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ON A GENERAL ZERO-SUM STOCHASTIC GAME WITH STOPPING STRATEGY FOR ONE PLAYER AND CONTINUOUS STRATEGY FOR THE OTHER

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Abstract. In the paper a general zero-sum game with a stopping strategy for the first player and a continuous one for the second player is considered. The author proves the existence of a value of the game and an optimal strategy for the first player under fairly general assumptions.

1. Introduction. There is a considerable number of papers dealing with general zero-sum stochastic games with optimal stopping [1, 2, 6, 8, 9]. A good survey on these results is given by Zabczyk in [10]. From another point of view Davis–Elliott [3] have studied a zero-sum game with continuous strategies. In this paper we consider the so-called mixed zero-sum game, where the first (resp. the second) player chooses a stopping time S (resp. a continuous strategy u) and looks for maximize (resp. minimize) a payoff $E_u(C_T^u + Y_T)$.

Section 2 gives a precise model of the game. In Section 3 we prove that the upper value function of the game $\hat{W}(T)$ is "aggregable" under right continuity assumptions on the processes, i.e. there exists a right continuous process $\hat{W}$ such that $\hat{W}(T) = \hat{W}_T$ P a.e. for every stopping time T.

The method is based on the results of Dellacherie–Lenglart [4]. The last section contains essential results of this paper: the value of the game and, with additional assumption on the left regularity of the processes, the existence of an optimal strategy for the first player.

2. Game model and basic assumptions.

Definition 1. We call *mixed game* the zero-sum game defined by the data of

$$(\Omega, F, F_t, P, P^u, u\in \mathscr{U}, \mathscr{T}, J(S, u)_{S\in\mathscr{T}, u\in\mathscr{U}}),$$

where (Ω, F, P) is a probability space, $(F_t)_{t\geqslant 0}$ — an increasing right continuous family of complete sub-σ-fields of F $(F_0 = (\Omega, \emptyset))$, T — the set of admissible strategies for the first player — is the set of F_t - stopping times, $\mathscr{U}$ — the set of admissible strategies for the second player — is the set of all V-valued F_t-predictable processes (V — compact metric space).

Under the strategy $u\in\mathscr{U}$ the probability P^u is defined by $dP^u/dP|F_t = L_t^u$, where L^u is a uniformly integrable martingale strictly positive with the following compatibility conditions:

If $u, v\in\mathscr{U}$ nd $u_t = v_t$ for every $t\in[S, T[$, $S, T\in\mathscr{T}$, then

$$\frac{L_t^u}{L_S^u} = \frac{L_t^v}{L_S^v} \quad \text{for every } t\in[S, T[\text{ a.e.}$$

Remark 1. Usually L^u is the exponential martingale associated with a family of stochastic integrals. Particularly, we consider the classical situation of the diffusions on R^n, where we control with the drift.

The payoff $J(S, u)$, where (S, u) is in $\mathscr{T}\times\mathscr{U}$, is $E_u(C_S^u + Y_S)$ (E_u is the expectation for P^u), where Y is optional bounded, $Y_\infty = 0$ and C^u is an F_t-adaptable process with integrable variations satisfying the compatibility conditions

(1) $$C_{t\wedge S}^u = C_{t\wedge S}^v \text{ if } v\in\mathscr{D}(u, S)$$

(i.e. $v = u$ on $[0, S[$),

(2) $$C_{t\vee S}^u - C_S^u = C_{t\vee S}^v - C_S^v \text{ if } u = v \text{ on } [S, +\infty[$$

and an assumption on the potentials generated by C^u:

(3) if $X_{S,T}^u = E_u(C_T^u - C_S^u/F_S)$, $T\geqslant S$, then X_T^u is uniformly bounded (in S and u) and non-negative.

Remark 2. Particularly, we consider the case

$$C_t^u = \int_0^t e^{-\alpha s} c(s, u_s)\, ds, \quad c\geqslant 0, \text{ bounded.}$$

Define for any F_t-stopping time T, u in $\mathscr{U}$:

$$\bar{X}(u, T) = P\text{-}\operatorname*{ess\,inf}_{v\in\mathscr{D}(u,T)} P\text{-}\operatorname*{ess\,sup}_{S\geqslant T} E_v(C_s^v + Y_S/F_T),$$

$$\bar{X}_0 = \inf_v \sup_S E_v(C_S^v + Y_S), \quad \underline{X}_0 = \sup_S \inf_v E_v(C_S^v + Y_S).$$

We show that if Y, C^u, $u \in \mathscr{D}$, are right continuous, then the mixed game has a value, i.e. $\bar{X}_0 = \underline{X}_0$. For this we need $\bar{X}(u, T) - C^u_T$ to be independent of u and aggregable in a right continuous process.

3. Aggregation of the upper value. First notice that

$$\bar{X}(u, T) = P\text{-}\operatorname*{ess\,inf}_{v \in \mathscr{D}(u,T)} P\text{-}\operatorname*{ess\,sup}_{S \geq T} E_v(C^v_S - C^v_T + Y_S/F_T) + C^u_T.$$

From the compatibility conditions on L^u and C^u we easily deduce that

$$\bar{X}(u, T) = P\text{-}\operatorname*{ess\,inf}_{v \in \mathscr{U}} P\text{-}\operatorname*{ess\,sup}_{S \geq T} E_v(C^v_S - C^v_T + Y_S/F_T) + C^u_T$$
$$= \hat{W}(T) + C^u_T \quad P \text{ a.e.}$$

The family $(\hat{W}(T), T \in \mathscr{T})$ is called *upper value* of the game.

To aggregate $\hat{W}$ we need the fundamental result of Dellacherie–Lenglart [4]. According to their terminology, we call *$\mathscr{T}$-system* any family $(X(T) | T \in \mathscr{T})$ of random functions such that

(i) $X(T) = X(T')$ a.e. on $T = T'$ for any T, T',

(ii) $X(T)$ is F_T-*measurable for any* T.

THEOREM 1 [4]. *Any $\mathscr{T}$-system X, upper right semicontinuous, i.e.*

$$X(T) \geq \limsup_n X(T_n) \text{ a.e. if } T_n \searrow T,$$

can be aggregated by an upper semicontinuous optional process.

It is obvious that $\hat{W}$ is a $\mathscr{T}$-system in this sense. We have

THEOREM 2. *Assume that, for all u, the process C^u is lower right semicontinuous. Then the $\mathscr{T}$-system $\hat{W}$ is upper right semicontinuous and there exists an optional upper-right semicontinuous process $\hat{W}$ such that*

$$\hat{W}(T) = \hat{W}_T \text{ a.e. for every } T \in \mathscr{T}.$$

Proof. Let, for all u in $\mathscr{U}$, Z^u be the P^u Snell's envelope of $C^u + Y$. We have

$$\hat{W}(T) = P\text{-}\operatorname*{ess\,inf}_u \big(P\text{-}\operatorname*{ess\,sup}_{S \geq T} E_u(C^u_S + Y_S/F_T) - C^u_T\big) = P\text{-}\operatorname*{ess\,inf}_u (Z^u - C^u)_T.$$

For all u of $\mathscr{U}$, Z^u is upper right semicontinuous as supermartingale. The assumption on C^u implies that $Z^u - C^u$ is also upper right semicontinuous for all u. Then, since an infimum of upper right semicontinuous functions is upper right semincontinuous, and the P-ess inf is always attained by a countable infimum, we easily deduce that the $\mathscr{T}$-system $\hat{W}$ is upper right semicontinuous, then aggregable.

We need a result based on the properties of increasing or decreasing filtration which allow to inverse ess inf or ess sup with conditional expectation. This kind of result has been already used (see [7] or [6]) for other zero-sum stochastic games.

LEMMA 1. *For all u in $\mathscr{U}$, for any stopping times T_1 and T_2, $T_1 \leqslant T_2$, we have*

$$E_u(\bar{X}^u_{T_2}/F_{T_1}) = P\text{-}\operatorname*{ess\,inf}_{v\in\mathscr{D}(u,T_2)} P\text{-}\operatorname*{ess\,sup}_{S\geqslant T_2} E_v(C^v_S + Y_S/F_{T_1}),$$

where $\bar{X}^u = C^u + \hat{W}$.

Proof. It is easy to see that, for all v in $\mathscr{D}(u, T_2)$, the family $(E_v(C^v_S + Y_S/F_{T_2}), S \geqslant T_2)$ is a lattice (for the supremum). Therefore, for all v of $\mathscr{D}(u, T_2)$, we have

$$(1)\quad E_u\big(P\text{-}\operatorname*{ess\,sup}_{S\geqslant T_2} E_v(C^v_S + Y_S/F_{T_2})/F_{T_1}\big)$$
$$= P\text{-}\operatorname*{ess\,sup}_{S\geqslant T_2} E_u\big(E_v(C^v_S + Y_S/F_{T_2})/F_{T_1}\big) = P\text{-}\operatorname*{ess\,sup}_{S\geqslant T_2} E_v(C^v_S + Y_S/F_{T_1}),$$

since P^u and P^v are the same on F_{T_2}.

On the other hand, the family

$$P\text{-}\operatorname*{ess\,sup}_{S\geqslant T_2} E_v(C^v_S + Y_S/F_{T_2}),\qquad v\in\mathscr{D}(u, T_2),$$

is also an infimum lattice. In fact, if v and v' are in $\mathscr{D}(u, T_2)$ and if

$$A = \{P\text{-}\operatorname*{ess\,sup}_{S\geqslant T_2} E_v(C^v_S + Y_S/F_{T_2}) \leqslant P\text{-}\operatorname*{ess\,sup}_{S\geqslant T_2} E_{v'}(C^{v'}_S + Y_S/F_{T_2})\},$$

then the strategy $w = v'/T_{2_A}/v$ (see [2]; w is the strategy v' on A^c bifurcating from v' to v on A at the time T_2) is in $\mathscr{D}(u, T_2)$ and we easily get that

$$\text{P-}\operatorname*{ess\,sup}_{s\geqslant T_2} E_w(C^w_S + Y_S/F_{T_2})$$
$$= P\text{-}\operatorname*{ess\,sup}_{S\geqslant T_2} E_v(C^v_S + Y_S/F_{T_2}) \wedge P\text{-}\operatorname*{ess\,sup}_{S\geqslant T_2} E_{v'}(C^{v'}_S + Y_S/F_{T_2}).$$

Therefore, by the property of inversion, we have

$$(2)\qquad E_u(\bar{X}^u_{T_2}/F_{T_1}) = P\text{-}\operatorname*{ess\,inf}_{v\in\mathscr{D}(u,T_2)} E_u\big(P\text{-}\operatorname*{ess\,sup}_{S\geqslant T_2} E_v(C^v_S + Y_S/F_{T_2})/F_{T_1}\big).$$

Summarizing (1) and (2) completes the proof.

We now prove under the assumption

(H) *C^u and Y are right continuous for all u of $\mathscr{U}$*

hat, for all u, the process $\bar{X}^u$ is lower right semincontinuous in expectation, then lower right semicontinuous and, finally, by Theorem 2, right continuous.

THEOREM 3. *For all u of $\mathscr{U}$, any stopping time T, any sequence (T_n) of* *stopping times decreasing to T, we have*

$$E_u(\bar{X}^u_T) \leqslant \liminf_n E_u(\bar{X}^u_{T_n}).$$

Proof. First, Lemma 1 for $T_1 = 0$ and $T_2 = T$ gives

$$(3) \qquad E_u(\bar{X}^u_T) = \inf_{v \in \mathscr{D}(u,T)} \sup_{S \geqslant T} E_v(C^v_S + Y_S) = \inf_{v \in \mathscr{D}(u,T_n)} \sup_{S \geqslant T} E_v(C^v_S + Y_S),$$

since $\mathscr{D}(u, T_n) \subseteq \mathscr{D}(u, T)$.

Let v be in $\mathscr{D}(u, T_n)$. Then

$$E_v(C^v_S + Y_S) = E_v\big(\mathbb{1}_{(S<T_n)}(C^v_S + Y_S)\big) + E_v\big(\mathbb{1}_{(S \geqslant T_n)}(C^v_S + Y_S)\big)$$

and, since v and u are the same until T_n,

$$\begin{aligned} E_v(C^v_S + Y_S) &= E_u\big(\mathbb{1}_{(S<T_n)}(C^u_S + Y_S)\big) + E_v\big(\mathbb{1}_{(S \geqslant T_n)}(C^v_{S \vee T_n} + Y_{S \vee T_n})\big) \\ &= E_u\big(\mathbb{1}_{(S<T_n)}(C^u_S + Y_S)\big) - E_v\big(\mathbb{1}_{(S<T_n)}(C^v_{S \vee T_n} + Y_{S \vee T_n})\big) + \\ &\qquad\qquad + E_v(C^v_{S \vee T_n} + Y_{S \vee T_n}) \\ &= E_u\big(\mathbb{1}_{(S<T_n)}(C^u_S + Y_S)\big) - E_v\big(\mathbb{1}_{(S<T_n)}(C^v_{T_n} + Y_{T_n})\big) + \\ &\qquad\qquad + E_v(C^v_{S \vee T_n} + Y_{S \vee T_n}) \\ &= E_u\big(\mathbb{1}_{(S<T_n)}(C^u_S + Y_S - C^u_{T_n} - Y_{T_n})\big) + E_v(C^v_{S \vee T_n} + Y_{S \vee T_n}). \end{aligned}$$

Then

$$\sup_{S \geqslant T} E_v(C^v_S + Y_S) \leqslant \sup_{S \geqslant T} E_u\big(\mathbb{1}_{(S<T_n)}(C^u_S + Y_S - C^u_{T_n} - Y_{T_n})\big) + \sup_{S \geqslant T_n} E_v(C^v_S + Y_S),$$

since $\{S \in \mathscr{T} \mid S \geqslant T_n\} = \{S \vee T_n / S \geqslant T\}$.

Taking the infimum over v of $\mathscr{D}(u, T_n)$ and applying to the left-hand side inequality (3) and to the right-hand one Lemma 1, we finally obtain

$$E_u(\bar{X}^u_T) \leqslant \sup_{S \geqslant T_n} E_u\big(\mathbb{1}_{(S<T_n)}(C^u_{S \wedge T_n} + Y_{S \wedge T_n} - C^u_{T_n} - Y_{T_n})\big) + E_u(\bar{X}^u_{T_n}).$$

Let now any $\varepsilon > 0$. We can choose $S_n \geqslant T$ such that

$$E_u(\bar{X}^u_T) \leqslant E_u\big(\mathbb{1}_{(S_n<T_n)}(C^u_{S_n \wedge T_n} + Y_{S_n \wedge T_n} - C^u_{T_n} - Y_{T_n})\big) + \varepsilon + E_u(\bar{X}^u_{T_n}).$$

Since $T_n \searrow T$ and $S_n \wedge T_n$ converges to T, we get, by the Lebesgue theorem, $E_u(\bar{X}^u_T) \leqslant \liminf E_u(\bar{X}^u_{T_n}) + \varepsilon$ for every $\varepsilon > 0$, and thus the final result.

We finally get the main result of this part:

THEOREM 4. *If* (H) *holds, then there exists a right continuous process* $\hat{W}$ *such that* $\hat{W}_T = \hat{W}(T)$ *a.e. for any stopping time* T.

We shall use $\hat{W}$ to construct stopping times which realize the ε-value and this leads us easily to the conclusion.

4. Existence of a value and an optimal strategy for the first player. Let, for all $\varepsilon > 0$ and any stopping time T,

$$D_T^\varepsilon = \inf(s \geq T, \hat{W}_s \leq Y_s + \varepsilon).$$

PROPOSITION 1. *If* (H) *holds, then for all u of $\mathscr{U}$ and for any stopping time T we have*

(1) $$\bar{X}_T^u \leq E(\bar{X}_{D_T^\varepsilon}/F_T) + \varepsilon.$$

Proof. For any stopping time $U \leq D_T^\varepsilon$, v of $\mathscr{D}(u, U)$ and Z^v being the P^v Snell's envelope of $C^v + Y$ we have

$$\bar{X}_U^u = P\text{-}\operatorname*{ess\,inf}_{v \in \mathscr{D}(u,U)} Z_U^v \leq P\text{-}\operatorname*{ess\,inf}_{v \in \mathscr{D}(u,D_T^\varepsilon)} Z_U^v$$

since $\mathscr{D}(u, U) \subseteq \mathscr{D}(u, D_T^\varepsilon)$. Then

(2) $$\bar{X}_U^u \leq Z_U^v$$

a.e. for all v of $\mathscr{D}(u, D_T^\varepsilon)$ and for any stopping time $U \leq D_T^\varepsilon$.

Let $T \leq t < D_T^\varepsilon$. By the definition of D_T^ε we have

$$\bar{X}_t^u > C_t^u + Y_t + \varepsilon = C_t^v + Y_t + \varepsilon$$

for all v of $\mathscr{D}(u, D_T^\varepsilon)$ since, from the compatibility conditions on C^u, we have $C^u = C^v$ until D_T^ε.

Using (2) we get

(3) $$Z_t^v > C_t^v + Y_t + \varepsilon \quad \text{for all } v \text{ of } \mathscr{D}(u, D_T^\varepsilon) \text{ on } \{T \leq t < D_T^\varepsilon\}.$$

If $D_T^{\varepsilon,v} = \inf(t \geq T, Z_t^v \leq Y_t + C_t^v + \varepsilon)$, we finally get

$$D_T^{\varepsilon,v} \geq D_T^\varepsilon \quad \text{for all } v \text{ of } \mathscr{D}(u, D_T^\varepsilon).$$

Then, using results of the optimal stopping [5], we obtain, for all v of $\mathscr{D}(u, D_T^\varepsilon)$ (since, from [5], $Z_{t \wedge D_T^{\varepsilon,v}}^v$ has the martingale property between T and $D_T^{\varepsilon,v}$),

$$Z_t^v = P\text{-}\operatorname*{ess\,sup}_{S \geq D_T^\varepsilon} E_v(C_S^v + Y_S/F_T),$$

and then, by Lemma 1,

(4) $$\bar{X}_T^u \leq P\text{-}\operatorname*{ess\,inf}_{v \in \mathscr{D}(u,D_T^\varepsilon)} Z_T^v = P\text{-}\operatorname*{ess\,inf}_{v \in \mathscr{D}(u,D_T^\varepsilon)} P\text{-}\operatorname*{ess\,sup}_{S \geq D_T^\varepsilon} E_v(C_S^v + Y_S/F_T) = E_u(X_{D_T^\varepsilon}^u/F_T).$$

From (4) we easily deduce the main result of this section.

THEOREM 5. *With assumption* (H), *the mixed game has a value.*

Furthermore, if processes Y *and* C^u, $u \in \mathcal{U}$, *are also upper left semicontinuous, then the first player has an optimal strategy* $D = \lim_{\varepsilon \to 0} D_0^\varepsilon$.

Proof. Since $\hat{W}$ and Y are right continuous, we have

$$(5) \qquad \hat{W}_{D_T^\varepsilon} \leqslant Y_{D_T^\varepsilon} + \varepsilon.$$

By (4) and (5) we easily get $\bar{X}_t^u \leqslant E_u(C_{D_T^\varepsilon}^u + Y_{D_T^\varepsilon}/F_T) + \varepsilon$ for all u of $\mathcal{U}$. Then, for $T = 0$,

$$(6) \qquad \bar{X}_0 \leqslant E_u(C_{D_0^\varepsilon}^u + Y_{D_0^\varepsilon}) + \varepsilon \quad \text{for all } u \text{ of } \mathcal{U}$$

and

$$\bar{X}_0 \leqslant \inf_{u \in \mathcal{U}} E_u(C_{D_0^\varepsilon}^u + Y_{D_0^\varepsilon}) + \varepsilon$$
$$\leqslant \sup_{T \in \mathcal{T}} \inf_{u \in \mathcal{U}} {}_u(C_T^u + Y_T) + \varepsilon = \underline{X}_0 + \varepsilon \quad \text{for all } \varepsilon > 0,$$

which implies that $\bar{X}_0 \leqslant \underline{X}_0$, hence $\bar{X}_0 = \underline{X}_0$ since the inverse inequality is always true.

Finally, if $\bar{D} = \lim_{\varepsilon \to 0} D_0^\varepsilon$, then by Fatou lemma and the upper left semicontinuity of the processes, letting $\varepsilon \to 0$ in (6), we have

$$\bar{X}_0 = \underline{X}_0 \leqslant E_u(C_{\bar{D}}^u + Y_{\bar{D}}) \quad \text{for all } u \text{ of } \mathcal{U}$$

and the stopping time $\bar{D}$ is optimal for the first player.

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