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LAW OF THE ITERATED LOGARITHM FOR WIENER PROCESSES WITH VALUES IN ORLICZ SPACES

BY

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Abstract. Wiener processes with values in separable Orlicz spaces are investigated. There is constructed an analogue of the abstract Wiener space of a Wiener measure on the space of all continuous functions defined on [0, 1] with values in Orlicz spaces. Moreover, the law of the iterated logarithm is proved for Wiener processes with values in p-homogeneous Orlicz spaces.

1. Consider independent Brownian motions $\{B^{(i)}(t): 0 \leqslant t < \infty\}$, $1 \leqslant i \leqslant k$. Let $B(t) = (B^{(1)}(t), \ldots, B^{(k)}(t))$, $\zeta_n(t) = (2n \log\log n)^{-1/2} B(nt)$, $n \geqslant 3$. Strassen [11] proved that the sequence $\{\zeta_n: n \geqslant 3\}$ is relatively compact subset of $C^{(k)}$ with probability 1, where $C^{(k)}$ is the space of all continuous functions defined for $0 \leqslant t \leqslant 1$ with values in R^k vanishing at 0 with the uniform topology. The set of its limit points coincides with the unit ball in the reproducing kernel of k-dimensional Brownian motion.

Kuelbs and Le Page [9] generalized this result for Wiener processes $\{W(t): 0 \leqslant t < \infty\}$ with values in separable Banach spaces E. They considered the net $\zeta_s(\cdot) = (2s \log\log s)^{-1/2} W(s\cdot)$ for $s > e$. In this case the reproducing kernel of Wiener processes is a tensor product $H \otimes H_0$, where H_0 is the kernel of a real Brownian motion and H is the kernel of the Gaussian distribution of $W(1)$ in E. It is worth pointing out that techniques applied there used essentially various properties of Banach spaces as, for example, the existence of non-trivial dual space.

However, there are metric linear spaces, natural from the point of view of the stochastic processes, having no non-zero continuous linear functionals at all. The best known examples are the space L_0 of all measurable functions defined on [0, 1] with Lebesgue measure, endowed with the convergence in measure, and spaces L_p, $0 < p < 1$ (more generally, Orlicz spaces L_Φ).

In this paper we investigate Wiener processes with values in L_Φ spaces.

We construct an analogue of the abstract Wiener space of a Wiener measure on the space of all continuous functions defined on [0, 1] with values in L_Φ. Moreover, we prove the law of the iterated logarithm (LIL) for Wiener processes with values in p-homogeneous Orlicz spaces. As an application we obtain LIL for symmetric Gaussian measures on L_Φ.

2. Let $(T, \mathscr{F}, m)$ be a positive σ-finite separable measure space. Let Φ be a *Young function*, i.e. a continuous non-decreasing function defined for $u \geqslant 0$ and such that $\Phi(u) = 0$ if and only if $u = 0$. Assume that Φ satisfies Δ_2-condition, i.e. $\Phi(2u) \leqslant C\Phi(u)$ for some $C > 0$ and for every $u \geqslant 0$. Let $\mathscr{L}_\Phi$ be the collection of all $\mathscr{F}$-measurable functions $x\colon T \to \boldsymbol{R}$ for which

$$\int_T \Phi(|x(t)|)\,dm(t) < \infty.$$

For $x \in \mathscr{L}_\Phi$ put

$$|x|_\Phi = \inf\{u > 0\colon \int_T \Phi\left(\frac{|x(t)|}{u}\right) dm(t) \leqslant u\}.$$

By L_Φ we denote the space of all equivalence classes of functions belonging to $\mathscr{L}_\Phi$ which are equal a.e. with respect to the measure m. Then L_Φ is a vector space and $|\cdot|_\Phi$ is a (usually non-homogeneous) pseudonorm on L_Φ. $(L_\Phi, |\cdot|_\Phi)$, called *Orlicz space*, is a complete measurable metric space.

An L_Φ-valued random variable X will be called *symmetric Gaussian* (in the sense of Fernique [4]) if for every pair X_1, X_2 of independent random variables having the same distributions as X, and for every pair of real numbers a, b such that $a^2+b^2 = 1$, the random variables aX_1+bX_2 and bX_1-aX_2 are independent and have the same distribution as X.

Consider now an L_Φ-valued symmetric Gaussian random variable X. A homogeneous L_Φ-valued stochastic process $\{W(t)\colon 0 \leqslant t \leqslant 1\}$ with independent increments and with a.s. continuous sample paths is called *Wiener process generated by* X if $W(t)$ has the same distribution as $t^{1/2}X$ for $0 \leqslant t \leqslant 1$. Such a process exists by [2].

Next, let $H \subset L_\Phi$ be the reproducing kernel Hilbert space for the Gaussian measure induced on L_Φ by X [6, 10]. Let $H_0 \subset C[0, 1]$ be the reproducing kernel Hilbert space for real Brownian motion on [0, 1] (see [5]). Denote by $(\cdot,\cdot)_H$, $\|\cdot\|_H$ and $(\cdot,\cdot)_{H_0}$, $\|\cdot\|_{H_0}$ the inner products and norms in H and H_0, respectively. Let $\mathscr{C}_\Phi$ denote the space of continuous functions defined on [0, 1] with values in L_Φ, vanishing at zero, with the uniform topology. Suppose that $\{a_n\colon n \in N\}$ is an orthonormal basis in H. We then define

$$\mathscr{H} = \{f \in \mathscr{C}_\Phi\colon f(0) = 0,\ f(t) \in H \text{ for } 0 \leqslant t \leqslant 1,\ (a_n, f(\cdot))_H \in H_0 \text{ for } n \in N,$$
$$\sum_{n=1}^{\infty} \int_0^1 \left(\frac{d}{dt}(a_n, f(t))_H\right)^2 dt < \infty\},$$

and

$$(f,g)_{\mathscr{H}} = \sum_{n=1}^{\infty} \int_0^1 \frac{d}{dt}(a_n, f(t))_H \frac{d}{dt}(a_n, g(t))_H\, dt \quad \text{for } f,g\in\mathscr{H}.$$

It is easy to see that $(\mathscr{H}, (\cdot,\cdot)_{\mathscr{H}})$ is an inner product space. Denote by $\|\cdot\|_{\mathscr{H}}$ the norm in $\mathscr{H}$ induced by the inner product $(\cdot,\cdot)_{\mathscr{H}}$.

Suppose that $f, h\in H$ and $\psi\in H_0$. Let $\psi^t = \min(t, \cdot)\in H_0$ for $t\in[0, 1]$. Let $\{a_i: i\in N\}$ and $\{\psi_k: k\in N\}$ be two orthonormal bases in H and in H_0, respectively. It is easy to verify that the space $\mathscr{H}$, as in the case of Banach spaces [9], has the following properties:

(i) $h\psi\in\mathscr{H}$, $\|h\psi\|_{\mathscr{H}} = \|h\|_H \|\psi\|_{H0}$, where $(h\psi)(t) = h\cdot\psi(t)$;
(ii) $(a_i\psi_k, a_j\psi_l)_{\mathscr{H}} = \delta_{ij}\delta_{kl}$;
(iii) $(f, h\psi)_{\mathscr{H}} = \big((h, f(\cdot))_H, \psi\big)_{H_0}$;
(iv) $\|(h, f(\cdot))_H\|_{H_0} \leqslant \|f\|_{\mathscr{H}}\|h\|_H$;
(v) $(f(t), h)_H = (f, h\psi^t)_{\mathscr{H}}$;
(vi) $\|f(t)\|_H \leqslant \|f\|_{\mathscr{H}}\sqrt{t}$.

PROPOSITION 1. *The canonical embedding of $\mathscr{H}$ into $\mathscr{C}_\Phi$ is continuous.*

Proof. Let $f_n\to 0$ in $\mathscr{H}$. Suppose, to the contrary, that there exist an $\varepsilon > 0$ and an infinite sequence $(n_k)\subset(n)$ such that

$$\sup_{0\leqslant t\leqslant 1} |f_{n_k}(t)|_\Phi > \varepsilon.$$

Then for every k there exists a $t_k\in[0, 1]$ such that

(1) $$|f_{n_k}(t_k)|_\Phi > \varepsilon.$$

Hence, for every k,

$$\int_T \Phi\left(\frac{|f_{n_k}(t_k)(s)|}{\varepsilon}\right) dm(s) > \varepsilon.$$

However, by (i)-(vi) we obtain that

$$\sup_{0\leqslant t\leqslant 1} \|f_n(t)\|_H \leqslant \sup_{0\leqslant t\leqslant 1} \|f_n\|_{\mathscr{H}}\sqrt{t} = \|f_n\|_{\mathscr{H}},$$

hence

$$\sup_{0\leqslant t\leqslant 1} \|f_n(t)\|_H \to 0 \quad \text{as } n\to\infty.$$

Therefore, $\|f_{n_k}(t_k)\|_H\to 0$ as $k\to\infty$ and, obviously, $|f_{n_k}(t_k)|_\Phi\to 0$. This contradicts (1) and completes the proof.

PROPOSITION 2. *$\mathscr{H}$ is a separable Hilbert space. Moreover, if $\{a_i: i\in N\}$ and $\{\psi_k: k\in N\}$ are two orthonormal bases in H and in H_0, respectively, then $\{a_i\psi_k: i, k\in N\}$ is an orthonormal basis in $\mathscr{H}$.*

Proof. First, we prove completeness of $\mathscr{H}$. Let $\{f_n: n\in N\}$ be a Cauchy sequence in $\mathscr{H}$, i.e. $\|f_n-f_m\|_{\mathscr{H}}\to 0$ as $n, m\to\infty$. Then $\{f_n: n\in N\}$ is also a Cauchy sequence in $\mathscr{C}_\Phi$. Since $\mathscr{C}_\Phi$ is complete, there exists an $f_0\in\mathscr{C}_\Phi$ such that

(2) $$\sup_{0\leqslant t\leqslant 1} |f_n(t)-f_0(t)|_\Phi\to 0 \quad \text{as } n\to\infty.$$

On the other hand, let

$$l_{2,H_0}=\{x=(x^{(1)}, x^{(2)}, \ldots): x^{(i)}\in H_0,\, i\in N,\, \|x\|_{2,H_0}=(\sum_{i=1}^{\infty}\|x^{(i)}\|_{H_0}^2)^{1/2}<\infty\}.$$

The space H_0 is complete, hence so is the space $(l_{2,H_0}, \|\cdot\|_{2,H_0})$. Write

$$x_n^{(i)}=(a_i, f_n(\cdot))_H\in H_0, \qquad x_n=(x_n^{(i)})_{n=1}^{\infty}\in l_{2,H_0}.$$

Since $\{f_n: n\in N\}$ is a Cauchy sequence in $\mathscr{H}$, it follows that $\{x_n: n\in N\}$ is Cauchy also in l_{2,H_0}. So there exists an $x\in l_{2,H_0}$ such that $\|x_n-x\|_{2,H_0}\to 0$ as $n\to\infty$ and, obviously, $\|x_n^{(i)}-x^{(i)}\|_{H_0}\to 0$ as $n\to\infty$, for every $i\in N$. Hence, for every $i\in N$

(3) $$\sup_{0\leqslant t\leqslant 1} |(a_i, f_n(t))_H-x^{(i)}(t)|\to 0 \quad \text{as } n\to\infty.$$

Since

$$\sum_{i=1}^{\infty}|x^{(i)}(t)|^2=\sum_{i=1}^{\infty}|(x^{(i)}, \psi^t)_{H_0}|^2\leqslant t\sum_{i=1}^{\infty}\|x^{(i)}\|_{H_0}^2<\infty,$$

the function

$$f(t)=\sum_{i=1}^{\infty}x^{(i)}(t)\,a_i$$

is well defined in H for every $t\in[0, 1]$. To prove the completeness of $\mathscr{H}$, it is enough to show that $f(t)=f_0(t)$. However, $f_n(t)\to f_0(t)$ in L_Φ. By (2), (3) and by the definition of f we have $(a_i, f_n(t)-f(t))_H\to 0$ as $n\to\infty$, for every $i\in N$. Since $\{a_i: i\in N\}$ is linearly dense in H and since, by (i)-(vi), $\{f_n(t): n\in N\}$ is bounded in H, it follows that $f_n(t)\to f(t)$ as $n\to\infty$, in the weak topology of H. It is easy to see that the injection of H with the weak topology into L_Φ is continuous, so $f_n(t)\to f(t)$ in L_Φ and hence $f(t)=f_0(t)$. Let now $f\in\mathscr{H}$ be such that $(f, a_i\psi_k)_{\mathscr{H}}=0$ for all $i, k\in N$. By (i)-(vi), $((a_i, f(\cdot))_H, \psi_k)_H=0$ for $i, k\in N$, so $(a_i, f(\cdot))_{H_0}=0$ in H for $i\in N$. Then $(a_i, f(t))_H=0$ for $t\in[0, 1]$, $i\in N$. Thus $f(t)=0$ in H for $t\in[0, 1]$, i.e. $f=0$ in $\mathscr{H}$. By (i)-(vi) this completes the proof.

Proposition 3. *The unit ball $\mathscr{K}$ of $\mathscr{H}$ is compact in $\mathscr{C}_\Phi$.*

Proof. Consider a sequence $\{f_n: n\in N\}$ from $\mathscr{K}$. Since $\mathscr{K}$ is weakly compact in $\mathscr{H}$, there exist a subsequence $(n') \subset (n)$ and an $f \in \mathscr{H}$ such that $f_{n'}$ tends weakly in $\mathscr{H}$ to f as $n' \to \infty$. In particular, for every $h \in \mathscr{H}$ and $t \in [0, 1]$, we have

$$(4) \qquad (f_{n'} - f, h\psi^t)_{\mathscr{H}} \to 0 \quad \text{as } n' \to \infty.$$

By (i)-(vi), $\big((h, f_{n'}(\cdot) - f(\cdot))_H, \psi^t\big)_{H_0} \to 0$ as $n' \to \infty$, thus, for every $h \in H$, $(h, f_{n'}(t) - f(t))_H \to 0$ as $n' \to \infty$. Since the identity mapping from H with the weak topology into L_Φ is continuous, we get, for every $t \in [0, 1]$,

$$(5) \qquad |f_{n'}(t) - f(t)|_\Phi \to 0 \quad \text{as } n' \to \infty.$$

Suppose now, to the contrary, that there exist an $\varepsilon > 0$ and a subsequence $(n'') \subset (n')$ such that

$$\sup_{0 \leqslant t \leqslant 1} |f_{n''}(t) - f(t)|_\Phi > \varepsilon.$$

Then for every n'' there exists a $t_{n''} \in [0, 1]$ such that $|f_{n''}(t_{n''}) - f(t_{n''})| > \varepsilon$. The subsequence (n'') can be chosen in such a way that $t_{n''}$ tends to some $t \in [0, 1]$ as $n'' \to \infty$. Then also $\psi^{t_{n''}} \to \psi^t$ in H_0 and, by (i)-(vi), we obtain that $h\psi^{t_n} \to h\psi^t$ in $\mathscr{H}$ for every $h \in H$. Hence

$$\begin{aligned}
|(f_{n''}(t_{n''}) - f(t_{n''}), h)_H| &= |(f_{n''} - f, h\psi^{t_n})_{\mathscr{H}}| \\
&\leqslant |(f_{n''} - f, h(\psi^{t_n} - \psi^t))_{\mathscr{H}}| + |(f_{n''} - f, h\psi^t)_{\mathscr{H}}| \\
&\leqslant \|f_{n''} - f\|_{\mathscr{H}} \|h\|_H \|\psi^{t_n} - \psi^t\|_{H_0} + |(f_{n''} - f, h\psi^t)_{\mathscr{H}}| \\
&\leqslant 2\|h\|_H \|\psi^{t_{n''}} - \psi^t\|_{H_0} + |(f_{n''} - f, h\psi^t)_{\mathscr{H}}|.
\end{aligned}$$

Then, by (4), $f_{n''}(t_{n''}) - f(t_{n''}) \to 0$ in H, so $|f_{n''}(t_{n''}) - f(t_{n''})|_\Phi \to 0$ as $n'' \to \infty$. This contradicts (5) and completes the proof.

3. In the sequel we assume that in L_Φ there exists a p-homogeneous pseudonorm $|\cdot|$, $0 < p \leqslant 1$, equivalent to $|\cdot|_\Phi$. Such a condition holds, for instance, if Φ is a p-convex Young function [8], i.e. $\Phi(at + bs) \leqslant a^p \Phi(t) + b^p \Phi(s)$ for all $a, b \geqslant 0$ such that $a + b \leqslant 1$ and for all $t, s \geqslant 0$. It also holds for $\Phi(t) = \Phi_0(t^r)$, where Φ_0 is a convex Young function $(0 < r < \infty)$. In these two cases the resulting p-homogeneous seminorm is $\|\cdot\|^p$, where

$$\|x\|_\Phi = \inf\{u > 0: \int_T \Phi\left(\frac{|x(t)|}{u}\right) dm(t) \leqslant 1\}, \qquad x \in L_\Phi.$$

The considered class of Orlicz spaces contains many spaces which are neither Banach nor even locally convex spaces, for example spaces L_p $(0 < p < 1)$.

Let now $\mathscr{W}$ be the random variable with values in $\mathscr{C}_\Phi$ induced by

Wiener process $\{W(t): 0 \leqslant t \leqslant 1\}$. Take now the following expansion of $\mathscr{W}$ in $\mathscr{C}_{\Phi}$ [7]:

$$(6) \qquad \mathscr{W} = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \lambda_n^{(j)} \varphi_j a_n = \sum_{n=1}^{\infty} B_n a_n,$$

where $\{a_n: n \in N\}$, $\{\varphi_j: j \in N\}$ are two orthonormal bases in reproducing kernels $H \subset L_{\Phi}$ and $H_0 \subset C[0, 1]$, respectively, $\{\lambda_n^{(j)}: n, j \in N\}$ is an appropriate standard normal sequence, and $\{B_n: n \in N\}$ is a sequence of independent real Brownian motions. These two series converge a.s. in $\mathscr{C}_{\Phi}$.

PROPOSITION 4. *$\mathscr{H}$ is dense in the topological support of a Gaussian measure μ_W induced on $(\mathscr{C}_{\Phi}, \mathscr{B}_{\Phi})$ by Wiener process $\{W(t): 0 \leqslant t \leqslant 1\}$, where $\mathscr{B}_{\Phi}$ denotes the Borel σ-field in $\mathscr{C}_{\Phi}$.*

Proof. Let us take an orthonormal basis $\{a_n: n \in N\}$ in the reproducing kernel $H \subset L_{\Phi}$ of $W(1)$. By (6) and by standard arguments it follows that μ_W is a Gaussian measure (in the sense of Fernique) on $(\mathscr{C}_{\Phi}, \mathscr{B}_{\Phi})$. Moreover, by the fact that the distribution of the series $\sum_{n,j=1}^{\infty} \lambda_n^{(j)} a_n \varphi_j$ is equal to μ_W, it follows that $\mathscr{H}$ is dense in supp μ_w.

Remark. By Propositions 1-4 the space $\mathscr{H}$ may be thought of as a reproducing kernel for Wiener measure μ_w on $\mathscr{C}_{\Phi}$.

Let $\{\tilde{W}(t): 0 \leqslant t < \infty\}$ denote the standard extension of a Wiener process $\{W(t): 0 \leqslant t \leqslant 1\}$ over all positive numbers. Write

$$\zeta_s(t) = (2s \log \log s)^{-1/2} \tilde{W}(st).$$

We are now able to formulate the main result of our paper.

THEOREM. *The net $\{\zeta_s: s > e\}$ is relatively compact in $\mathscr{C}_{\Phi}$ with probability 1 and the set of its limit points coincides with the unit ball $\mathscr{K}$ of $\mathscr{H}$.*

We prove this theorem using similar arguments as in the proof of LIL for Wiener processes with values in Banach spaces [9]. Let $(E, \mathscr{B})$ be a measurable linear space, let X be a symmetric Gaussian random variable with values in E and let $|\cdot|$ be a measurable p-homogeneous pseudonorm in E, $0 < p \leqslant 1$. In our situation we need the following version of Fernique's estimate which is known, at present, only for p-homogeneous pseudonorm [3, 8]:

$$(7) \qquad E\exp(\beta |X|^{2/p}) < \infty \quad \text{for } \beta < \frac{\log[r/(1-r)]}{4s^{2/p}}(2^{p/2}-1)^{2/p},$$

where $P\{|X| \leqslant s\} = r > 1/2$.

Proofs of the next three propositions are modifications of those in [9].

Write

$$\mathcal{K}_\varepsilon = \{f \in \mathcal{C}_\Phi : |f - \mathcal{K}|_{\mathcal{C}_\Phi} < \varepsilon\},$$

$$l_s = (2 \log \log s)^{1/2}, \quad s > e,$$

$$|f|_{\mathcal{C}_\Phi} = \sup_{0 \leqslant t \leqslant 1} |f(t)|,$$

$$\mathcal{W}^{(M,J)} = \sum_{n=1}^{M} \sum_{j=1}^{J} \lambda_n^{(j)} \varphi_j a_n, \qquad \mathcal{W}^{(k)} = \sum_{n=1}^{k} B_n a_n.$$

PROPOSITION 5. *For every* $\varepsilon > 0$ *and* $r \sim 1$ *we have*

$$P\{l_s^{-1} \mathcal{W} \notin \mathcal{K}_\varepsilon\} \leqslant \exp(-r^2 l_s^2/2)$$

for all sufficiently large s.

Proof. Suppose that $\varepsilon > 0$, $r > 1$, $M, J \geqslant 1$, $s > e$. For $r_0 > r$

$$(8) \quad P\{l_s^{-1} \mathcal{W} \notin \mathcal{K}_\varepsilon\} \leqslant P\{l_s^{-1} r_0^{-1} \mathcal{W}^{(M,J)} \notin \mathcal{K}\} + \\ + P\{l_s^{-1} r_0^{-1} \mathcal{W}^{(M,J)} \in \mathcal{K}, |l_s^{-1} r_0^{-1} \mathcal{W}^{(M,J)} - l_s^{-1} \mathcal{W}|_{\mathcal{C}_\Phi} \geqslant \varepsilon\}.$$

The first term on the right-hand side of (8) is equal to $P\{\chi^2_{N(J+1)} > r_0^2 l_s^2\}$, where $\chi^2_{N(J+1)}$ is a χ^2 random variable with $N(J+1)$ degrees of freedom, so for all sufficiently large n and for $r < r_0$ this is less than $(1/2)\exp(-r^2 l_s^2/2)$. To estimate the second term, let $l_s^{-1} r_0^{-1} \mathcal{W}^{(M,J)}(\omega) = f(\omega) \in \mathcal{K}$ and $|f(\omega) - l_s^{-1} \mathcal{W}(\omega)|_{\mathcal{C}_\Phi} \geqslant \varepsilon$. Since $\mathcal{K}$ is compact in $\mathcal{C}_\Phi$ and the scalar multiplication is uniformly continuous on compact sets, we can find an $r_0 > 1$ independent of ω so that $|(r_0 - 1) f(\omega)|_{\mathcal{C}_\Phi} < \varepsilon/2$. Hence, and by (7), we have

$$P\{f \in \mathcal{K}, |f - l_s^{-1} \mathcal{W}|_{\mathcal{C}_\Phi} \geqslant \varepsilon\} \leqslant \exp\left(-\alpha \left(\frac{\varepsilon}{2}\right)^{2/p} l_s^2\right) E \exp(\alpha |\mathcal{W} - \mathcal{W}^{(M,J)}|_{\mathcal{C}_\Phi}^{2/p})$$

and the last integral is finite for

$$\alpha < \frac{(2^{p/2} - 1)^{2/p}}{4t^{2/p}} \log \frac{P\{|\mathcal{W} - \mathcal{W}^{(M,J)}|_{\mathcal{C}_\Phi} \leqslant t\}}{P\{|\mathcal{W} - \mathcal{W}^{(M,J)}|_{\mathcal{C}_\Phi} > t\}},$$

where $P\{|\mathcal{W} - \mathcal{W}^{(M,J)}|_{\mathcal{C}_\Phi} \leqslant t\} > 1/2$. Since, by (6),

$$\lim_{M \to \infty} \lim_{J \to \infty} P\{|\mathcal{W} - \mathcal{W}^{(M,J)}|_{\mathcal{C}_\Phi} > t\} = 0,$$

we can choose such M, J and α that $2\alpha(\varepsilon/2)^{2/p} > r^2$ and $E \exp(\alpha |\mathcal{W} - \mathcal{W}^{(M,J)}|_{\mathcal{C}_\Phi}^{2/p}) < \infty$. For these M, J and α the second term of (8) is estimated by $(1/2)\exp(-r^2 l_s^2/2)$ for sufficiently large s.

PROPOSITION 6. *For every $\varepsilon > 0$ and $r > 1$ there exists a k such that, for all sufficiently large s,*

$$P\{|l_s^{-1}(\mathscr{W}-\mathscr{W}^{(k)})|_{\mathscr{C}_\Phi} \geqslant \varepsilon\} \leqslant \exp(-r^2 l_s^2/2).$$

Proof. By (6), for $t > 0$, $P\{|\mathscr{W}-\mathscr{W}^{(k)}|_{\mathscr{C}_\Phi} > t\} \to 0$ as $k \to \infty$. Hence and by (7) we can choose a k and an α such that $2\alpha\varepsilon^{2/p} > r^2$ and

$$E\exp(\alpha|\mathscr{W}-\mathscr{W}^{(k)}|_{\mathscr{C}_\Phi}^{2/p}) < \infty.$$

Since

$$P\{|l_s^{-1}(\mathscr{W}-\mathscr{W}^{(k)})_{\mathscr{C}_\Phi} \geqslant \varepsilon\} \leqslant \exp(-\alpha\varepsilon^{2/p} l_s^2) E\exp(\alpha|\mathscr{W}-\mathscr{W}^{(k)}|_{\mathscr{C}_\Phi}^{2/p}),$$

the left-hand side of this inequality is less than $\exp(-r^2 l_s^2/2)$ for all s large enough.

PROPOSITION 7. *For every $\varepsilon > 0$ it is possible to choose a $c > 1$ sufficiently close to 1, so that, for every continuous function $\tilde{f}\colon [0, \infty) \to L_\Phi$ satisfying $\tilde{f}([c^{n+1}]\cdot)/([c^{n+1}]^{1/2} l_{[c^{n+1}]}) \in \mathscr{K}_\varepsilon$, we have $\tilde{f}(s\cdot)/(s^{1/2} l_s) \in \mathscr{K}_{2\varepsilon}$ for all sufficiently large n, when $[c^n] \leqslant s < [c^{n+1}]$.*

Proof. This result can be proved in the same way as Lemma 6 in [9]. We use here, by Proposition 2, the boundedness of $\mathscr{K}$ in $\mathscr{C}_\Phi$ and p-homogeneity of $|\cdot|_{\mathscr{C}_\Phi}$.

Our theorem now follows by Propositions 1-7 and by application of standard arguments.

COROLLARY (cf. [6]). *Let μ be a symmetrie Gaussian measure on a separable Orlicz space $(L_\Phi, \mathscr{B}_{L_\Phi})$ with topology generated by p-homogeneous pseudonorm $(0 < p \leqslant 1)$. Let $\{X_i\colon i \in N\}$ be a sequence of independent random variables with values in L_Φ with distributions μ. Then the sequence $\{\eta_n\colon n \geqslant 3\}$, defined as*

$$\eta_n = (2n\log\log n)^{-1/2} \sum_{i=1}^{n} X_i,$$

is relatively compact in L_Φ with probability 1, and the set of its limit points coincides with the unit ball K of the reproducing kernel H of μ.

Proof. Let $\{\tilde{W}(t)\colon 0 \leqslant t < \infty\}$ be a Wiener process with values in L_Φ such that the distribution of $W(1)$ is equal to μ. For $f \in \mathscr{C}_\Phi$ let $\Theta(f) = f(1)$. Observe that finite-dimensional distributions of sequences $\{\eta_n\colon n \geqslant 3\}$ and $\{\zeta_n\colon n \geqslant 3\}$ are equal. Hence, and by the continuity of Θ, the sequence $\{\eta_n\colon n \geqslant 3\}$ is relatively compact with probability 1 in L_Φ. The set of its limit points coincides with $\Theta(\mathscr{K})$.

Note that $\Theta(\mathscr{K})$ is compact in L_Φ and $\Theta(\mathscr{K}) = K$. Indeed, by virtue of (i)-(vi) for every $f \in \mathscr{K}$ we have $\|f(1)\|_{\mathscr{H}} \leqslant \|f\|_{\mathscr{H}} \leqslant 1$. On the other hand, let $f(t) = th$ for some $h \in K$ and for all $t \in [0, 1]$. Then $f \in \mathscr{H}$ and $\|f\|_{\mathscr{H}} = \|h\|_H = 1$, which completes the proof of Corollary.

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