accordingly, it is clear that P belongs to $L_{c,\alpha}(X)$. Let q be an arbitrary element of $L_{c,\alpha}(X)$ and $\{n_k\}$ be a sequence of natural numbers such that the sequence $\{P_{n_k}\}$ converges to q. Further, we put $t_k = [a^{n_k^2}]$ and

$$V_k = A^{n_k^3} P^{t_k} * \delta_{x_{n_k}} \quad (k = 1, 2, \ldots). \tag{5.16}$$

Our further aim is to prove that $\{V_k\}$ converges to q which should finish the proof of the Theorem.

For every $k = 1, 2, \ldots,$ we put

$$N_k^1 = \sum_{n > n_k} t_k [a^{n^2}]^{-1} A^{n_k^3 - n^3} G_n, \tag{5.17}$$

$$N_k^2 = \sum_{n < n_k} t_k [a^{n^2}]^{-1} A^{n_k^3 - n^3} G_n \tag{5.18}$$

and

$$H_k^i = \sum_{m=0}^{\infty} r_{m,\alpha} T_{c^m} N_k^i \quad (i = 1, 2). \tag{5.19}$$

(For a similar setting of N_k^i $(i = 1, 2)$ see formulas (3.6) and (3.7) in [16]).

It is evident that N_k^i and H_k^i $(i = 1, 2)$ are Levy's measures and

$$V_k = P_{n_k} * [H_k^1] * [H_k^2] \quad (k = 1, 2, \ldots). \tag{5.20}$$

It is the same as in the proof of Theorem 3.4 [16] the following facts hold:

$$\lim_{k\to\infty} N_k^1(X) = 0 \tag{5.21}$$

and

$$\lim_{k\to\infty} \int_X \|x\| N_k^2(dx) = 0. \tag{5.22}$$

Moreover, for each $s > 0$ we have

$$\lim_{k\to\infty} \int_{B_1'} \log^s \|x\| N_k^i(dx) = 0 \quad (i = 1, 2). \tag{5.23}$$

Further, by (5.14) and Lemma 2.2 it follows that

$$H_k^i(B_1') = \int_{B_1'} \Big(\sum_{k=0}^{[\log\|x\|/\log c^{-1}]} r_{k,\alpha} \Big) N_k^i(dx) \tag{5.24}$$

$$\leqslant \frac{A_2(\alpha)}{\log^\alpha c^{-1}} \int_{B_1'} \log^\alpha \|x\| N_k^i(dx),$$

where the constant $A_2(\alpha)$ is the same as in Lemma 2.2, which by virtue of (5.18) implies that

$$\lim_{k\to\infty} H_k^i(B_1') = 0 \quad (i = 1, 2). \tag{5.25}$$

By the same manner we get the equation

$$\lim_{k\to\infty} H_k^i(B_r') = 0 \quad (i = 1, 2) \tag{5.26}$$

for every $r > 0$.

Proceeding successively by (5.19), Lemma 5.3, (5.21), (5.22) and (5.23) it follows that

$$\lim_{k\to\infty} \int_{B_1} \|x\| H_k^i(dx) = 0 \quad (i = 1, 2). \tag{5.27}$$

Similarly, we get the equation

$$\lim_{k\to\infty} \int_{B_r} \|x\| H_k^i(dx) = 0 \quad (i = 1, 2) \tag{5.28}$$

for each $r > 0$.

Since every Banach space X is Rademacher type 1, the equations (5.26) and (5.28) together imply, by Corollary 1.8 [2], that $[H_k^i]$ converges to

δ_0 $(i = 1, 2)$. Hence and by (5.20) it follows that V_k converges to q. The Theorem is thus fully proved.

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Hanoi Institute of Mathematics
P.O. Box 631, Bo ho
Hanoi, Vietnam

Received on 6. 5. 1983