PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 6, Fasc. 1 (1985), p. 29-41

FUNCTIONAL RANDOM CENTRAL LIMIT THEOREMS FOR RANDOM WALKS CONDITIONED TO STAY POSITIVE

BY

A. SZUBARGA AND D. SZYNAL (LUBLIN)

Abstract. The aim of this note is to investigate the limiting behaviour of the random function Y_{N_n} conditioned on $[T > N_n]$, where $\{N_n, n \geq 0\}$ is a sequence of positive integer-valued random variables. The results obtained are extensions of results [7] under the additional assumption that $E|X_1|^3 < +\infty$, and X_1 is non-lattice or integer-valued with span 1.

1. Introduction. Let $\{X_k, k \geq 1\}$ be a sequence of independent, identically distributed random variables (i. i. d. r. v.) with

$$\mathrm{E} X_1 = 0, \quad \mathrm{E} X_1^2 = \sigma^2, \quad 0 < \sigma^2 < \infty.$$

Define the random function Y_n by

$$Y_n(t) = S_{[nt]}/\sigma\sqrt{n}, \quad 0 \leq t \leq 1,$$

where $S_0 = 0$, $S_n = X_1 + \ldots + X_n$, $n \geq 1$. Next, let T be the hitting time of the set $(-\infty, 0]$ by the random walk $\{S_n, n \geq 1\}$,

$$T = \inf\{n > 0: S_n \leq 0\},$$

where the infimum of the empty set is taken to be $+\infty$.

We assume that $\{X_k, k \geq 1\}$ are the coordinate functions defined on the product space $(\Omega, \mathscr{A}, P)$. Let Λ_n stand for $[T > n]$, $D \equiv D[0, 1]$ stand for the space of real-valued right continuous functions on $[0, 1]$ having left limits and $\mathscr{D}$ stand for the σ-field of Borel sets generated by the open sets of the Skorokhod $\mathscr{J}_1$-topology. For $g, f \in D$ let

$$\varrho(f, g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)| \quad \text{and} \quad K(\theta, \sqrt[4]{\varepsilon}) = \{f \in D: \varrho(\theta, f) < \sqrt[4]{\varepsilon}\},$$

where $\theta(x) \equiv 0$ for $x \in [0, 1]$ and $\varepsilon > 0$.

Put $D_+ = \{f \in D: f \geqslant \theta\}$ and $\mathscr{D}_+ = D_+ \cap \mathscr{D}$. The measurable mapping $Y_n^+ : (\Lambda_n, \Lambda_n \cap \mathscr{A}) \to (D_+, \mathscr{D}_+)$ is defined by

$$Y_n^+(\cdot, \omega) = S_{[n\cdot]}(\omega)/\sigma\sqrt{n}, \qquad \omega \in \Lambda_n.$$

In [3] it is given a complete proof of the functional conditioned central limit theorem, i. e. it is shown that $Y_n^+ \Leftarrow W^+$, $n \to \infty$, if $\mathrm{E}|X_1|^3 < \infty$, and X_1 is nonlattice or integer-valued with span 1, where W^+ is Brownian meander.

2. Results.

THEOREM 1. *Let $\{X_k, k \geqslant 1\}$ be a sequence of i.i.d.r.v. with* $\mathrm{E}X_1 = 0$, $\mathrm{E}X_1^2 = \sigma^2 < +\infty$, $\mathrm{E}|X_1|^3 < +\infty$, *$X_1$ being nonlattice or integer-valued with span* 1.

If $\{N_n, n \geqslant 0\}$, $N_0 = 0$ a.s., is a sequence of positive integer-valued random variables independent of $\{X_k, k \geqslant 1\}$ and $\{\alpha_n, n \geqslant 1\}$ is a sequence of positive real numbers such that, for any given $\varepsilon > 0$,

$$\text{(1)} \qquad \mathrm{P}[|N_n/\alpha_n - \lambda| \geqslant \varepsilon] = o\big(E(1/\sqrt{N_n})\big)$$

with $\alpha_n \to \infty$ as $n \to \infty$, and λ is a random variable such that

$$\text{(2)} \qquad \mathrm{P}[\lambda \geqslant a] = 1 \quad \textit{for a constant } a > 0,$$

then

$$\text{(3)} \qquad Y_{N_n}^+ \Leftarrow W^+, \quad n \to \infty.$$

Remark. Note that if λ is a degenerate random variable at a, then (2) is satisfied. In this case we can use instead of (1) the condition

$$\text{(1')} \qquad \mathrm{P}[|N_n/\alpha_n - a| \geqslant \varepsilon] = o(1/\sqrt{\alpha_n}).$$

In general, (1') cannot be replaced by the weaker condition

$$N_n/\alpha_n \xrightarrow{\mathrm{P}} a, \quad n \to \infty$$

(P – in probability), which is shown by the following example.

Let $\mathrm{P}[N_n = 1] = 1/\sqrt{n}$, $\mathrm{P}[N_n = [an]] = 1 - 1/\sqrt{n}$ $(n = 1, 2, \ldots)$, $a > 0$, where $[x]$ denotes the integral part of x. Then, for any given $\varepsilon > 0$,

$$\mathrm{P}[|N_n/n - a| \geqslant \varepsilon] = 1/\sqrt{n} \to 0, \quad n \to \infty.$$

Let $\{X_k, k \geqslant 1\}$ be a sequence of i.i.d.r.v. of Theorem 1, independent of N_n, $n \geqslant 1$. In this case we have

$$\mathrm{E}N_n = 1/\sqrt{n} + [an](1 - 1/\sqrt{n}),$$

and, for sufficiently large n,

$$\mathrm{P}[S_1 > 0, \ldots, S_{N_n} > 0] \sim \mathrm{P}[S_1 > 0]/\sqrt{n} + c(1 - 1/\sqrt{n})/\sqrt{[an]}$$

as (see [6])

$$\text{(4)} \qquad \mathrm{P}[S_1 > 0, \ldots, S_n > 0] \sim c/\sqrt{n}, \quad n \to \infty.$$

Therefore, taking into account that in this case (3) reduces to

$$\text{(3')} \quad \lim_{n\to\infty} \mathrm{P}[S_n/\sigma\sqrt{n} < x \mid S_1 > 0, \ldots, S_n > 0] = 1 - \exp\left(\frac{-x^2}{2}\right), \quad x \geqslant 0,$$

we have

$$\mathrm{P}[S_{N_n}/\sigma\sqrt{N_n} < x \mid S_1 > 0, \ldots, S_{N_n} > 0]$$

$$= \frac{\mathrm{P}[S_1 < \sigma x,\, S_1 > 0]/\sqrt{n}}{\mathrm{P}[S_1 > 0]/\sqrt{n} + \mathrm{P}[S_1 > 0, \ldots, S_{[an]} > 0]\,\bar{T}_n} +$$

$$+ \frac{\mathrm{P}\left[\dfrac{S_{[an]}}{\sigma\sqrt{[an]}} < x \mid S_1 > 0, \ldots, S_{[an]} > 0\right] \mathrm{P}[S_1 > 0, \ldots, S_{[an]} > 0]\,\bar{T}_n}{\mathrm{P}[S_1 > 0]/\sqrt{n} + \mathrm{P}[S_1 \downarrow 0, \ldots, S_{[an]} > 0]\,\bar{T}_n}$$

$$\to \frac{\mathrm{P}[X_1 < \sigma x,\, X_1 > 0] + \dfrac{c}{\sqrt{a}}\left(1 - \exp\left(\dfrac{-x^2}{2}\right)\right)}{\mathrm{P}[X_1 > 0] + c/\sqrt{a}} \neq 1 - \exp\left(\frac{-x^2}{2}\right),$$

where $\bar{T}_n = 1 - 1/\sqrt{n}$. Obviously, in this case (2) is trivially satisfied.

We now show that, in general, assumption (2) cannot be omitted in proving (3) when λ is a nondegenerate random variable. Assume that $(\langle 0, 1\rangle, \mathscr{B}(\langle 0, 1\rangle), P)$ is a probability space, where P is the Lebesgue measure and $\mathscr{B}(\langle 0, 1\rangle)$ is the σ-field of Borel subsets of $\langle 0, 1\rangle$. Assume that $\{X_k, k \geqslant 1\}$ is a sequence of random variables satisfying the assumptions of Theorem 1, independent of $\{N_n, n \geqslant 1\}$, where $\{N_n, n \geqslant 1\}$ is defined by

$$N_n(\omega) = \begin{cases} 1 & \text{if } \omega \in \langle 0, 1/\sqrt{n}\rangle, \\ n+1 & \text{if } \omega \in (1/\sqrt{n}, 1/n - (1/\sqrt{n} - [\sqrt{n}]/n)\rangle, \\ k & \text{if } \omega \in ((k-1)/n, k/n\rangle,\ k = [\sqrt{n}] + 2, \ldots, n. \end{cases}$$

It is not difficult to see that for any $\varepsilon > 0$ there exists an n_0 such that

$$\mathrm{P}[|N_n/n - \lambda| \geqslant \varepsilon] = 0 \quad \text{for } n \geqslant n_0 > [1/\sqrt{\varepsilon}] + 1,$$

where λ is uniformly distributed on $\langle 0, 1\rangle$. Thus (1) is satisfied but (2) does not hold.

Next we have

$$P[S_1>0,\dots,S_{N_n}>0] = P[S_1>0]/\sqrt{n}+\sum_{k=[\sqrt{n}]+2}^{n} P[S_1>0,\dots,S_k>0]/n+$$

$$+P[S_1>0,\dots,S_{[\sqrt{n}]+1}>0]\left(\frac{1}{n}-\left(\frac{1}{\sqrt{n}}-\frac{[\sqrt{n}]}{n}\right)\right)$$

$$\approx \frac{1}{\sqrt{n}}P[S_1>0]+$$

$$+\frac{1}{\sqrt{n}}\int\limits_{([\sqrt{n}]+2)/n}^{1}\frac{c}{\sqrt{n}}dx+c(\sqrt{[\sqrt{n}]+1})\left(\frac{1}{n}-\left(\frac{1}{\sqrt{n}}-\frac{[\sqrt{n}]}{n}\right)\right)$$

$$= T_n = O(1/\sqrt{n}) \quad \text{as } n\to\infty.$$

Hence, using (3′), we get

$$P[Y_{N_n}(1)<x\,|\,S_1>0,\dots,S_{N_n}>0] \approx \frac{P[X_1<\sigma x,\ X_1>0]}{nT_n}+$$

$$+\sum_{k=[\sqrt{n}]+1}^{n}\frac{P[Y_k(1)<x\,|\,S_1>0,\dots,S_k>0]\,P[S_1>0,\dots,S_k>0]}{nT_n}+$$

$$+P[Y_{[\sqrt{n}]+1}(1)<x\,|\,S_1>0,\dots,S_{[\sqrt{n}]+1}>0]\times$$

$$\times P[S_1>0,\dots,S_{[\sqrt{n}]+1}>0]\frac{1}{T_n}\left(\frac{1}{n}-\left(\frac{1}{\sqrt{n}}-\frac{[\sqrt{n}]}{n}\right)\right)$$

$$\to \frac{P[X_1<\sigma x,\ X_1>0]+\left(1-\exp\left(\frac{-x^2}{2}\right)\right)2c}{P[X_1>0]+2c} \neq 1-\exp\left(\frac{-x^2}{2}\right).$$

We have seen that, in general, (2) cannot be omitted in proving (3). However, we are able to give more general conditions than (1) and (2), under which (3) holds.

THEOREM 2. *Let* $\{X_n, n\geqslant 1\}$ *be a sequence of i.i.d.r.v. of Theorem* 1. *Suppose that* $\{N_n, n\geqslant 1\}$ *is a sequence of positive integer-valued random variables independent of* $\{X_k, k\geqslant 1\}$ *and* $\{\alpha_n, n\geqslant 1\}$ *is a sequence of positive real numbers such that* $\lim_{n\to\infty}\alpha_n=\infty$.

If λ is a positive random variable such that

$$\mathrm{P}[|N_n/\alpha_n - \lambda| > \varepsilon_n] = o\big(E(1/\sqrt{N_n})\big), \tag{5}$$

$$\mathrm{P}[\lambda - 2\alpha_n] = o\big(E(1/\sqrt{N_n})\big), \tag{6}$$

where $\{\varepsilon_n, n \geqslant 1\}$ is a sequence of positive real numbers such that $0 < \varepsilon_n \to 0$, $\alpha_n \varepsilon_n \to \infty$ as $n \to \infty$, then (3) *holds.*

Note that assumptions similar to (5) and (6) were used in [4] to give the rate of convergence in the functional central limit theorem.

A functional random central limit theorem for random walks conditioned to stay positive without the assumption of independence $\{X_k, k \geqslant 1\}$ and $\{N_n, n \geqslant 1\}$ is given in the following

THEOREM 3. *Let $\{X_k, k \geqslant 1\}$ be a sequence of i.i.d.r.v. of Theorem* 1. *If $\{N_n, n \geqslant 1\}$ is a sequence of positive integer-valued random variables and $\{\alpha_n, n \geqslant 1\}$ is a sequence of positive real numbers such that, for any given $\varepsilon > 0$,*

$$\mathrm{P}[|N_n/\alpha_n - a| \geqslant \varepsilon] = o(1/\sqrt{\alpha_n}) \tag{7}$$

with $\alpha_n \to \infty$ as $n \to \infty$, where a is a positive constant, then (3) *holds.*

3. Proofs of the results.

Proof of Theorem 1. Let ε, $0 < \varepsilon < 1$, be fixed, and $a_n = [(a-\varepsilon)\alpha_n]$. By (1), (2) and the assumption $\alpha_n \to \infty$ we can choose an n such that

$$0 \leqslant \sum_{k=1}^{a_n} \mathrm{P}[S_1 > 0, \ldots, S_k > 0]\,\mathrm{P}[N_n = k] \leqslant \sum_{k=1}^{a_n} \mathrm{P}[N_n = k]$$
$$\leqslant \mathrm{P}[|N_n/\alpha_n - \lambda| \geqslant \varepsilon] = o\big(E(1/\sqrt{N_n})\big)$$

and, at the same time, by (4),

$$\sum_{k=a_n+1}^{\infty} \mathrm{P}[S_1 > 0, \ldots, S_k > 0]\,\mathrm{P}[N_n = k] \quad \sum_{k=a_n+1}^{\infty} (c/\sqrt{k})\,\mathrm{P}[N_n = k]$$
$$= c\big(E(1/\sqrt{N_n})\big) - \sum_{k=1}^{a_n} (c/\sqrt{k})\,\mathrm{P}[N_n = k].$$

But

$$0 \leqslant c \sum_{k=1}^{a_n} \frac{1}{\sqrt{k}} \mathrm{P}[N_n = k] \leqslant c \sum_{k=1}^{a_n} \mathrm{P}[N_n = k] \leqslant c\,\mathrm{P}\left[\left|\frac{N_n}{\alpha_n} - \lambda\right| \geqslant \varepsilon\right]$$
$$= o\big(E(1/\sqrt{N_n})\big).$$

Hence

$$\mathrm{P}[S_1 > 0, \ldots, S_{N_n} > 0] \approx cE(1/\sqrt{N_n}) \tag{8}$$

Put now

$$C_{n,k} = \frac{\mathrm{P}[S_1 > 0, \ldots, S_k > 0]\,\mathrm{P}[N_n = k]}{\mathrm{P}[S_1 > 0, \ldots, S_{N_n} > 0]} \quad (k \geqslant 1,\ n \geqslant 1). \tag{9}$$

We see that $\sum_{k=1}^{\infty} C_{n,k} = 1$ and, for fixed k, by (1) and (8),

$$0 \leqslant C_{n,k} \leqslant \frac{\sum_{k=1}^{a_n} \mathrm{P}[N_n = k]}{\mathrm{P}[S_1 > 0, \ldots, S_{N_n} > 0]} \approx \frac{o(E(1/\sqrt{N_n}))}{cE(1/\sqrt{N_n})} \to 0, \quad n \to \infty,$$

which proves that $[C_{n,k}]$ is a Toeplitz matrix. Therefore, by [5], p. 472, and (3′), we have

$$\begin{aligned} &\mathrm{P}[Y_{N_n}(1) < x \mid S_1 > 0, \ldots, S_{N_n} > 0] \\ &= \sum_{k=1}^{\infty} C_{n,k}\,\mathrm{P}[Y_k(1) < x \mid S_1 > 0, \ldots, S_k > 0] \to 1 - \exp\left(\frac{-x^2}{2}\right), \\ &\qquad n \to \infty,\ x \geqslant 0. \end{aligned} \tag{10}$$

Now we need the notations

$$g(t, x_1, x_2) = (2\pi t)^{-1/2}[\exp(-(x_2 - x_1)^2/2t) - \exp(-(x_1 + x_2)^2/2t)],$$
$$x_1, x_2 > 0, \quad 0 < t \leqslant 1,$$
$$p(0, 0, t, x) = t^{-3/2} x \exp(-x^2/2t)\,|N|(x/(1-t)^{1/2}),$$

where

$$|N|(x) = (2/\pi)^{1/2} \int_0^x \exp\left(\frac{-u^2}{2}\right) du,$$

and

$$p(t_1, x_1, t_2, x_2) = g(t_2 - t_1, x_1, x_2)\,|N|(x_2/(1-t_2)^{1/2})\,|N|(x_1/(1-t_1)^{1/2}),$$
$$x_1, x_2 > 0, \quad 0 < t_1 < t_2 \leqslant 1.$$

It is known [3] that for $x \geqslant 0$

$$\lim_{n\to\infty} \mathrm{P}[Y_n(t) < x \mid T > n] = \int_0^x p(0, 0, t, y)\,dy, \tag{11}$$

whence

(12) $$P[Y_{N_n}(t) < x \mid S_1 > 0, \ldots, S_{N_n} > 0]$$

$$= \sum_{k=1} C_{n,k} P[Y_k(t) < x \mid S_1 > 0, \ldots, S_k > 0] \to \int_0^x p(0, 0, t, y)\, dy,$$

$n \to \infty$ as (11) holds and $[C_{n,k}]$ is a Toeplitz matrix. Moreover, since

(13) $$\lim_{n\to\infty} P[Y_n(t_1) < x_1, Y_n(t_2) < x_2, \ldots, Y_n(t_k) < x_k \mid T > n]$$

$$= \int_0^{x_1} \ldots \int_0^{x_k} p(0, 0, t_1, y_1) p(t_1, y_1, t_2, y_2) \ldots p(t_{k-1}, y_{k-1}, t_k, y_k)\, dy_1 \ldots dy_k$$

for all $k \geqslant 1$, $x_1, \ldots, x_k > 0$ and $0 < t_1 < t_2 < \ldots < t_k \leqslant 1$, we have

(14) $$P[Y_{N_n}(t_1) < x_1, \ldots, Y_{N_n}(t_k) < x_k \mid S_1 > 0, \ldots, S_{N_n} > 0]$$

$$= \sum_{j=1}^{\infty} C_{n,j} P[Y_j(t_1) < x_1, \ldots, Y_j(t_k) < x_k \mid S_1 > 0, \ldots, S_j > 0]$$

$$\to \int_0^{x_1} \ldots \int_0^{x_k} p(0, 0, t_1, y_1) p(t_1, y_1, t_2, y_2) \ldots p(t_{k-1}, y_{k-1}, t_k, y_k)\, dy_1 \ldots dy_k$$

for all $k \geqslant 1$, $x_1, \ldots, x_k > 0$ and $0 < t_1 < t_2 < \ldots < t_k \leqslant 1$.

We now prove that $\{Y_{N_n}^+\}$ is tight.

Taking into account that for $\varepsilon > 0$

(15) $$\lim_{\delta \searrow 0} \limsup_{n\to\infty} P[\omega_{y_n}(\delta, 0, 1) \geqslant \varepsilon \mid S_1 > 0, \ldots, S_n > 0] = 0,$$

where

$$\omega_x(\delta, a, b)$$
$$= \sup_{s,t} \{|x(s) - x(t)|:\ 0 \leqslant a \leqslant b \leqslant 1,\ 0 < \delta < 1,\ a \leqslant s \leqslant t \leqslant b,\ |t-s| < \delta\},$$

we obtain, by the above arguments,

$$\lim_{\delta \searrow 0} \limsup_{n\to\infty} P[\omega_{Y_{N_n}}(\delta, 0, 1) \geqslant \varepsilon \mid S_1 > 0, \ldots, S_{N_n} > 0] = 0,$$

which, by theorems 15.1 and 15.5 of [1], proves that $\{Y_{N_n}^+\}$ is tight. Therefore, by (14) and (15), we have proved (3).

Proof of Theorem 2. By (4) and (5) we have, for sufficiently large n,

(16) $$P[S_1 > 0, \ldots, S_{N_n} > 0] = \sum_{k=1}^{[\alpha_n \varepsilon_n]} P[S_1 > 0, \ldots, S_k > 0] P[N_n = k] +$$

$$+ \sum_{k=[\alpha_n \varepsilon_n]+1}^{\infty} P[S_1 > 0, \ldots, S_k > 0] P[N_n = k] \approx cE(1/\sqrt{N_n}).$$

Now we note that, for $n \geqslant 1$ and $j \geqslant 1$,

$$C_{n,j} = \frac{P[S_1 > 0, \ldots, S_j > 0] P[N_n = j]}{P[S_1 > 0, \ldots, S_{N_n} > 0]}$$

is a Toeplitz matrix. Indeed, we have

$$C_{n,j} \geqslant 0, \quad \sum_{j=1}^{\infty} C_{n,j} = 1,$$

$$C_{n,j} \frac{\sum_{k=1}^{[\alpha_n, \varepsilon_n]} P[N_n = k]}{cE(1/\sqrt{N_n})} \leqslant \frac{P\left[\left|\frac{N_n}{\alpha_n} - \lambda\right| \geqslant \varepsilon_n\right] + P[\lambda < 2\varepsilon_n]}{cE(1/\sqrt{N_n})} \to 0,$$

$n \to \infty$, by (5), (6) and (16), since $j < \alpha_n \varepsilon_n$ for sufficiently large n.

Following the considerations of the proof of Theorem 1 we get (3).

Proof of Theorem 3. Let ε, $0 < \varepsilon < a$, be fixed and put $a_n = [(a-\varepsilon)\alpha_n]$, $b_n = [(a+\varepsilon)\alpha_n]$, $c_n = b_n - a_n$, $u_n = (a_n/b_n)^{1/2}$, $A_n = \{k\colon a_n \leqslant k \leqslant b_n\}$ and A_n^c is the complement of A_n.

Set

$$r_k = P[S_1 > 0, \ldots, S_k > 0], \quad \hat{r}_n = P[S_1 > 0, \ldots, S_{N_n} > 0].$$

From (4) and (7) we get

$$\frac{c}{\sqrt{b_n}} - o(1/\sqrt{b_n}) \leqslant \hat{r}_n \leqslant \frac{c}{\sqrt{a_n}} + o(1/\sqrt{a_n}). \tag{17}$$

We see that

$$\begin{aligned} (18) \quad & P[Y_{N_n}(1) < x \mid S_1 > 0, \ldots, S_{N_n} > 0] \\ &= P[Y_{N_n}(1) < x,\ S_1 > 0, \ldots, S_{N_n} > 0]/\hat{r}_n \\ &\sim \sum_{k \in A_n} P[Y_k(1) < x,\ S_1 > 0, \ldots, S_k > 0,\ N_n = k]/\hat{r}_n + \\ &\quad + P[Y_{N_n}(1) < x,\ S_1 > 0, \ldots, S_{N_n} > 0,\ N_n \in A_n^c]/\hat{r}_n. \end{aligned}$$

But, by (7) and (17) we have

$$P[Y_{N_n}(1) < x,\ S_1 > 0, \ldots, S_{N_n} > 0,\ N_n \in A_n^c]/\hat{r}_n \to 0, \quad n \to \infty.$$

Therefore, to prove that for $x \geqslant 0$

$$P[Y_{N_n}(1) < x \mid S_1 > 0, \ldots, S_{N_n} > 0] \to 1 - \exp\left(\frac{-x^2}{2}\right), \quad n \to \infty, \tag{19}$$

it is enough to consider

$$P[Y_{N_n}(1) < x,\ S_1 > 0, \ldots, S_{N_n} > 0,\ N_n \in A_n]/\hat{r}_n.$$

Put now, for $0 \leqslant t \leqslant 1$,

$$Z_n(t) = \max_{k \in A_n} \frac{S_{[kt]} - S_{[(a+\varepsilon)\alpha_n t]}}{\sigma\sqrt{b_n}}, \qquad Z_n^*(t) = \max_{k \in A_n} \frac{S_{[kt]} - S_{[(a-\varepsilon)\alpha_n t]}}{\sigma\sqrt{a_n}}.$$

Then we have

$$\text{(20)} \quad \begin{aligned} & P[Y_{N_n}(1) < x, S_1 > 0, \ldots, S_{N_n} > 0, N_n \in A_n] \\ & \geqslant P[Y_{N_n}(1) < x, S_1 > 0, \ldots, S_{b_n} > 0, N_n \in A_n] \\ & \geqslant P[Y_{b_n}(1) + Z_n(1) < xu_n, S_1 > 0, \ldots, S_{b_n} > 0, N_n \in A_n] \\ & \geqslant P[Y_{b_n}(1) + Z_n(1) < xu_n, S_1 > 0, \ldots, S_{b_n} > 0, Z_n(1) < \sqrt[4]{\varepsilon}] - P[N_n \in A_n^c] \\ & \geqslant P[Y_{b_n}(1) < xu_n - \sqrt[4]{\varepsilon} \mid S_1 > 0, \ldots, S_{b_n} > 0] r_{b_n} - P[Z_n(1) \geqslant \sqrt[4]{\varepsilon}] r_{a_n} - \\ & \qquad - P[N_n \in A_n^c], \end{aligned}$$

as $Z_n(1)$ does not depend on $S_1, S_2, \ldots, S_{a_n}$.

The similar evaluations lead us to

$$\text{(21)} \quad \begin{aligned} & P[Y_{N_n}(1) < x, S_1 > 0, \ldots, S_{N_n} > 0, N_n \in A_n] \\ & \leqslant P[Y_{a_n}(1) < x/u_n - \sqrt[4]{\varepsilon} \mid S_1 > 0, \ldots, S_{a_n} > 0] r_{a_n} + P[Z_n(1) \geqslant \sqrt[4]{\varepsilon}] r_{a_n}. \end{aligned}$$

Note now that, by Kolmogorov's inequality,

$$\text{(22)} \quad \begin{aligned} P[Z_n(t) \geqslant \sqrt[4]{\varepsilon}] & \leqslant P\left[\max_{1 \leqslant k \leqslant [(a+\varepsilon)\alpha_n t] - [(a-\varepsilon)\alpha_n t]} \left|\frac{S_k}{\sigma\sqrt{b_n}}\right| \geqslant \sqrt[4]{\varepsilon}\right] \\ & \leqslant \frac{[(a+\varepsilon)\alpha_n t] - [(a-\varepsilon)\alpha_n t]}{\sqrt{\varepsilon}\, b_n} \to \frac{2t\sqrt{\varepsilon}}{a+\varepsilon} \quad \text{as } n \to \infty \end{aligned}$$

and

$$\text{(23)} \quad P[Z_n^*(t) \geqslant \sqrt[4]{\varepsilon}] \leqslant \frac{[(a+\varepsilon)\alpha_n t] - [(a-\varepsilon)\alpha_n t]}{\sqrt{\varepsilon}\, a_n} \to \frac{2t\sqrt{\varepsilon}}{a-\varepsilon} \quad \text{as } n \to \infty.$$

Therefore, by (18), (20)-(23), we obtain

$$\text{(24)} \quad \begin{aligned} & P[Y_{b_n}(1) < xu_n - \sqrt[4]{\varepsilon} \mid S_1 > 0, \ldots, S_{b_n} > 0](r_{b_n}/\hat{r}_n) - \\ & - \frac{c_n}{\sqrt{\varepsilon}\, b_n}(r_{a_n}/\hat{r}_n) - P[N_n \in A_n^c]/\hat{r}_n \leqslant P[Y_{N_n}(1) < x \mid S_1 > 0, \ldots, S_{N_n} > 0] \\ & \leqslant P[Y_{a_n}(1) < x/u_n + \sqrt[4]{\varepsilon} \mid S_1 > 0, \ldots, S_{a_n} > 0](r_{a_n}/\hat{r}_n) + \frac{c_n}{\sqrt{\varepsilon}\, b_n}(r_{a_n}/\hat{r}_n) + \\ & \qquad + P[N_n \in A_n^c]/\hat{r}_n. \end{aligned}$$

But by (3′) we have

$$\mathrm{P}[Y_{b_n}(1) < xu_n - \sqrt[4]{\varepsilon} \mid S_1 > 0, \ldots, S_{b_n} > 0] \tag{25}$$

$$\to 1 - \exp\left(-\frac{\left(x\sqrt{\frac{a+\varepsilon}{a-\varepsilon}} - \sqrt[4]{\varepsilon}\right)^2}{2}\right)$$

and

$$\mathrm{P}[Y_{a_n}(1) < x/u_n - \sqrt[4]{\varepsilon} \mid S_1 > 0, \ldots, S_{a_n} > 0] \tag{26}$$

$$\to 1 - \exp\left(-\frac{\left(x\sqrt{\frac{a-\varepsilon}{a+\varepsilon}} + \sqrt[4]{\varepsilon}\right)^2}{2}\right)$$

as $n \to \infty$.

Moreover, by (4) and (17) we get

$$\liminf_{n\to\infty} (r_{b_n}/\hat{r}_n) = \sqrt{\frac{a-\varepsilon}{a+\varepsilon}} \tag{27}$$

and

$$\limsup_{n\to\infty} (r_{a_n}/\hat{r}_n) = \sqrt{\frac{a+\varepsilon}{a-\varepsilon}}. \tag{28}$$

Therefore, for any given ε, $0 < \varepsilon < a$, by (20)-(28) we get, for $x \geqslant 0$,

$$\frac{-2\sqrt{\varepsilon}}{a+\varepsilon}\sqrt{\frac{a+\varepsilon}{a-\varepsilon}} + \sqrt{\frac{a-\varepsilon}{a+\varepsilon}}\left(-\exp\left(-\frac{\left(x\sqrt{\frac{a-\varepsilon}{a+\varepsilon}} - \sqrt[4]{\varepsilon}\right)^2}{2}\right)\right)$$

$$\leqslant \liminf_{n\to\infty} \mathrm{P}[Y_{N_n}(1) < x \mid S_1 > 0, \ldots, S_{N_n} > 0]$$

$$\leqslant \limsup_{n\to\infty} \mathrm{P}[Y_{N_n}(1) < x \mid S_1 > 0, \ldots, S_{N_n} > 0]$$

$$\leqslant \frac{2\sqrt{\varepsilon}}{a-\varepsilon}\sqrt{\frac{a+\varepsilon}{a-\varepsilon}} + \sqrt{\frac{a+\varepsilon}{a-\varepsilon}}\left(1 - \exp\left(-\frac{\left(x\sqrt{\frac{a+\varepsilon}{a-\varepsilon}} + \sqrt[4]{\varepsilon}\right)^2}{2}\right)\right).$$

Letting now $\varepsilon \to 0$ for $x \geqslant 0$, we obtain (19).

Note that $Z_n(t)$ does not depend on $S_1, \ldots, S_{[(a-\varepsilon)\alpha_n t]}$. This fact and the same arguments as above show that, for $x \geqslant 0$,

$$(29)\quad \mathrm{P}[Y_{b_n}(t) < xu_n - \sqrt[4]{\varepsilon} \mid S_1 > 0, \ldots, S_{b_n} > 0]\, r_{b_n}/\hat{r}_n - \\ - \frac{[(a+\varepsilon)\alpha_n t] - [(a-\varepsilon)\alpha_n t]}{\sqrt{\varepsilon}\, b_n} (r_{[(a-\varepsilon)\alpha_n t]}/\hat{r}_n) - \mathrm{P}[N_n \in A_n^c]/\hat{r}_n \\ \leqslant \mathrm{P}[Y_{N_n}(t) < x \mid S_1 > 0, \ldots, S_{N_n} > 0] \\ \leqslant \mathrm{P}[Y_{a_n}(t) < x/u_n + \sqrt[4]{\varepsilon} \mid S_1 > 0, \ldots, S_{a_n} > 0](r_{a_n}/\hat{r}_n) + \\ + \frac{[(a+\varepsilon)\alpha_n t] - [(a-\varepsilon)\alpha_n t]}{\sqrt{\varepsilon}\, b_n} (r_{[(a-\varepsilon)\alpha_n t]}/\hat{r}_n) + \mathrm{P}[N_n \in A_n^c]/\hat{r}_n.$$

Letting now $n \to \infty$, next $\varepsilon \to 0$ for $x \geqslant 0$ and $t \in (0, 1)$ we obtain

$$(30)\quad \lim_{n\to\infty} \mathrm{P}[Y_{N_n}(t) < x \mid S_1 > 0, \ldots, S_{N_n} > 0] = \int_0^x p(0, 0, t, y)\, dy.$$

In the same way for all $k \geqslant 1$, $x_1, \ldots, x_k > 0$ and $0 < t_1 < \ldots < t_k \leqslant 1$ we have

$$(31)\quad \mathrm{P}[Y_{b_n}(t_1) < x_1 u_n - \sqrt[4]{\varepsilon}, \ldots, Y_{b_n}(t_k) \\ < x_k u_n - \sqrt[4]{\varepsilon} \mid S_1 > 0, \ldots, S_{b_n} > 0](r_{b_n}/\hat{r}_n) - \mathrm{P}[Z_n(t_1) \geqslant \sqrt[4]{\varepsilon}] \times \\ \times \mathrm{P}[S_1 > 0, \ldots, S_{[(a-\varepsilon)\alpha_n t]} > 0]/\hat{r}_n - \ldots - \mathrm{P}[Z_n(t_k) \geqslant \sqrt[4]{\varepsilon}] \times \\ \times \mathrm{P}[S_1 > 0, \ldots, S_{[(a-\varepsilon)\alpha_n t]} > 0]/\hat{r}_n - \mathrm{P}[N_n \in A_n^c]/\hat{r}_n \\ \leqslant \mathrm{P}[Y_{N_n}(t_1) < x_1, \ldots, Y_{N_n}(t_k) < x_k \mid S_1 > 0, \ldots, S_{N_n} > 0] \\ \leqslant \mathrm{P}[Y_{a_n}(t_1) < x_1/u_n + \sqrt[4]{\varepsilon}, \ldots, Y_{a_n}(t_k) < x_k/u_n + \sqrt[4]{\varepsilon} \mid S_1 > 0, \ldots, S_{a_n} > 0] \times \\ \times (r_{a_n}/\hat{r}_n) + \mathrm{P}[Z_n^*(t_1) \geqslant \sqrt[4]{\varepsilon}](r_{[(a-\varepsilon)t_1\alpha_n]}/\hat{r}) + \ldots + \\ + \mathrm{P}[Z_n^*(t_k) \geqslant \sqrt[4]{\varepsilon}](r_{[a-\varepsilon)t_k\alpha_n]}/\hat{r}_n) + P[N_n \in A_n^c]/\hat{r}_n.$$

Hence, putting $n \to \infty$ and using (13), (4), (17), (22), (23), and next letting $\varepsilon \to 0$, we obtain

$$(32)\quad \lim_{n\to\infty} \mathrm{P}[Y_{N_n}(t_1) < x_1, \ldots, Y_{N_n}(t_k) < x_k \mid S_1 > 0, \ldots, S_{N_n} > 0] \\ = \int_0^{x_1} \ldots \int_0^{x_k} p(0, 0, t_1, y_1)\, p(t_1, y_1, t_2, y_2) \ldots p(t_{k-1}, y_{k-1}, t_k, y_k)\, dy_1 \ldots dy_k$$

for all $k \geqslant 1$, $x_1, \ldots, x_k > 0$ and $0 < t_1 < t_2 < \ldots < t_k < 1$.

To complete the proof of the weaker convergence of $\{Y_{N_n}^+\}$ to W^+ it suffices (cf. Theorems 15.1 and 15.3 of [1]) to show that for every $v > 0$

$$(33)\quad \lim_{\delta\searrow 0} \limsup_{n\to\infty} \mathrm{P}[\omega_{Y_{N_n}}(\delta, 0, 1) > v \mid S_1 > 0, \ldots, S_{N_n} > 0] = 0.$$

We can see that $\omega_f(\delta, 0, 1) < 2\sqrt[4]{\varepsilon}$ whenever $f \in K(\theta, \sqrt[4]{\varepsilon})$. Hence, for a fixed $\varepsilon > 0$ such that $2\sqrt[4]{\varepsilon} < \nu$, we have

(34) $$\mathrm{P}[\omega_{Y_{N_n}}(\delta, 0, 1) \geqslant \nu, S_1 > 0, \ldots, S_{N_n} > 0]/\hat{r}_n$$

$$\leqslant \mathrm{P}[\omega_{Y_{a_n}}(\delta, 0, 1) + \omega_{(Y_{N_n} - Y_{a_n})}(\delta, 0, 1)$$

$$\geqslant \nu, S_1 > 0, \ldots, S_{N_n} > 0, N_n \in A_n, (Y_{N_n} - Y_{a_n}) \in K(\theta, \sqrt[4]{\varepsilon})]/\hat{r}_n +$$

$$+ \mathrm{P}[(Y_{N_n} - Y_{a_n}) \in K(\theta, \sqrt[4]{\varepsilon}), S_1 > 0, \ldots, S_{a_n} > 0, N_n \in A_n]/\hat{r}_n + \mathrm{P}[N_n \in A_n^c]/\hat{r}_n$$

$$\leqslant \mathrm{P}[\omega_{Y_{a_n}}(\delta, 0, 1) \geqslant \nu - \sqrt[4]{\varepsilon} \mid S_1 > 0, \ldots, S_{a_n} > 0](r_{a_n}/\hat{r}_n) +$$

$$+ \mathrm{P}[\max_{k \in A_n} |Y_k - Y_n| \notin K(\theta, \sqrt[4]{\varepsilon}), S_1 > 0, \ldots, S_{a_n} > 0]/\hat{r}_n + \mathrm{P}[N_n \in A_n^c]/\hat{r}_n$$

$$\leqslant P[\omega_{Y_{a_n}}(\delta, 0, 1) \geqslant \nu - 2\sqrt[4]{\varepsilon} \mid S_1 > 0, \ldots, S_{a_n} > 0](r_{a_n}/\hat{r}_n) +$$

$$+ \mathrm{P}\left[\max_{k \in A_n} \left|\frac{S_{[k\cdot]}}{\sigma\sqrt{a_n}} - \frac{S_{[a_n\cdot]}}{\sigma\sqrt{a_n}}\right| \notin K(\theta, \sqrt[4]{\varepsilon}), S_1 > 0, \ldots, S_{a_n} > 0\right]\Big/\hat{r}_n + \mathrm{P}[N_n \in A_n^c]/\hat{r}_n.$$

Knowing that

(35) $$\lim_{\delta \searrow 0} \limsup_{n \to \infty} \mathrm{P}[\omega_{Y_{a_n}}(\delta, 0, 1) \geqslant \nu - 2\sqrt[4]{\varepsilon} \mid S_1 > 0, \ldots, S_{a_n} > 0](r_{a_n}/\hat{r}_n) = 0$$

and

(36) $$\lim_{n \to \infty} \mathrm{P}[N_n \in A_n^c]/\hat{r}_n = 0$$

and taking into account that

(37) $$\mathrm{P}\left[\max_{k \in A_n} \left|\frac{S_{[k\cdot]}}{\sigma\sqrt{a_n}} - \frac{S_{[a_n\cdot]}}{\sigma\sqrt{a_n}}\right| \notin K(\theta, \sqrt[4]{\varepsilon}), S_1 > 0, \ldots, S_{a_n} > 0\right]\Big/\hat{r}_n$$

$$\leqslant \mathrm{P}\left[\max_{k \in A_n} \left|\frac{S_k - S_{a_n}}{\sigma\sqrt{a_n}}\right| > \sqrt[4]{\varepsilon}, S_1 > 0, \ldots, S_{a_n} > 0\right]\Big/\hat{r}_n$$

$$= \mathrm{P}[Z_n(1) > \sqrt[4]{\varepsilon}, S_1 > 0, \ldots, S_{a_n} > 0]/\hat{r}_n,$$

we conclude, by (37), (22), (35) and (36), that (33) holds. This completes the proof of Theorem 3.

REFERENCES

[1] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York 1968.

[2] E. Bolthausen, *On a functional central limit theorem for random walks conditioned to stay positive*, Ann. Probability, 4 (1976), p. 480-485.

[3] D. L. Iglehart, *Functional central limit theorem for random walks conditioned to stay positive*, ibidem 2 (1974), p. 608-619.

[4] D. Landers and L. Rogge, *The exact approximation order in the central limit theorem for random summation*, Z. Wahrsch. Verw. Gebiete 36 (1976), p. 269-283.

[5] A. Renyi, *Probability Theory*, North Holland, Amsterdam-London 1970.

[6] F. Spitzer, *A Tauberian theorem and its probability interpretation*, Trans. Amer. Math. Soc. 94 (1960), p. 150-169.

[7] A. Szubarga and D. Szynal, *Random limit theorems for random walks conditioned to stay positive*, Probability and Math. Stat. 5.1 (1985), p. 83-89.

Mathematical Institute
Maria Curie-Skłodowska University
ul. Nowotki 10
20-031 Lublin, Poland

Received on 17. 2. 1984