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BIAS-ROBUST ESTIMATION OF THE SCALE PARAMETER

BY

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Abstract. The paper deals with the concept of robustness given by Zieliński (see [17] and [18]). The uniformly most bias-robust estimates of the scale parameter, based on order statistics and spacings, for some statistical models are obtained. Violation of these models are generated by ordering relations in the set of distributions like stochastic ordering, dispersive ordering, convex and star-shaped orderings and others.

1. PRELIMINARIES

Throughout the paper we identify probability distributions with its distribution functions and assume that all considered distributions are continuous and strictly increasing on their supports which will be intervals. We also assume that all expectations being considered exist and are finite.

Let random variables X and Y have the distributions F and G with the supports S_F and S_G, respectively. Denote by $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ and $Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}$ order statistics of samples from the distributions F and G. Define $X_{0:n} = \inf\{x\colon F(x) > 0\}$ and $Y_{0:n} = \inf\{x\colon G(x) > 0\}$ if they are finite. The random variables $V_{i:n} = X_{i:n} - X_{i-1:n}$ and $U_{i:n} = Y_{i:n} - Y_{i-1:n}$ are called *spacings* from the distributions F and G, respectively. We recall some partial orderings of distributions which will be used in the sequel.

1.1. Stochastic ordering. We say that F is *stochastically less than* G ($F \overset{\text{st}}{\leqslant} G$) if and only if $F(x) \geqslant G(x)$ for every x. We shall also use the notation $X \overset{\text{st}}{\leqslant} Y$ if and only if $F \overset{\text{st}}{\leqslant} G$. It is well known that if $X \overset{\text{st}}{\leqslant} Y$, then $X_{i:n} \overset{\text{st}}{\leqslant} Y_{i:n}$ and hence $EX_{i:n} \leqslant EY_{i:n}$, $i = 1, 2, \ldots, n$.

1.2. Dispersive ordering. Distributions F and G are said to be *ordered in dispersion* ($F \overset{\text{disp}}{<} G$) if and only if $F^{-1}(\beta) - F^{-1}(\alpha) \leqslant G^{-1}(\beta) - G^{-1}(\alpha)$

whenever $0 < \alpha < \beta < 1$. We shall also use the notation $X \overset{\text{disp}}{<} Y$ if and only if $F \overset{\text{disp}}{<} G$.

Many authors have studied properties of this ordering, e.g. Saunders and Moran [14], Lewis and Thompson [11], Shaked [15]. Deshpande and Kochar [6] have noticed that this ordering is the same as tail-ordering, introduced by Doksum [7]. Bickel and Lehmann [5] have called this relation "G is more spread out than F". Shaked [15] has proved that under the above general assumptions on F and G, $F \overset{\text{disp}}{<} D$ if and only if the so-called shift function $\Delta(x) = G^{-1}F(x) - x$ (see [8] and [12]) is nondecreasing for $x \in S_F$. Hence it follows that if $X \overset{\text{disp}}{<} Y$, then $X_{i:n} \overset{\text{disp}}{<} Y_{i:n}$, $i = 1, 2, \ldots, n$. One can also prove that if F and G are symmetric about the origin, i.e. $F(x) = 1 - F(-x)$ for all x (for G analogously), then $X \overset{\text{disp}}{<} Y$ implies $|X| \overset{\text{disp}}{<} |Y|$. The following other properties of the dispersive ordering will be applied in next sections.

Lemma 1 (Shaked [15]). *Let $S_F = [0, a_F]$ and $S_G = [0, a_G]$, $a_F \leqslant a_G \leqslant \infty$. If $F \overset{\text{disp}}{<} G$, then $F \overset{\text{st}}{\leqslant} G$.*

Lemma 2 (Oja [12]). *Let $S_F = [0, a_F]$ and $S_G = [0, a_G]$, $a_F \leqslant \infty$, $a_G \leqslant \infty$. If $F \overset{\text{disp}}{<} G$, then $V_{i:n} \overset{\text{st}}{\leqslant} U_{i:n}$, $i = 1, 2, \ldots, n$.*

Lemma 3 (Bickel and Lehmann [5]). *Let F and G have the densities f and g, respectively. Then $F \overset{\text{disp}}{<} G$ if and only if $gG^{-1}(u) \leqslant fF^{-1}(u)$, $u \in (0, 1)$.*

1.3. Convex and star-shaped orderings. Let $F(0) = 0 = G(0)$. Van Zwet [20] has introduced the convex ordering relation: F is *convex with respect to* G ($F \overset{\text{c}}{<} G$) if and only if $G^{-1}F$ is convex on S_F. Barlow and Proschan [1] have considered the weaker relation: F is *star-shaped with respect to* G ($F \overset{*}{<} G$) if and only if $G^{-1}F$ is star-shaped on S_F, i.e. $G^{-1}F(x)/x$ is nondecreasing in $x \in S_F$. It is easy to see that $F \overset{\text{c}}{<} G$ implies $F \overset{*}{<} G$. These relations are partial orderings of the scale equivalent classes of distributions. The following properties of the convex and star-shaped orderings will be used in the sequel.

Lemma 4 (Barlow and Proschan [1]). *If $F \overset{*}{<} G$, then $EX_{i:n}/EY_{i:n}$ is nonincreasing in $i = 1, 2, \ldots, n$.*

Lemma 5 (van Zwet [21]). *If $F \overset{\text{c}}{<} G$, then $EV_{i:n}/EU_{i:n}$ is nonincreasing in $i = 1, 2, \ldots, n$.*

Lemma 6 (Sathe [13]). *If $F \overset{\text{st}}{\leqslant} G$ and $F \overset{*}{<} G$, then $F \overset{\text{disp}}{<} G$.*

LEMMA 7 (León and Lynch [10]). *Let G have a density g on $[0, a_G]$, $a_G \leqslant \infty$, which is positive and continuously differentiable on $(0, a_G)$. Then the class of continuous distributions $\{F\colon G \overset{*}{<} F\}$ is closed under mixtures if and only if $ug'(u)/g(u)$ is decreasing on $(0, a_G)$.*

1.4. s-ordering and r-ordering. The analogues of convex and star-shaped orderings for symmetric distributions have been studied by van Zwet [20], Doksum [7] and Lawrance [9]. Assume that F and G are symmetric about the origin. We say that F and G are *ordered with respect to the s-ordering* ($F \overset{s}{<} G$) if and only if $G^{-1}F$ is convex on $S_F \cap [0, \infty)$ and concave on $S_F \cap (-\infty, 0]$. We say that F and G are *ordered with respect to the r-ordering* ($F \overset{r}{<} G$) if and only if $G^{-1}F$ is star-shaped on $S_F \cap [0, \infty)$ and $-G^{-1}F$ is star-shaped on $S_F \cap (-\infty, 0]$. It is easy to notice that $F \overset{s}{<} G$ implies $F \overset{r}{<} G$. These relations are partial orderings of the scale equivalent classes of symmetric distributions. The following lemmas give properties of these orderings which will be needed in next sections. The first of them is obvious.

LEMMA 8. *Let F and G be symmetric about the origin and let $F^{(\alpha)}$ denote the distribution of $|X|^\alpha$, $\alpha \geqslant 1$, for $G^{(\alpha)}$ analogously. Then:* (i) *$F \overset{s}{<} G$ implies $F^{(\alpha)} \overset{c}{<} G^{(\alpha)}$;* (ii) *$F \overset{r}{<} G$ implies $F^{(\alpha)} \overset{*}{<} G^{(\alpha)}$.*

LEMMA 9. *If $F \overset{r}{<} G$, then $E|X_{i:n}|/E|Y_{i:n}|$ is nondecreasing in i for $i \leqslant (n+1)/2$ and nonincreasing in i for $i \geqslant (n+1)/2$.*

The proof of Lemma 9 is quite similar to the proof of Theorem 4.6 in [9].

After simple modifications of Lemma 7 we obtain the following result:

LEMMA 10. *Let G be symmetric about the origin and have a density g on $[-c_G, c_G]$, $c_G \leqslant \infty$, which is positive and continuously differentiable on $(-c_G, c_G)$. Then the class of continuous and symmetric about the origin distributions $\{F\colon G \overset{r}{<} F\}$ is closed under mixtures if and only if $ug'(u)/g(u)$ is decreasing on $(0, c_G)$ (and hence increasing on $(-c_G, 0)$).*

An interesting fact noticed by Oja [12] is that all above-mentioned orderings may be characterized by the convexity properties of the shift function Δ. It is obvious that if X has the distribution F, then $X+\Delta(X)$ is distributed according to G.

2. ROBUST ESTIMATION OF THE SCALE PARAMETER FOR DISTRIBUTIONS ON R_+

2.1. Estimates based on order statistics. Let $G_\lambda(\cdot) = G(\cdot/\lambda)$, $\lambda > 0$, be a specified continuous distribution with the scale parameter λ, having the support $S_{G_\lambda} = [a_\lambda, b_\lambda]$, $0 \leqslant a_\lambda < b_\lambda \leqslant \infty$, and $G(0) = 0$. We are interested in an unbiased estimation of λ based on a sample of size n. The appropriate statistical model is

$$M_0 = (R_+, \mathscr{B}_+, \{G_\lambda, \lambda > 0\})^n.$$

Suppose that the original model M_0 is violated in such a way that in fact the underlying random variables are distributed according to $F_\lambda(\cdot) = F(\cdot/\lambda)$ from a set of distributions $\Pi_{H,K}(\lambda)$ satisfying the following conditions:

(i) $H_\lambda \overset{\mathrm{st}}{\leqslant} F_\lambda \overset{\mathrm{st}}{\leqslant} K_\lambda$ for every $F_\lambda \in \Pi_{H,K}(\lambda)$, where H_λ and K_λ are fixed continuous distributions with the scale parameter λ and $H(0) = 0 = K(0)$;

(ii) $H_\lambda \in \Pi_{H,K}(\lambda)$, $K_\lambda \in \Pi_{H,K}(\lambda)$;

(iii) $\Pi_{H,K}(\lambda') \cap \{G_\lambda, \lambda > 0\} = \{G_{\lambda'}\}$ for every $\lambda' > 0$.

The set $\Pi_{H,K}(\lambda)$ will be called a *violation of* M_0 (see [17] and [18]) *generated by stochastic ordering.*

Let T be an unbiased estimate of λ in the model M_0. Let F_λ run through the set $\Pi_{H,K}(\lambda)$ and let $E_{F_\lambda} T$ denote the expected value of T if the underlying distribution is F_λ ($E_F T$ if $\lambda = 1$). Then

$$b_T(\lambda) = \sup_{F_\lambda \in \Pi_{H,K}(\lambda)} (E_{F_\lambda} T - \lambda) - \inf_{F_\lambda \in \Pi_{H,K}(\lambda)} (E_{F_\lambda} T - \lambda)$$

is the oscillation of the bias of T over $\Pi_{H,K}(\lambda)$ and gives us a measure of robustness of the estimate T with respect to its bias under the violation $\Pi_{H,K}$. The function $\lambda \to b_T(\lambda)$, $\lambda > 0$, is called the *bias-robustness* of T (see [17] and [18]).

The problem is to find T_0 such that

$$b_{T_0}(\lambda) \leqslant b_T(\lambda) \quad \text{for every } \lambda > 0 \tag{1}$$

and every T in a given class of statistics. The estimate T_0 for which (1) holds, is called the *uniformly most bias-robust estimate* (UMBRE) of λ in the given class of statistics.

For our problem of estimation consider the class of statistics

$$\mathscr{T}^+ = \{T(\alpha) = \sum_{i=1}^{n} \alpha_i X_{i:n},\ \alpha = (\alpha_1, \ldots, \alpha_n) \in R^n_+,\ E_{G_\lambda} T(\alpha) = \lambda,\ \lambda > 0\},$$

i.e. the class of all nonnegative linear combinations of order statistics which are unbiased estimates of λ in M_0. Notice that if $T \in \mathscr{T}^+$, then $b_T(\lambda) = \lambda b_T(1)$ and the problem of finding the UMBRE of λ in $\mathscr{T}^+$ reduces to that of finding T_0 which minimizes $b_T(1)$ in $\mathscr{T}^+$. We can state the following theorem:

THEOREM 1. *Under the violation $\Pi_{H,K}$ of the model M_0:*

(i) *if $H \overset{*}{<} G \overset{*}{<} K$, then $X_{1:n}/E_G X_{1:n}$ is the UMBRE of λ in the class $\mathscr{T}^+$;*

(ii) *if $K \overset{*}{<} G \overset{*}{<} H$, then $X_{n:n}/E_G X_{n:n}$ is the UMBRE of λ in the class $\mathscr{T}^+$.*

Proof. The idea of the proof is the same as in [19] (see also [4]). From the properties of the stochastic ordering it follows that if $F \in \Pi_{H,K}$, then $E_H X_{i:n} \leqslant E_F X_{i:n} \leqslant E_K X_{i:n}$, $i = 1, 2, \ldots, n$. Hence for every $T \in \mathcal{T}^+$ we have

$$\sup_{F \in \Pi_{H,K}} E_F T(\alpha) = \sum_{i=1}^{n} \alpha_i E_K X_{i:n} \quad \text{and} \quad \inf_{F \in \Pi_{H,K}} E_F T(\alpha) = \sum_{i=1}^{n} \alpha_i E_H X_{i:n}.$$

The problem of finding T_0 reduces to that of finding $\alpha_i \geqslant 0$, $i = 1, 2, \ldots, n$, which minimize $\sum_{i=1}^{n} \alpha_i (E_K X_{i:n} - E_H X_{i:n})$ under the condition of T_0 being unbiased in M_0, i.e.

$$\sum_{i=1}^{n} \alpha_i E_G X_{i:n} = 1.$$

This is a simple linear programming problem with a single constraint, the solution is therefore $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ with exactly one nonzero coordinate. Hence $T_0 = X_{i^*:n}/E_G X_{i^*:n}$, where i^* minimizes $(E_K X_{i:n} - E_H X_{i:n})/E_G X_{i:n}$. From Lemma 4 it follows that if $H \overset{*}{<} G \overset{*}{<} K$, then $i^* = 1$, and if $K \overset{*}{<} G \overset{*}{<} H$, then $i^* = n$.

Example 1. *Exponential model.* Let $G_\lambda(x) = 1 - e^{-x/\lambda}$, $x \geqslant 0$, $\lambda > 0$. The relation $H \overset{*}{<} G \overset{*}{<} K$ is equivalent to that H is an IFRA distribution and K is a DFRA distribution. Some families of distributions regarded as violations of the exponential model have been considered by Zieliński [19], Bartoszewicz [2] and Bartoszewicz and Zieliński [4]. Among them two parametric families have some particular interest:

(2) $$\Pi(\lambda; p_1, p_2) = \{G_{\lambda,p}: p_1 \leqslant p \leqslant p_2\},$$

where $G_{\lambda,p}$ is the exponential power distribution with the density function

$$g_{\lambda,p}(x) = \exp(-x/\lambda)^p/[\lambda^p \Gamma(1+1/p)], \quad x \geqslant 0,\ 0 < p_1 \leqslant 1 \leqslant p_2 \leqslant 2.16,$$

and

(3) $$\Pi^*(\lambda; p_1, p_2) = \{G^*_{\lambda,p}: p_1 \leqslant p \leqslant p_2\},$$

where $G^*_{\lambda,p}$ is the gamma distribution with the density function

$$g^*_{\lambda,p}(x) = x^{p-1} \exp(-x/\lambda)/[\lambda^p \Gamma(p)], \quad x \geqslant 0,\ 0 < p_1 \leqslant 1 \leqslant p_2 < \infty.$$

Under the violation (2) the UMBRE of λ in $\mathcal{T}^+$ is $nX_{1:n}$ and under the violation (3) the UMBRE of λ in $\mathcal{T}^+$ is $X_{n:n}/(1+1/2+\ldots+1/n)$.

Example 2. *Uniform model.* Let $G_\lambda(x) = x/\lambda$, $0 \leqslant x \leqslant \lambda$. If a distribution H has an increasing density, then $H \overset{c}{<} G$ and hence $H \overset{*}{<} G$; similarly, if K has a decreasing density, then $G \overset{c}{<} K$ and hence $G \overset{*}{<} K$. As a violation of the uniform model generated by stochastic ordering consider the parametric family of distributions

(4) $$\{\beta(p, 1): p_1 \leqslant p \leqslant p_2\},$$

where $\beta(p, 1)$ is the beta distribution with the density px^{p-1}, $0 \leqslant x \leqslant 1$, and $0 < p_1 \leqslant 1 \leqslant p_2 < \infty$. Evidently, $\beta_\lambda(1, 1) = G_\lambda$. It is easy to see that for each $\lambda > 0$ the family of distributions (4) is stochastically increasing in p, i.e. $\beta_\lambda(p, 1) \overset{st}{\leqslant} \beta_\lambda(p', 1)$ for $p < p'$ and also $\beta(p_2, 1) \overset{c}{<} \beta(1,1) \overset{c}{<} \beta(p_1, 1)$. Hence $(n+1)X_{n:n}/n$ is the UMBRE of λ in the class $\mathscr{T}^+$ under violation (4). It is well known that $(n+1)X_{n:n}/n$ is the uniformly minimum variance unbiased estimate (UMVUE) of λ in the original uniform model.

Similarly one can easily obtain that this estimate is also the UMBRE of λ in $\mathscr{T}^+$ under the violation $\{\beta_\lambda(1, q)\colon q_1 \leqslant q \leqslant q_2\}$, where $\beta(1, q)$ is the beta distribution with the density $q(1-x)^{q-1}$, $0 \leqslant x \leqslant 1$, and $0 < q_1 \leqslant 1 \leqslant q_2 < \infty$.

Define now the following set of distributions:

$$\{F_{\lambda,\varepsilon}\colon -\xi \leqslant \varepsilon \leqslant \eta\}, \tag{5}$$

where $F_{\lambda,\varepsilon}$ has the density

$$f^*_{\lambda,\varepsilon}(x) = \begin{cases} [2\varepsilon(1+\varepsilon)^{-2}x/\lambda + (1-\varepsilon)/(1+\varepsilon)]/\lambda & \text{if } 0 \leqslant x \leqslant (1+\varepsilon)\lambda, \\ 0 & \text{otherwise,} \end{cases}$$

and ξ, η are fixed numbers from (0, 1). Of course, $F_{\lambda,0} = G_\lambda$. It is not difficult to show that if $-\xi \leqslant \varepsilon < \varepsilon' \leqslant \eta$, then $F_{\lambda,\varepsilon} \overset{st}{\leqslant} F_{\lambda,\varepsilon'}$ and also $F_{\lambda,\varepsilon} \overset{c}{<} F_{\lambda,\varepsilon'}$ for each $\lambda > 0$. Thus the set (5) is a violation generated by stochastic ordering with $H_\lambda = F_{\lambda,-\xi}$ and $K_\lambda = F_{\lambda,\eta}$ and Theorem 1 implies that $(n+1)X_{1:n}$ is the UMBRE of λ in $\mathscr{T}^+$.

Consider another violation of the uniform model

$$\{F^*_{\lambda,\varepsilon}\colon -\xi \leqslant \varepsilon \leqslant \eta\}, \tag{6}$$

where $F^*_{\lambda,\varepsilon}$ has the density

$$f^*_{\lambda,\varepsilon}(x) = \begin{cases} [2\varepsilon(1+\varepsilon)^{-2}x/\lambda + (1-\varepsilon)/(1+\varepsilon)]/\lambda & \text{if } 0 \leqslant x \leqslant (1+\varepsilon)\lambda, \\ 0 & \text{otherwise,} \end{cases}$$

and ξ, η are fixed numbers from (0, 1). If $-\xi \leqslant \varepsilon < \varepsilon' \leqslant \eta$, then $F^*_{\lambda,\varepsilon} \overset{st}{\leqslant} F^*_{\lambda,\varepsilon'}$ and $F^*_{\lambda,\varepsilon'} \overset{c}{<} F_{\lambda,\varepsilon}$. Hence it follows from Theorem 1 that $(n+1)X_{n:n}/n$ is the UMBRE of λ in $\mathscr{T}^+$ under the violation (6).

Example 3. *Pareto model.* Let $G_\lambda(x) = 1-(x/\lambda)^{-r}$, $0 < \lambda \leqslant x < \infty$, where $r > 1$ is known. Consider the following set of distributions:

$$\{F^\varepsilon_\lambda\colon F^\varepsilon_\lambda(x) = 1-(x/\lambda)^{-r-\varepsilon},\ \lambda \leqslant x,\ -\gamma \leqslant \varepsilon \leqslant \delta\}, \tag{7}$$

where γ and δ are positive fixed numbers and $\gamma < r-1$. It is easy to verify that if $\varepsilon < \varepsilon'$, then $F^{\varepsilon'}_\lambda \overset{st}{\leqslant} F^\varepsilon_\lambda$ and also $F^{\varepsilon'}_\lambda \overset{c}{<} F^\varepsilon_\lambda$ for each $\lambda > 0$. Hence the set (7) is a violation generated by stochastic ordering with $H_\lambda = F^\delta_\lambda$ and $K_\lambda = F^\gamma_\lambda$. Theorem 1 implies that the statistic $(rn-1)X_{1:n}/rn$ is the UMBRE of λ in the class $\mathscr{T}^+$ under the violation (7). It is easily seen that this statistic is the UMVUE of λ in the original Pareto model.

Example 4. *Contaminated model.* This example has a more general character than previous ones. Let G_λ be a distribution on $[0, c_\lambda]$, $c_\lambda \leqslant \infty$, with continuously differentiable density g_λ such that $ug'(u)/g(u)$ is decreasing on $(0, c)$. Let L_λ be a distribution with the scale parameter λ such that $L_\lambda \overset{\mathrm{st}}{\leqslant} G_\lambda$ for each $\lambda > 0$ and also $G \overset{*}{<} L$. Consider the following set of distributions

$$\Pi(\lambda; \varepsilon_0) = \{G_{\lambda,\varepsilon} = (1-\varepsilon) G_\lambda + \varepsilon L_\lambda,\ 0 \leqslant \varepsilon \leqslant \varepsilon_0\},$$

where $\varepsilon_0 \in (0, 1)$ is fixed. It is easy to verify that $\Pi(\lambda; \varepsilon_0)$ is a stochastically decreasing family of distributions, i.e. $G_\lambda^{\varepsilon'} \overset{\mathrm{st}}{\leqslant} G_\lambda^{\varepsilon}$ for $0 \leqslant \varepsilon < \varepsilon' \leqslant 1$ and every $\lambda > 0$ and hence $\Pi(\lambda; \varepsilon_0)$ is a violation generated by stochastic ordering with $H_\lambda = (1-\varepsilon_0) G_\lambda + \varepsilon_0 L_\lambda$ and $K_\lambda = G_\lambda$. From Lemma 7 it follows that $G \overset{*}{<} (1-\varepsilon_0) G + \varepsilon_0 L$. Hence Theorem 1 implies that $X_{n:n}/E_G X_{n:n}$ is the UMBRE of λ in the class $\mathscr{T}^+$ under the violation $\Pi(\lambda; \varepsilon_0)$ of the original model. The particular case of this example, when $G_\lambda(x) = 1 - e^{-x/\lambda}$, $x \geqslant 0$, and L_λ is the gamma distribution with a fixed shape parameter $p < 1$, has been considered by Bartoszewicz [2].

2.2. Robust estimates based on spacings. Consider again the model M_0. Now suppose that the model M_0 is violated in such a way that the underlying random variables have an unknown distribution $F_\lambda(\cdot) = F(\cdot/\lambda)$ from a set of distributions $\tilde{\Pi}_{H,K}(\lambda)$ satisfying the following conditions:

(i) $H_\lambda \overset{\mathrm{disp}}{<} F_\lambda \overset{\mathrm{disp}}{<} K_\lambda$ for every $F_\lambda \in \tilde{\Pi}_{H,K}(\lambda)$ where H_λ and K_λ are fixed continuous distributions with the scale parameter λ and $H(0) = 0 = K(0)$;

(ii) $H_\lambda \in \tilde{\Pi}_{H,K}(\lambda)$, $K_\lambda \in \tilde{\Pi}_{H,K}(\lambda)$;

(iii) $\tilde{\Pi}_{H,K}(\lambda') \cap \{G_\lambda, \lambda > 0\} = \{G_{\lambda'}\}$ for every $\lambda' > 0$.

The set $\tilde{\Pi}_{H,K}(\lambda)$ will be called a *violation of M_0 generated by dispersive ordering*. If $S_F = [0, a_F]$, $S_H = [0, a_H]$, $S_K = [0, a_K]$ and $a_H \leqslant a_F \leqslant a_K \leqslant \infty$ for each $F \in \tilde{\Pi}_{H,K}$, then from Lemma 1 it follows that $\tilde{\Pi}_{H,K}(\lambda)$ is also a violation generated by stochastic ordering.

Consider the class of statistics

$$\mathscr{S}^+ = \{S(\alpha) = \sum_{i=1}^{n} \alpha_i V_{i:n},\ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{R}_+^n,\ E_{G_\lambda} S(\alpha) = \lambda,\ \lambda > 0\},$$

where $V_{i:n} = X_{i:n} - X_{i-1:n}$, $i = 1, 2, \ldots, n$, $X_{0:n} = \inf\{x\colon F_\lambda(x) > 0\}$, are spacings. Thus $\mathscr{S}^+$ is the class of all nonnegative linear combinations of spacings which are unbiased estimates of λ in the model M_0. It is easy to see that $\mathscr{T}^+ \subset \mathscr{S}^+$. Notice that if $S \in \mathscr{S}^+$, then $b_S(\lambda) = \lambda b_S(1)$, where $b_S(\lambda)$ is the bias-robustness of S defined similarly as previously. Thus the problem of finding the UMBRE of λ in the class $\mathscr{S}^+$ reduces to minimizing $b_S(1)$ in $\mathscr{S}^+$.

Similar to Theorem 1, from Lemmas 2 and 5 we have the following theorem:

THEOREM 2. *Under the violation $\bar{\Pi}_{H,K}$ of the model M_0:*
(i) *if $H \overset{c}{<} G \overset{c}{<} K$, then $V_{1:n}/E_G V_{1:n}$ is the UMBRE of λ in the class $\mathscr{S}^+$;*
(ii) *if $K \overset{c}{<} G \overset{c}{<} H$, then $V_{n:n}/E_G V_{n:n}$ is the UMBRE of λ in the class $\mathscr{S}^+$.*

Example 5. *Exponential model.* Let $G_\lambda(x) = 1-e^{-x/\lambda}$, $x \geqslant 0$, $\lambda > 0$. The relation $H \overset{c}{<} G \overset{c}{<} K$ means that H is an IFR distribution and K is a DFR distribution. Violations (2) and (3) satisfy the assumptions of Theorem 2 (see [3]) and hence, in the class $\mathscr{S}^+$ under the violation (2), $nV_{1:n}$ is the UMBRE of λ and, under the violation (3), $V_{n:n}$ is the UMBRE of λ.

Example 6. *Uniform model.* Let G_λ be the uniform distribution on $[0, \lambda]$. Consider the violation (5) of the uniform model. Since for $-\xi \leqslant \varepsilon < \varepsilon' \leqslant \eta$ we have $F_{\lambda,\varepsilon} \overset{st}{\leqslant} F_{\lambda,\varepsilon'}$ and $F_{\lambda,\varepsilon} \overset{c}{<} F_{\lambda,\varepsilon'}$, from Lemma 6 it follows that $F_{\lambda,\varepsilon} \overset{disp}{<} F_{\lambda,\varepsilon'}$. This can be also proved using Lemma 3. Thus the set (5) is a violation generated by dispersive ordering with $H_\lambda = F_{\lambda,-\xi}$ and $K_\lambda = F_{\lambda,\eta}$. Theorem 2 implies that $(n+1)V_{1:n}$ is the UMBRE of λ in $\mathscr{S}^+$.

Consider now the violation (6) of the uniform model. It easily follows from Lemma 3 that $F_{\lambda,\varepsilon} \overset{disp}{<} F_{\lambda,\varepsilon'}$ for $-\xi \leqslant \varepsilon < \varepsilon' \leqslant \eta$. Since $F_{\lambda,\eta} \overset{c}{<} G_\lambda \overset{c}{<} F_{\lambda,-\xi}$, from Theorem 2 we obtain that $(n+1)V_{n:n}$ is the UMBRE of λ in $\mathscr{S}^+$.

3. ROBUST ESTIMATES OF THE SCALE PARAMETER FOR SYMMETRIC DISTRIBUTIONS

Consider the statistical model $M_0 = (\boldsymbol{R}, \mathscr{B}, \{G_\lambda, \lambda > 0\})^n$, where $G_\lambda(\cdot) = G(\cdot/\lambda)$ is a continuous and symmetric about the origin distribution on the support $S_G = [-c_\lambda, c_\lambda]$, $c_\lambda \leqslant \infty$. As a violation of M_0 consider a set $\bar{\Pi}_{H,K}(\lambda)$ of symmetric about the origin distributions with the scale parameter λ satisfying the following conditions:

(i) $H_\lambda(x) \geqslant F_\lambda(x) \geqslant K_\lambda(x)$ for every $F_\lambda \in \bar{\Pi}_{H,K}(\lambda)$ and each $x \geqslant 0$ and $\lambda > 0$ where H_λ and K_λ are fixed continuous symmetric about the origin distributions with the scale parameter λ;

(ii) $H_\lambda \in \Pi_{H,K}(\lambda)$, $K_\lambda \in \Pi_{H,K}(\lambda)$;

(iii) $\Pi_{H,K}(\lambda') \cap \{G_\lambda, \lambda > 0\} = \{G_{\lambda'}\}$ for every $\lambda' > 0$.

For the family of distributions $\{G_\lambda, \lambda > 0\}$ the vector $(|X_{1:n}|, |X_{2:n}|, \ldots, |X_{n:n}|)$ is a sufficient statistic (see [9]). Define the class $\mathscr{W}^+$ of linear estimates of λ, unbiased in the original model $\bar{M}_0$ and based on the sufficient statistic, such that

$$\mathscr{W}^+ = \{W(\alpha) = \sum_{i=1}^{n} \alpha_i |X_{i:n}|, (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \boldsymbol{R}^n_+, E_{G_\lambda} W(\alpha) = \lambda, \lambda > 0\}.$$

The problem consists of finding the UMBRE of λ in the class $\mathscr{W}^+$

under the violation $\bar{\Pi}_{H,K}$. Similarly as previously, if $W \in \mathscr{W}^+$, then $b_W(\lambda) = \lambda b_W(1)$ and it suffices to find a $W_0 \in \mathscr{W}^+$ which minimizes $b_W(1)$.

Similar to Theorems 1 and 2, from Lemma 9 we have the following theorem:

THEOREM 3. *Under the violation $\bar{\Pi}_{H,K}$ of the model $\bar{M}_0$:*

(i) *if $H \overset{r}{<} G \overset{r}{<} K$, then for arbitrary $\gamma \in [0, 1]$*

$$W_0 = \begin{cases} \dfrac{|X_{(n+1)/2:n}|}{E_G |X_{(n+1)/2:n}|}, & \text{if } n \text{ is odd,} \\ \gamma \dfrac{|X_{n/2:n}|}{E_G |X_{n/2:n}|} + (1-\gamma) \dfrac{|X_{n/2+1:n}|}{E_G |X_{n/2+1:n}|}, & \text{if } n \text{ is even,} \end{cases} \tag{8}$$

is the UMBRE of λ in the class $\mathscr{W}^+$;

(ii) *if $K \overset{r}{<} G \overset{r}{<} H$, then for arbitrary $\gamma \in [0, 1]$*

$$W_0 = \gamma \frac{|X_{1:n}|}{E_G |X_{1:n}|} + (1-\gamma) \frac{|X_{n:n}|}{E_G |X_{n:n}|}$$

is the UMBRE of λ in the class $\mathscr{W}^+$.

It is easy to notice that the vector $(|X|_{1:n}, |X|_{2:n}, \ldots, |X|_{n:n})$ is also a sufficient statistic for the family $\{G_\lambda, \lambda > 0\}$. Consider the following class of linear estimates of λ, unbiased in the original model $\bar{M}_0$:

$$\mathscr{V}^+ = \{V(\alpha) = \sum_{i=1}^{n} \alpha_i |X|_{i:n}, (\alpha_1, \ldots, \alpha_n) \in R_+^n, E_{G_\lambda} V(\alpha) = \lambda, \lambda > 0\}.$$

Similarly, using Lemmas 4 and 8, one can prove the following result:

THEOREM 4. *Under the violation $\bar{\Pi}_{H,K}$ of the model $\bar{M}_0$:*

(i) *if $H \overset{r}{<} G \overset{r}{<} K$, then $|X|_{1:n}/E_G |X|_{1:n}$ is the UMBRE of λ in the class $\mathscr{V}^+$;*

(ii) *if $K \overset{r}{<} G \overset{r}{<} H$, then $|X|_{n:n}/E_G |X|_{n:n}$ is the UMBRE of λ in the class $\mathscr{V}^+$.*

Consider now the model $\bar{M}_0$ and its violation $\hat{\Pi}_{H,K}(\lambda)$ generated by dispersive ordering, i.e. $\hat{\Pi}_{H,K}(\lambda)$ satisfies conditions (i) and (ii) from the definition of $\bar{\Pi}_{H,K}(\lambda)$ and also it holds $H_\lambda \overset{\text{disp}}{<} F_\lambda \overset{\text{disp}}{<} K_\lambda$ for every $F_\lambda \in \hat{\Pi}_{H,K}(\lambda)$. It is easy to see that if $S_H \subset S_F \subset S_K$ for all $F \in \hat{\Pi}_{H,K}$, then $\hat{\Pi}_{H,K}$ is also of the type $\bar{\Pi}_{H,K}$.

Define the class $\mathscr{Z}^+$ of linear estimates of λ, unbiased in the original model $\bar{M}_0$, such that

$$\mathscr{Z}^+ = \{Z(\alpha) = \sum_{i=1}^{n} \alpha_i (|X|_{i:n} - |X|_{i-1:n}), (\alpha_1, \ldots, \alpha_n) \in R_+^n, E_{G_\lambda} Z(\alpha) = \lambda, \lambda > 0\},$$

where $|X|_{0:n} = 0$. Notice that $\mathscr{V}^+ \subset \mathscr{Z}^+$. Since for symmetric about the origin distribution $X \overset{\text{disp}}{<} Y$ implies $|X| \overset{\text{disp}}{<} |Y|$, using Lemmas 2, 5 and 8 one can easily prove the following result:

THEOREM 5. *Under the violation $\bar{\Pi}_{H,K}$ of the model $\bar{M}_0$:*

(i) *if $H \overset{s}{<} G \overset{s}{<} K$, then $|X|_{1:n}/E_G|X|_{1:n}$ is the UMBRE of λ in the class $\mathscr{X}^+$;*

(ii) *if $K \overset{s}{<} G \overset{s}{<} H$, then $(|X|_{n:n}-|X|_{n-1:n})/E_G(|X|_{n:n}-|X|_{n-1:n})$ is the UMBRE of λ in the class $\mathscr{X}^+$.*

Example 7. *Contaminated normal model.* Let $G_\lambda(\cdot) = \Phi(\cdot/\lambda)$, where Φ is the distribution function of the normal distribution $N(0, 1)$, and suppose that in fact the underlying random variables have a distribution F_λ^ε from the set

$$\bar{\Pi}(\lambda;\varepsilon_0) = \{F_\lambda^\varepsilon(\cdot) = (1-\varepsilon)\Phi(\cdot/\lambda)+\varepsilon\Phi(\cdot/3\lambda),\ 0 \leqslant \varepsilon \leqslant \varepsilon_0\}, \quad \lambda > 0,$$

where $\varepsilon_0 < 1$ is fixed.

It is easy to notice that the violation $\bar{\Pi}(\lambda;\varepsilon_0)$ is of the type $\bar{\Pi}_{H,K}(\lambda)$ with $H_\lambda = \Phi(\cdot/\lambda)$ and $K_\lambda = (1-\varepsilon_0)\Phi(\cdot/\lambda)+\varepsilon_0\Phi(\cdot/3\lambda)$. Since for the standard normal density φ we have $u\varphi'(u)/\varphi(u) = -u^2$ and $\Phi(\cdot) \overset{r}{<} \Phi(\cdot/3)$, from Lemma 10 it follows that

$$\Phi(\cdot) \overset{r}{<} (1-\varepsilon_0)\Phi(\cdot)+\varepsilon_0\Phi(\cdot/3).$$

Now from Theorem 3 we obtain that the statistic W^*, being of the form (8) with $G = \Phi$, is the UMBRE of λ in the class $\mathscr{W}^+$ under the violation $\bar{\Pi}(\lambda;\varepsilon_0)$ of the normal model.

From Theorem 4 it follows that the statistics $V^* = |X|_{1:n}/E_\Phi|X|_{1:n}$ is the UMBRE of λ in the class $\mathscr{V}^+$ under the violation $\bar{\Pi}(\lambda;\varepsilon_0)$ of the normal model.

Consider two classical unbiased estimators of the standard deviation λ in the original normal model:

$$S_n = \Gamma(n/2)\left(\frac{1}{2}\sum_{i=1}^{n} X_i^2\right)^{1/2} / \Gamma(n/2+1/2),$$

i.e. the UMVUE of λ, and

$$S_n^* = (2/n)\sum_{i=1}^{n} |X_i|/(2/\pi)^{1/2}.$$

For the considered contaminated normal model Tukey [16] has studied the asymptotic relative efficiency of these estimators. Using our definition of bias-robustness one can easily check that under the violation $\bar{\Pi}(\lambda;\varepsilon_0)$ the bias-robustness of these estimates are the same, namely $b_{S_n}(\lambda) = b_{S_n^*}(\lambda) = 2\lambda\varepsilon_0$. However, notice that $S_n^* \in \mathscr{W}^+$ and also $S_n^* \in \mathscr{V}^+$, so the estimates W^* and V^* are more bias-robust than S_n and S_n^* under the violation $\bar{\Pi}(\lambda;\varepsilon_0)$.

REFERENCES

[1] R. E. Barlow and F. Proschan, *Inequalities for linear combinations of order statistics from restricted families*, Ann. Math. Statist. 37 (1966), p. 1574-1592.

[2] J. Bartoszewicz, *On the most bias-robust linear estimates of the scale parameter of the exponential distribution*, Zastos. Mat. 18 (1984), p. 251-255.

[3] — *Bias-robust estimates based on order statistics and spacings in the exponential model*, ibidem (to appear).

[4] — and R. Zieliński, *A bias-robust estimate of the scale parameter of the exponential distribution under violation of the hazard function*, ibidem 18 (1985), p. 609-612.

[5] P. J. Bickel and E. L. Lehmann, *Descriptive statistics for nonparametric models IV. Spread*, p. 33-40 in: *Contributions to Statistics*, Hájek Memorial Volume, Academia, Prague 1979.

[6] J. V. Deshpande and S. C. Kochar, *Dispersive ordering is the same as tail-ordering*, Adv. Appl. Prob. 15 (1983), p. 686-687.

[7] K. Doksum, *Starshaped transformations and the power of rank tests*, Ann. Math. Statist. 40 (1969), p. 1167-1176.

[8] — *Empirical probability plots and statistical inference for nonlinear models in the two-sample case*, Ann. Statist. 2 (1974), p. 267-277.

[9] M. J. Lawrance, *Inequalities of s-ordered distributions*, ibidem 3 (1975), p. 413-428.

[10] R. V. León and J. Lynch, *Preservation of life distribution classes under reliability operations*, Math. Oper. Res. 8 (1983), p. 159-169.

[11] T. Lewis and J. W. Thompson, *Dispersive distributions and the connection between dispersivity and strong unimodality*, J. Appl. Prob. 18 (1981), p. 76-90.

[12] H. Oja, *On location, scale, skewness and kurtosis of univariate distributions*, Scand. J. Statist. 8 (1981), p. 154-168.

[13] Y. S. Sathe, *A comment on star-ordering and tail-ordering*, Adv. Appl. Prob. 16 (1984), p. 692.

[14] I. W. Saunders and P. A. P. Moran, *On the quantiles of the gamma and F distributions*, J. Appl. Prob. 15 (1978), p. 426-432.

[15] M. Shaked, *Dispersive ordering of distributions*, ibidem 19 (1982), p. 310-320.

[16] J. W. Tukey, *A survey of sampling from contaminated distributions*, p. 448-485 in: *Contributions to Probability and Statistics*, Essays in Honor of Harold Hotelling, Stanford University Press, Stanford 1960.

[17] R. Zieliński, *Robustness: a quantitative approach*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys. 25 (1977), p. 1281-1286.

[18] — *Robust statistical procedures: a general approach*, *Stability Problems of Stochastic Models*, Springer Verlag Lecture Notes on Mathematics 982 (1983), p. 283-295.

[19] — *A most bias-robust linear estimate of the scale parameter of the exponential distribution*, Zastos. Mat. 18 (1983), p. 73-77.

[20] W. R. van Zwet, *Convex Transformations of Random Variables*, Mathematisch Centrum, Amsterdam 1964.

[21] — *An inequality for expected values of sample quantiles*, Ann. Math. Statist. 38 (1967), p. 1817-1821.

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