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ON A THEOREM OF SALISBURY

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Abstract. Salisbury proved that there exist a domain and attainable minimal Martin boundary points x and y such that no h-process can start at x and terminate at y. A new, less computational proof is supplied in this note.

1. Introduction. Salisbury [4, 5] has recently proved that in some Greenian domains there exist attainable Martin boundary points x and y such that there does not exist an h-process starting from x and terminating at y.

His proof is based on two lemmas: Theorem 3.3 and Theorem 3.4. An alternative proof is presented below. Theorem 3.4 is generalized and Theorem 3.3 is replaced by a similar one. New proofs use many ideas taken from the original ones but are less computational in nature, for example they do not require Schwarz-Christoffel formulae or estimates of Cranston and McConnel [1].

The reader is referred to Doob [2] for the definitions of an h-process, Martin boundary and related concepts.

The author would like to thank Thomas Salisbury for the interesting and stimulating discussion of Martin boundaries.

2. The Martin boundary. It will be convenient to use the complex notation. Let

$$A_n = \{z \in C\colon \operatorname{Im} z = a_n,\ b_n < \operatorname{Re} z < 1-b_n\},\ n \geqslant 1,$$

and

$$D = \{z \in C\colon 0 < \operatorname{Re} z < 1,\ 0 < \operatorname{Im} z < 1\} \setminus \bigcup_{n=1}^{\infty} A_n.$$

Assume that

$$0 < a_n < 1,\ 0 < b_n < 1/2 \quad \text{for } n \geqslant 1,$$

$$a_n < a_m \quad \text{for } n < m,$$

$$\lim_{n\to\infty} a_n = a_\infty = 1, \quad \limsup_{n\to\infty} b_n = b_\infty < 1/2.$$

Write $a_{n+1}-a_n = e_n$. Let $\{z_n\}$ be a sequence of points such that $z_n \in D$, $\operatorname{Re} z_n = 1/2$ for $n \geqslant 1$ and

$$\lim_{n\to\infty} \operatorname{Im} z_n = 1.$$

Then there exists a subsequence z_{n_k} which converges (in the Martin topology) to a Martin boundary point z_0 of D. Let h be a Martin function in D corresponding to z_0.

THEOREM 2.1. *The function h is not minimal.*

Remarks 2.1. (i) Theorem 2.1 generalizes Theorem 3.4 of Salisbury [5], who assumed in addition that $b_n/e_n < c < \infty$ for all n.

(ii) The idea of the proof is the following. Brownian motion in D is unlikely to travel thin canals. This property is inherited by the h-process. The h-paths are therefore likely to cluster near the points i or $i+1$. By symmetry these events have probability 1/2 and the tail σ-field is not trivial. This implies that h is not minimal.

The proof of the theorem will be preceded by some more notation and two lemmas.

Let $b = (b_\infty + 1/2)/2$. It will be assumed WLOG that $b_n < b$ for all $n \geqslant 1$. Define B_n^k for $n \geqslant 1$ and $k = 1, 2, \ldots, 6$ by

$$B_n^k = \left\{ z \in C\colon\ a_n < \operatorname{Im} z < a_{n+1},\ \operatorname{Re} z = \frac{6-k}{5} b + \frac{k-1}{5}(1-b) \right\}.$$

The distribution of the Brownian motion in D (h-process) starting from $x \in D$ will be denoted by $P^x(P_h^x)$. If $x = 1/2 + ia_1/2$, then the superscript will be suppressed.

The paths of processes will be denoted by $X(t)$ and the lifetime R can be written as

$$R = \inf\{t\colon\ \liminf_{s\to t-} \operatorname{dist}(X(s), \partial D) = 0\}.$$

The hitting time of a set B will be called T_B. For sets $B_1, B_2, \ldots, B_{2k} \subset D$ and $j = 1, 2, \ldots, \infty$ define events

$$F_j(B_1(B_2), B_3(B_4), \ldots, B_{2k-1}(B_{2k}))$$
$$\stackrel{\mathrm{df}}{=} \{T_1 \stackrel{\mathrm{df}}{=} T_{B_1} < \infty \text{ and } T_1 < T_{B_2} \text{ and } T_2 \stackrel{\mathrm{df}}{=} \inf\{t > T_1\colon X(t) \in B_3\} < \infty$$
$$\text{and } T_2 < \inf\{t > T_1\colon X(t) \in B_4\} \ldots$$
$$\text{and } T_k \stackrel{\mathrm{df}}{=} \inf\{t > T_{k-1}\colon X(t) \in B_{2k-1}\} < \infty$$
$$\text{and } T_k < \inf\{t > T_{k-1}\colon X(t) \in B_{2k}\} \text{ and } T_k < \inf\{t \geqslant 0\colon \operatorname{Im} X(t) \geqslant a_j\}\}.$$

If one of the sets $B_2, B_4, \ldots, B_{2k}$ is equal to ∂D, then it is suppressed in the notation.

LEMMA 2.1. *There exists a constant $c_0 < \infty$ such that for all $n \geqslant 1$ and $j \geqslant n+1$*

$$P(F_j(B_n^3, B_n^4)) < c_0 \cdot e_n \cdot P(F_j(B_n^3)).$$

Proof. There exists a constant $c_1 < \infty$ such that if K is an interval of the real line of the length $a > 0$, then the chance that Brownian motion in $\{\mathrm{Im}\, z > 0\}$ starting from $i(1-2b)/5$ will terminate at K is less than $a \cdot c_1$. It follows that for $k = 2, 3, 4, 5$ and $x \in B_n^k$

$$P^x(F_j(B_n^{k-1}(B_n^{k+1}))) < e_n \cdot c_1 \tag{2.1a}$$

and

$$P^x(F_j(B_n^{k+1}(B_n^{k-1}))) < e_n \cdot c_1. \tag{2.1b}$$

The event $F_j(B_n^4)$ is a union of an event N such that $P^x(N) = 0$ for all $x \in B_n^3$ and a countable union from $m = 0$ to ∞ of the events

$$F_j(\underbrace{B_n^2 \cup B_n^6(B_n^4),\ B_n^3 \cup B_n^5,\ \ldots,\ B_n^2 \cup B_n^6(B_n^4),\ B_n^3 \cup B_n^5}_{m \text{ times}},\ B_n^4(B_n^2 \cup B_n^6)).$$

The P^x-probability of such an event is less than $(e_n \cdot c_1)^{m+1}$ for all $x \in B_n^3$, which follows from (2.1a, b) and the repeated use of the strong Markov property. Thus, for $x \in B_n^3$,

$$P^x(F_j(B_n^4)) \leqslant \sum_{m=0}^{\infty} (e_n \cdot c_1)^{m+1} < c_0 \cdot e_n. \tag{2.2}$$

The last inequality holds for some $c_0 < \infty$ and all e_n small enough. It will be assumed WLOG that it holds for all e_n.

By the strong Markov property at $T_{B_n^3}$ and (2.2) one obtains

$$P(F_j(B_n^3, B_n^4)) < c_0 \cdot e_n \cdot P(F_j(B_n^3)),$$

which completes the proof.

LEMMA 2.2. *Let μ be a measure on B_n^3 (or on B_n^4)). Then P^μ-distribution of $X(T_{B_n^2 \cup B_n^5})$ has a density $g_n^\mu(x)$, $x \in B_n^2 \cup B_n^5$. There exists a function $g_n(x)$ such that, for every μ,*

$$g_n^\mu(x)/g_n(x) = c(\mu, n)\, k_n^\mu(x) \quad \textit{for all } x \in B_n^2 \cup B_n^5. \tag{2.3}$$

Here $c(\mu, n)$ does not depend on x and

$$0 < 1/c_2 < k_n^\mu(x) < c_2 < \infty \quad \textit{for all } x \in B_n^2 \cup B_n^5.$$

The constant c_2, $1 < c_2 < \infty$, does not depend on μ or n.

Proof. Let f be a conformal bijection of the rectangle U_n bounded by A_n, A_{n+1}, B_n^2 and B_n^5 onto the disc $D_1 = \{|z| < 1\}$. Assume that the midpoint of B_n^3 is mapped onto $0 \in D_1$ and B_n^2 and B_n^5 are mapped onto arcs symmetric wrt real axis. For small e_n the P^x-chance, $x \in B_n^3$, of hitting ∂U_n to the right from B_n^3 is arbitrarily close to 1/2, so B_n^3 is mapped on an arc close to the imaginary axis.

A conformal mapping of Brownian motion is a time-changed Brownian motion so the hitting probabilities are preserved. Therefore it is enough to prove (2.3) with $g_n^\mu(x)$ replaced by $\tilde{g}_n^\mu(x)$, $x \in f(B_n^2 \cup B_n^5)$, where $\tilde{g}_n^\mu(x)$ is the density of $X(R-)$ for Brownian motion in D_1 with the initial distribution $\mu \circ f^{-1}$. If e_n is small, then $f(B_n^2 \cup B_n^5)$ consists of two small arcs. The hitting distribution for Brownian motion in D_1 may be written down explicitly (see p. 102 of [3]) and it is easy to verify (2.3) for $\tilde{g}_n^\mu(x)$ directly.

Proof of Theorem 2.1. Let $C_j = \{\operatorname{Im} z = a_j\} \cap D$. The lemmas and the strong Markov property applied at $T_{B_n^2 \cup B_n^5}$ imply that the density of the P-distribution of

$$\left(X(T_{C_j}) \in \cdot,\ F_j(B_n^3, B_n^4)\right), \qquad j \geqslant n+1,$$

is at most $e_n \cdot c_0 \cdot c_2^2$ times the P-density of

$$\left(i \operatorname{Im} X(T_{C_j}) + \left(1 - \operatorname{Re} X(T_{C_j})\right) \in \cdot,\ F_j(B_n^3)\right)$$

for all the points of C_j. Therefore formula (2.1) of [2] (p. 672) and the symmetry of h imply that

$$P_h\left(F_j(B_n^3, B_n^4)\right) < e_n \cdot c_0 \cdot c_2^2 \cdot P_h\left(F_j(B_n^3)\right) \leqslant e_n \cdot c_0 \cdot c_2^2.$$

If $j \to \infty$, then $F_j(B_n^3, B_n^4) \uparrow F_\infty(B_n^3, B_n^4)$ and, therefore,

$$P_h\left(F_\infty(B_n^3, B_n^4)\right) \leqslant e_n \cdot c_0 \cdot c_2^2.$$

By symmetry $P_h\left(F_\infty(B_n^4, B_n^3)\right) \leqslant e_n \cdot c_0 \cdot c_2^2$. Thus

$$\sum_{n=1}^{\infty} P_h\left(F_\infty(B_n^3, B_n^4) \cup F_\infty(B_n^4, B_n^3)\right) \leqslant 2 \cdot c_0 \cdot c_2^2 \sum_{n=1}^{\infty} e_n < \infty.$$

It follows that P_h-a.s. only finitely many events $F_\infty(B_n^3, B_n^4) \cup F_\infty(B_n^4, B_n^3)$ happen and this implies that P_h-a.s.

$$\limsup_{t \to R} \operatorname{Re} X(t) < \frac{2}{5} b + \frac{3}{5}(1-b) \quad \text{or} \quad \liminf_{t \to R} \operatorname{Re} X(t) > \frac{3}{5} b + \frac{2}{5}(1-b).$$

By symmetry the P_h-probability of each of these events is 1/2. Since these events are in the tail σ-field, it follows from [2], p. 730, that h is not minimal.

3. Estimates of the Naim kernel. By Theorem 2.1 there exist at least two minimal Martin boundary points x_1, x_2, such that if $x \to x_1$ or $x \to x_2$, then $\operatorname{Im} x \to 1$. The points x_1 and x_2 are attainable by results of Cranston and McConnell [1]. Salisbury [5] (Corollary 3.5) has shown under some assumptions that there does not exist an h-process which starts at x_1 and terminates at x_2. A new proof of this result will be given below. Salisbury's basic lemma (Theorem 3.3) will be replaced by Proposition 3.1.

$K(x_1, x)$, $x \in D$, will denote the Martin function and $G_D(x, y)$, $x, y \in D$, will be the Green function. Let $z_0 = 1/2 + ia_1/2$. Fix a_n's for the rest of this section.

PROPOSITION 3.1. *If $b_n \to 0$ sufficiently fast as $n \to \infty$, then*

$$\limsup_{t \to 1-} K(x_1, \Gamma(t))/G_D(z_o, \Gamma(t)) = \infty$$

for every continuous path $\Gamma = \{\Gamma(t), 0 < t < 1\} \subset D$ such that

$$\lim_{t \to 1-} \operatorname{Im} \Gamma(t) = 1.$$

Proof. The functions $K(x_1, \cdot)$ and $G_D(z_o, \cdot)$ have limits 0 at the parts of the boundary $\partial D \cap \{\operatorname{Im} z < a_n\}$ for every $1 \leqslant n < \infty$. The set $D_n = D \cap \{a_n < \operatorname{Im} z < a_{n+1}\}$ is a Lipschitz domain. An easy variation of Theorem 1 of Wu [6] applied to the subset D_n^1 of D_n,

$$D_n^1 = D \cap \left\{\frac{2}{3}a_n + \frac{1}{3}a_{n+1} < \operatorname{Im} z < \frac{1}{3}a_n + \frac{2}{3}a_{n+1}\right\}$$

shows that if $K(x_1, x)/G_D(z_0, x) = d$ for some $x \in D$, $\operatorname{Im} x = (a_n + a_{n+1})/2$, then

$$\frac{K(x_1, y)}{G_D(z_0, y)} \geqslant d \cdot c_n \text{ for all } y \in D, \quad \operatorname{Im} y = (a_n + a_{n+1})/2. \tag{3.1}$$

The constants $c_n > 0$ do not depend on d or b_n's.

It is easy to see that b_1 can be chosen so small that $G_D(z_0, x) < 1$ for all $x \in D$, $\operatorname{Im} x \geqslant a_1$.

For each $n \geqslant 2$ choose b_n so small that if a harmonic function g in $D \cap \{\operatorname{Im} z < (a_n + a_{n+1})/2\}$ has the boundary limit 0 for each

$$x \in \partial D \cap \{\operatorname{Im} z < (a_n + a_{n+1})/2\}$$

and is bounded by 1 on $\{\operatorname{Im} z = (a_n + a_{n+1})/2\}$, then $g(z_0) < c_n/n$.

Normalize $K(x_1, y)$ so that $K(x_1, z_0) = 1$. By the choice of b_n's for $n \geqslant 2$ we have $K(x_1, z_n) \geqslant n/c_n$ for some $z_n \in D$, $\operatorname{Im} z_n = (a_n + a_{n+1})/2$. It follows from (3.1) that

$$\frac{K(x_1, y)}{G_D(z_0, y)} \geqslant c_n \cdot \frac{K(x_1, z_n)}{G_D(z_0, z_n)} \geqslant c_n \cdot \frac{n/c_n}{1} = n$$

for all $y \in D$, $\operatorname{Im} y = (a_n + a_{n+1})/2$ and this completes the proof.

COROLLARY 3.1 (Salisbury). *If $b_n \to 0$ sufficiently fast as $n \to \infty$, then there does not exist an h-process in D starting from x_1 and terminating at x_2.*

Proof. Use Proposition 3.1 and Theorem 2.3 (c) of Salisbury [5].

Remark 3.1 How fast is "fast" in the last corollary? The above method of proof does not provide an answer, see however Salisbury [5].

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