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ON CONVERGENCE OF SOME RANDOM SERIES AND INTEGRALS

BY
NGUYEN NAM HONG (HANOI)

Abstract. The purpose of the present paper is to prove some convergent criterions for random series $\sum_{n=1}^{\infty} a(n)\xi_n$ and random integral $\int_0^{\infty} a(t)\,d\xi_t$, where $a(\cdot)\colon R_+ \to R_+$ is a strictly decreasing function and (ξ_t), $t \in R_+$, a homogeneous process with independent increments. In particular, we obtain an extension of the logarithmic criterion due to Zakusilo [3].

Let $\eta_1, \eta_2, \ldots$ be a sequence of i.i.d. nonnegative random variables (r.v.'s) with $P\{\eta_1 = 0\} < 1$ and $a(\cdot)\colon R_+ = [0, \infty] \to R_+$ be a strictly decreasing function.

In the first part of this paper we shall be concerned with the convergence of the following series:

$$\sum_{n=1}^{\infty} a(n)\eta_n, \tag{1}$$

$$\sum_{n=1}^{\infty} \mathrm{E}\{a(n)\eta_n^1\}, \tag{2}$$

$$\sum_{n=1}^{\infty} \mathrm{D}\{a(n)\eta_n^1\}, \tag{3}$$

$$\sum_{n=1}^{\infty} \mathrm{P}\{a(n)\eta_n \geqslant 1\}, \tag{4}$$

where

$$\eta_n^1 = \begin{cases} \eta_n & \text{if } \eta_n \leqslant 1/a(n), \\ 0 & \text{if } \eta_n > 1/a(n), \end{cases}$$

and E and D denote the expectation and variance, respectively.

We start the study by proving the following result:

LEMMA 1. *Series* (1) *is convergent with* P.1 *if and only if series* (4) *is convergent and* $\sum_{n=1}^{\infty} a(n) < +\infty$.

Proof. First of all, by virtue of the Three-Series Theorem ([2], Theorem 8, p. 323) it follows from the convergence of series (1) that series (2), (3), and (4) are convergent. It also follows from the convergence of series (1) that there exists a positive number c such that $P\{\eta_1 \leqslant c\} = 1$. Then $0 < \mathrm{E}\eta_1 < \infty$, which together with the convergence of series (2) implies $\sum_{n=1}^{\infty} a(n) < \infty$.

Conversely, if series (4) is convergent, then it follows from the Borel-Cantelli Lemma ([2], Lemma 4, p. 320) that there exists a $c > 0$ such that $\mathrm{P}\{\eta_1 \leqslant c\} = 1$. Hence $\mathrm{E}\eta_1 < \infty$ and $\mathrm{D}\eta_1 < \infty$, which together with $\sum_{n=1} a(n) < \infty$ implies the convergence of series (2) and (3). Now, it is sufficient to apply the Three-Series Theorem once more to get the convergence of (1). Thus the proof is complete.

Next, let $b(\cdot)$ denote the inverse function of $1/a(\cdot)$. Then we get

LEMMA 2. *Series* (4) *is convergent if and only if* $\mathrm{E}(b(\eta_1)) < \infty$.

Proof. We have

$$1 + \sum_{n=0}^{\infty} \mathrm{P}\left\{\eta_1 \geqslant \frac{1}{a(n)}\right\} = 1 + \sum_{n=0}^{\infty} \mathrm{P}\{b(\eta_1) \geqslant n\}$$

$$= \sum_{n=0}^{\infty} (1+n) P\{n+1 > b(\eta_1) \geqslant n\} \geqslant \mathrm{E}\{b(\eta_1)\}$$

and

$$\sum_{n=1}^{\infty} \mathrm{P}\left\{\eta_1 \geqslant \frac{1}{a(n)}\right\} = \sum_{n=1}^{\infty} \mathrm{P}\{b(\eta_1) \geqslant n\}$$

$$= \sum_{n=1}^{\infty} n\mathrm{P}\{n+1 > b(\eta_1) \geqslant n\} \leqslant \mathrm{E}b(\eta_1),$$

which completes the proof of Lemma 2.

As a consequence of Lemmas 1 and 2 we obtain the following theorem:

THEOREM 1. *The series* $\sum_{n=1}^{\infty} a(n)\eta_n$ *converges with* P.1 *if and only if* $\sum_{n=1}^{\infty} a(n) < \infty$ *and* $\mathrm{E}b(\eta_1) < \infty$.

Let $\xi_1, \xi_2, \ldots$ be a sequence of i.i.d. random variables with $\mathrm{P}\{\xi_1 = 0\} < 1$. Applying the above results we get

LEMMA 3. *The series* $\sum_{n=1}^{\infty} a(n)|\xi_n|$ *converges with* P.1 *if and only if* $\sum_{n=1}^{\infty} a(n) < \infty$ *and* $\sum_{n=1}^{\infty} a(n)\xi_n$ *converges with* P.1.

Proof. It is easy to show that if the series $\sum_{n=1}^{\infty} a(n)|\xi_n|$ converges with P.1, then $\sum_{n=1}^{\infty} a(n) < \infty$ and $\sum_{n=1}^{\infty} a(n)\xi_n$ converges with P.1.

Conversely, if $\sum_{n=1}^{\infty} a(n) < \infty$ and $\sum_{n=1}^{\infty} a(n)\xi_n$ converges with P.1, then, by the Borel-Cantelli Lemma, it follows that

$$\sum_{n=1}^{\infty} \mathrm{P}\{a(n)|\xi_n| \geqslant 1\} < \infty.$$

By virtue of Lemma 1, the series $\sum_{n=1}^{\infty} a(n)|\xi_n|$ converges with P.1. Thus the proof is completed.

Now, by Lemmas 1, 2, and 3 we get the following extension of the logarithmic criterion obtained by Zakusilo [3] (putting $a(n) = c^n$, $0 < c < 1$):

THEOREM 2. *If* $\sum_{n=1}^{\infty} a(n) < \infty$, *then* $\sum_{n=1}^{\infty} a(n)\xi_n$ *converges with* P.1. *if and only if* $\mathrm{E}b(|\xi_1|) < \infty$.

In the rest of this paper we consider some random integrals.

Let $\{\tau(t), t \geqslant 0\}$ be a nonnegative random process with nonnegative, independent and stationary increments, and $\mathrm{P}\{\tau(0) = 0\} = 1$, $\mathrm{P}\{\tau(1) = 0\} < 1$.

By Theorem 1 and by the fact that

(P.1) $$\sum_{n=0}^{\infty} a(n+1)\eta_{n+1} \leqslant \int_0^{\infty} a(t)\,d\tau(t) \leqslant \sum_{n=0}^{\infty} a(n)\eta_n,$$

where $\eta_0 = 0$, $\eta_n = \tau(n)-\tau(n-1)$, $n = 1, 2, \ldots$, one can prove the following

THEOREM 3. *The integral* $\int_0^{\infty} a(t)\,d\tau(t)$ *converges with* P.1 *if and only if* $\int_0^{\infty} a(t)\,dt < \infty$ *and* $\mathrm{E}b\big(\tau(1)\big) < \infty$.

Finally, let $\{\xi(t), t \geqslant 0\}$ be a random process with independent and stationary increments and $\mathrm{P}\{\xi(0) = 0\} = 1$, $\mathrm{P}\{\xi(1) = 0\} < 1$. Applying Lemma 3 and Theorem 3 we get the following result, which is an extension of that in [1] (when putting $a(t) = e^{-t^{1/\alpha}}$, $\alpha > 0$):

THEOREM 4. *If* $\int_0^{\infty} a(t)\,dt < \infty$, *then the integral* $\int_0^{\infty} a(t)\,d\xi(t)$ *converges with* P.1 *if and only if* $\mathrm{E}b\big(|\xi(1)|\big) < \infty$.

Note. All the results in this paper can be extended to the case of Banach space valued random sequences and processes.

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Institute of Mathematics
Vien Toan hoc
P. O. Box 631, Bo ho
Hanoi, Vietnam

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