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RANDOM OPERATORS IN BANACH SPACES*

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Abstract. The aim of this paper is to examine the notion of random operators from a Fréchet space into a Banach one. Characteristic function, convergence and decomposability of random operators are studied.

0. INTRODUCTION

Suppose, in decribing an experiment, that X, A and Y stand for the set of inputs, the set of actions to be performed and the set of possible outcomes, respectively. Ax denotes the outcome corresponding to the input x and the action A. There are many situations, however, in which even the exact knowledge of inputs and actions does not allow to predict the outcome exactly. Under such circumstances, instead of considering Ax as an element in Y, we shall consider Ax as a Y-valued random variable.

A correspondence that associates to each element x in X a Y-valued random variable Ax is called the *random mapping* from X into Y.

The aim of this paper is to examine the notion of random operators from a Fréchet space into a Banach space. Section 1 contains the definition, examples and some general theorems on random operators. Section 2 is devoted to the notion of the characteristic function of random operators. Theorem 2.3 gives a necessary and sufficient condition for a function to be the characteristic function of some random operator. In Section 3 we define four modes of convergence of random operators and study their relationships.

Up to now, the important problem of extendibility to Radon measures of cylindrical measures has been studied by several authors (cf. [4], [5], [6], [13] and references therein). This problem can be stated in terms of the decomposability of certain random linear functionals. In Section 4 the notion

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of decomposability is extended to random operators. Theorem 4.5 shows that there is the difference between the case of random linear functionals and that of random operators taking the values in an infinite-dimensional Banach space.

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1. DEFINITION, EXAMPLES AND SOME GENERAL THEOREMS

Throughout this paper Y denotes a Banach space with the dual Y'.

Let $(\Omega, \mathscr{F}, P)$ be a probability space. A Y-valued random variable is a measurable mapping from Ω into Y. $L_0(\Omega, Y)$ denotes the set of all Y-valued r.v.'s. $L_0(\Omega, Y)$ is a Fréchet space with the F-norm $\|\varphi\|_0 = E\|\varphi\|/(1+\|\varphi\|)$.

The convergence in $L_0(\Omega, Y)$ is equivalent to the convergence in probability. By $\mathscr{L}(\varphi)$ we denote the distribution of the Y-valued r.v. φ and by $\mathscr{L}(\varphi_1, \ldots, \varphi_n)$ – the distribution of the Y^n-valued r.v. $(\varphi_1, \ldots, \varphi_n)$. The characteristic function of a Y-valued r.v. φ is defined by

$$\hat{\varphi}(y) = \mathrm{E}\exp\{i\langle\varphi, y\rangle\}, \quad y \in Y'.$$

1.1. Definition. Let X be a Fréchet space. A linear mapping A from X into $L_0(\Omega, Y)$ is called a *random linear mapping* from X into Y. A linear continuous mapping from X into $L_0(\Omega, Y)$ is called a *random operator* from X into Y.

A random operator from X into the real line R is called a *random linear functional* on X.

1.2. Examples. (a) If the Y-valued r.v. Ax is concentrated at a point for all x, then the random operator A is the non-random ordinary linear operator.

(b) Let $L(X, Y)$ be the space of linear continuous operators from X into Y. Then with every $L(X, Y)$-valued r.v. B we may correspond a random operator A from X into Y be setting

$$Ax(\omega) = B(\omega)x. \tag{1.1}$$

We say that the random operator A in (1.1) is *generated* by an $L(X, Y)$-valued r.v. B or A is *decomposable*.

There exist random operators which are not decomposable. The problem of decomposability of random operators will be discussed in Section 4.

(c) Especially interesting examples of random operators are given by random integrals of Banach-valued functions. Let us recall the definition of random integral (see [9]). Let (T, Σ, μ) be a finite measurable space. A random mapping $M: \Sigma \to R$ is called the *random measure* on (T, Σ, μ) if for every sequence $A_1, A_2, \ldots$ of disjoint sets from Σ the random variables $M(A_1), M(A_2), \ldots$ are independent and

$$M\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} M(A_n) \text{ P-a.s.}$$

Let $f: T \to Y$ be a *simple function*, i.e. $f = \sum_{i=1}^{n} x_i 1_{A_i}$, where $A_i \in \Sigma$ are pairwise disjoint and $x_i \in Y$. For every $B \in \Sigma$ we set

$$\int_B f dM = \sum_{i=1}^{n} x_i M(A_i \cap B).$$

A function $f: T \to Y$ is said to be *integrable with respect to M* (shortly: *M-integrable*) if there exists a sequence $\{f_n\}$ of simple functions such that $f_n \to f$ in μ-measure and for each $B \in \Sigma$ the sequence of Y-valued r.v.'s $\{\int_B f_n dM\}$ converges in probability. Then we put

$$\int_B f dM = \text{P-lim} \int_B f_n dM.$$

The set of all Y-valued M-integrable functions is denoted by $\mathscr{L}_Y(M)$.

Set $|||f|||_0 = \|f\|_0 + \|\int_T f\, dM\|_0$, where $\|\cdot\|_0$ denotes the F-norms in $L_0(T, Y)$ and $L_0(\Omega, Y)$, respectively. By the definition, $(\mathscr{L}_Y(M), |||\cdot|||_0)$ forms a Fréchet space.

Define a random mapping A from $\mathscr{L}_Y(M)$ into Y by means of $Af = \int_T f dM$. It is easy to see that A is a random operator from $\mathscr{L}_Y(M)$ into Y.

1.3. Some general theorems. By definition, a random operator from X into Y is a linear continuous operator from X into $L_0(\Omega, Y)$. Because X and $L_0(\Omega, Y)$ are Fréchet spaces, the theory of linear continuous operators in Fréchet spaces becomes available for the study of random operators. The following theorems are consequences of the corresponding theorems in the theory of linear continuous operators in Fréchet spaces (cf. e.g. [8]).

1.3a. THEOREM. *Let A be a random linear mapping from X into Y. Then A is a random operator if and only if*

$$\lim_{t \to \infty} \sup_{\|x\| \leqslant 1} \mathrm{P}\{\|Ax\| > t\} = 0.$$

1.3b. THEOREM (Closed graph theorem for random operators). *Let A be a random linear mapping from X into Y. Then A is a random operator if and only if, for every sequence $(x_n) \subset X$ such that $x_n \to x$ in X and $Ax_n \to \varphi$ in probability, we have $Ax = \varphi$ P-a.s.*

1.3c. Theorem (Principle of uniform boundedness for random operators). *Let $(A_i)_{i\in I}$ be a family of random operators from X into Y such that, for each $x\in X$,*

$$\lim_{t\to\infty} \sup_{i\in I} \mathrm{P}\{\|A_i x\| > t\} = 0.$$

Then we have

$$\lim_{t\to\infty} \sup_{\|x\|\leqslant 1} \sup_{i\in I} \mathrm{P}\{\|A_i x\| > t\} = 0.$$

1.3d. Theorem (Theorem of Banach-Steinhaus for random operators). *Let (A_n) be random operators from X into Y such that, for each $x\in X$, $A_n x$ converges in probability. Then the random mapping A from X into Y, given by $Ax = \text{P-lim}\, A_n x$, is a random operator.*

2. CHARACTERISTIC FUNCTION OF RANDOM OPERATORS

Let us recall the concept of the tensor product of vector spaces E and F (see [7]).

Given two vector spaces E and F let $E\square F$ be a vector space whose elements are finite formal linear combinations $\Sigma\, a_k(x_k, y_k)$, $x_k\in E$ and $y_k\in F$. Let N denote the subspace of $E\square F$ spanned on all vectors of the form

$$(x, y_1+y_2)-(x, y_1)-(x, y_2), \quad (x, ty)-t(x, y),$$
$$(x_1+x_2, y)-(x_1, y)-(x_2, y), \quad (tx, y)-t(x, y).$$

The tensor product $E\otimes F$ is defined as the quotient space $E\otimes F = E\square F/N$.

Let φ be the restriction of the canonical map $\psi: E\square F\to E\otimes F$ to the space $E\times F$. Then $\varphi(x, y)$ will be denoted by $(x\otimes y)$. The role of the tensor product is emphasized by the fact that it enables us to replace a bilinear b: $E\times F\to W$ from the Cartesian product $E\times F$ into a linear space W by a linear map l: $E\otimes F\to W$ such that $b(x, y) = l(x\otimes y)$.

Now suppose that A is a random operator from X into Y. Define the map b_A from $X\times Y'$ into $L_0(\Omega, R)$ by $b_A(x, y) = (Ax, y)$. It is evident that b_A is bilinear. Hence, by the property of the tensor product, b_A determines a unique linear map l_A: $X\otimes Y'\to L_0(\Omega, R)$ such that $l_A(x\otimes y) = b_A(x, y) = (Ax, y)$.

2.1. Definition. Let A be a random operator from X into Y. Then the *characteristic function* (ch. f.) of A is a function with the domain $X\otimes Y'$ and range C. It is defined by

$$\hat{A}h = \mathrm{E}\exp\{il_A(h)\}, \quad h\in X\otimes Y'.$$

Two random operators A and B are said to be *equivalent* (denoted by $A \sim B$) if, for every finite sequence (x_i) in X, $\mathcal{L}(Ax_1, Ax_2, \ldots, Ax_n) = \mathcal{L}(Bx_1, Bx_2, \ldots, Bx_n)$.

The following proposition explains why the function $\hat{A}$ is called characteristic.

2.2. PROPOSITION. *Let A and B be two random operators from X into Y. Then A and B are equivalent if and only if they have the same characteristic function.*

Proof. By definition, $A \sim B$ if and only if

$$\mathrm{E}\exp\{i\sum_{k=1}^{n} t_k(Ax_k, y_k)\} = \mathrm{E}\exp\{i\sum_{k=1}^{n} t_k(Bx_k, y_k)\}$$

for all $t_1, t_2, \ldots, t_n \in R$ and $(x_1, y_1), \ldots, (x_n, y_n) \in X \times Y'$. Since every element $h \in X \otimes Y'$ has the form $h = \sum_{k=1}^{n} t_k(x_k \otimes y_k)$ and

$$\hat{A}h = \mathrm{E}\exp\{i\sum_{k=1}^{n} t_k(Ax_k, y_k)\} \quad \text{if } h = \sum_{k=1}^{n} t_k(x_k \otimes y_k),$$

it follows that $A \sim B$ if and only if $\hat{A}h = \hat{B}h$ for all $h \in X \otimes Y'$.

The following theorem gives the criterion for a function from $Y \otimes Y'$ into C to be the characteristic function of a random operator.

2.3. THEOREM. *For a function $f\colon X \otimes Y' \to C$ to be the characteristic function of a random operator it is necessary and sufficient that it satisfies the following conditions:*

(i) $f(0) = 1$;

(ii) *f is positive definite;*

(iii) *the function $B(x, y) = f(x \otimes y)$ is continuous on $X \times Y'$;*

(iv) *for each $x \in X$ the function $H_x\colon Y' \to C$, given by $H_x(y) = f(x \otimes y)$, is the ch. f. of some probability measure on Y.*

Proof. Suppose that A is a random operator from X into Y. For $c_1, c_2, \ldots, c_n \in C$ and $h_1, h_2, \ldots, h_n \in X \otimes Y'$ we have

$$\sum_{i,j} c_i \bar{c}_j \hat{A}(h_i - h_j) = \sum_{i,j} c_i \bar{c}_j \mathrm{E}\exp\{il_A(h_i - h_j)\}$$

$$= \sum_{i,j} c_i \bar{c}_j \mathrm{E}\exp\{il_A(h_i)\}\exp\{il_A(h_j)\} = \mathrm{E}\,|\sum_{i=1}^{n} c_i \exp\{il_A(h_i)\}|^2 \geqslant 0,$$

hence $\hat{A}$ is positive definite.

Since $\lim(Ax_n, y_n) = (Ax, y)$ in probability as $(x_n, y_n) \to (x, y)$, it follows that

$$\lim \hat{A}(x_n \otimes y_n) = \lim \mathrm{E} \exp\{i(Ax_n, y_n)\} = \mathrm{E} \exp\{i(Ax, y)\} = \hat{A}(x \otimes y)$$

as $(x_n, y_n) \to (x, y)$. Hence $H(x, y) = \hat{A}(x \otimes y)$ is continuous. The function $H_x(y) = \hat{A}(x \otimes y) = \mathrm{E}\exp\{i(Ax, y)\}$ is the ch. f. of $\mathscr{L}(Ax)$.

Conversely, suppose that $f: X \otimes Y' \to C$ is a function satisfying conditions (i)-(iv). For each finite set $I = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ we define a function $F(t_1, t_2, \ldots, t_n)$ on R^n by

$$F(t_1, t_2, \ldots, t_n) = f[\sum_{k=1}^{n} t_k (x_k \otimes y_k)]. \tag{2.1}$$

In view of (i)$\to$(iii), F is positive definite and continuous with $F(0, 0, \ldots, 0) = 1$. By the Bochner theorem, a measure μ_I on R^n with ch. f. (2.1) is defined. The family $\{\mu_I\}$ is consistent and by the Kolmogorov theorem there exists a random function $B(x, y)$ on $X \times Y'$ such that

$$f[\sum_{k=1}^{n} t_k (x_k \otimes y_k)] = \mathrm{E}\exp\{i \sum_{k=1}^{n} t_k B(x_k, y_k)\}.$$

$B(x, y)$ is bilinear. Indeed, for example we have

$$\mathrm{E}\exp\{itB(x_1+x_2, y) - B(x_1, y) - B(x_2, y)\}$$
$$= f[t(x_1+x_2)\otimes y - t(x_1 \otimes y) - t(x_2 \otimes y)] = f[0] = 1 \quad \text{for all } t \in R.$$

This shows that $B(x_1+x_2, y) = B(x_1, y) + B(x_2, y)$ P-a.s.

B is continuous by (iii). By (iv), for each $x \in X$, the random linear function $y \to B(x, y)$ is decomposed by a Y-valued random variable denoted by Ax, i.e., for all $y \in Y'$, $B(x, y) = (Ax, y)$ P-a.s.

The decomposition of r.v. Ax is uniquely determined. So the random mapping $x \to Ax$ is well-defined. To complete the proof it only remains to show that A is linear and continuous.

A is linear. Let $x_1, x_2 \in X$. We have $B(x_1+x_2, y) = B(x_1, y) + B(x_2, y) = (Ax_1, y) + (Ax_2, y) = (Ax_1 + Ax_2, y)$ P-a.s. for all $y \in Y'$. This shows that $A(x_1+x_2) = Ax_1 + Ax_2$ P-a.s.

A is continuous. Suppose that $x_n \to x$ in X and $Ax_n \to \varphi$ in probability. Then $B(x, y) = \text{P-lim}\, B(x_n, y) = \text{P-lim}(Ax_n, y) = (\varphi, y)$ for all $y \in Y'$, which shows that $\varphi = Ax$ (P-a.s.). By Theorem 1.3b we conclude that A is continuous.

3. CONVERGENCE OF RANDOM OPERATORS

Let $\{A_n\}_{n \geq 0}$ be random operators from X into Y. We define four modes of convergence of the sequence $\{A_n\}$ as follows:

3.1. Definition. (1) We say that A_n *converges to* A_0 if, for each $x \in X$, $A_n x \to A_0 x$ in probability.

(2) We say that A_n *converge weakly to* A_0 if, for each pair (x, y) in $X \times Y'$, $(A_n x, y) \to (A_0 x, y)$ in probability.

(3) We say that A_n *converges to* A_0 *in distribution* if, for each $k \in N$ and $x_1, x_2, \ldots, x_k$ in X, we have

$$\mathscr{L}(A_n x_1, A_n x_2, \ldots, A_n x_k) \Rightarrow \mathscr{L}(A_0 x_1, A_0 x_2, \ldots, A_0 x_k).$$

(4) We say that A_n *converges weakly to* A_0 *in distribution* if, for each $k \in N$ and $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ in $X \times Y'$,

$$\mathscr{L}[(A_n x_1, y_1), \ldots, (A_n x_k, y_k)] \Rightarrow \mathscr{L}[(A_o x_1, y_1), \ldots, (A_0 x_k, y_k)].$$

The following implications are obvious:

convergence	⇒	weak convergence		
⇓		⇓		
convergence in distribution	⇒	weak convergence in distribution	⇒	convergence of their ch. f.'s

The convergence in distribution implies the convergence in the following sense:

3.2. Theorem. *Let $\{A_n\}_{n \geqslant 0}$ be random operators from a separable Fréchet space X into Y and suppose A_n converge to A_0 in distribution. Then there exist random operators B_n, $n \geqslant 0$, such that $A_n \sim B_n$ for each $n \geqslant 0$ and B_n converge to B_0.*

Proof. Let $Z = (x_i)$ be the countable set dense in X. Consider the Y^{∞}-valued r.v.'s: $X_n = [A_n x_i]_{i=1}^{\infty}$, $n = 0, 1, 2, \ldots$

Because operators A_n converge to A_0 in distribution, it follows that $\mathscr{L}(X_n) \Rightarrow \mathscr{L}(X_0)$. By Skorokhod theorem [11] there exist Y^{∞}-valued r.v.'s $\tilde{X}_n = [X_n^{(i)}]_{i=1}^{\infty}$, $n = 0, 1, 2, \ldots$, such that $\mathscr{L}(\tilde{X}_n) = \mathscr{L}(X_n)$ for each $n \geqslant 0$ and $\tilde{X}_n$ converge to X_0 in probability. This implies that

(i) for each $i = 1, 2, \ldots, \tilde{X}_n^{(i)}$ converge to $\tilde{X}_0^{(i)}$ in probability;

(ii) $\mathscr{L}(\tilde{X}_n^{(1)}, \ldots, \tilde{X}_n^{(k)}) = \mathscr{L}(A_n x_1, \ldots, A_n x_k)$ *for each* $n \geqslant 0$ and each $k \geqslant 1$.

For each $n \geqslant 0$ we define a random mapping B_n from Z into Y by means of $B_n x_k = \tilde{X}_n^{(k)}$, $k = 1, 2, \ldots$

B_n can be extended over the entire space X. Indeed, let $x \in X$ and $(x_k)_1^{\infty}$ be a sequence in Z such that $x_k \to x$. Since $A_n x_k$ converge to $A_n x$ in $L_0(\Omega, Y)$ as $k \to \infty$ by (ii), $(B_n x_k)_1^{\infty}$ is a Cauchy sequence in $L_0(\Omega, Y)$. Hence $\lim_{k \to \infty} B_n x_k$ exists in $L_0(\Omega, Y)$.

It is not difficult to show that $A_n \sim B_n$. This fact implies that B_n is a random operator. It remains to prove that B_n converge to B_0. As $\mathscr{L}(B_n x)$

converges weakly for each x, by Prokhorov theorem we have

$$\lim_{t\to\infty}\sup_{n\geqslant 0} \mathrm{P}\{\|B_n x\|\} > t\} = 0.$$

By Theorem 1.3c it follows that

$$\lim_{t\to\infty}\sup_{\|x\|\leqslant 1}\sup_{n\geqslant 0} \mathrm{P}\{\|B_n x\| > t\} = 0.$$

Given $x \in X$, we choose a sequence (x_k) in Z converging to x. For each $\delta > 0$ we have

$$\mathrm{P}\{\|B_n x - B_0 x\| > \delta\}$$
$$\leqslant \mathrm{P}\left\{\|B_n x - B_n x_k\| > \frac{\delta}{3}\right\} + \mathrm{P}\left\{\|B_0 x_k - B_0 x\| > \frac{\delta}{3}\right\} + \mathrm{P}\left\{\|B_n x_k - B_0 x_k\| > \frac{\delta}{3}\right\}$$
$$\leqslant 2\sup_{\|x\|\leqslant 1}\sup_{n\geqslant 0} \mathrm{P}\left\{\|B_n x\| > \frac{\delta}{3}\|x - x_k\|\right\} + \mathrm{P}\left\{\|B_n x_k - B_0 x_k\| > \frac{\delta}{3}\right\}.$$

Let $n \to \infty$. Then $k \to \infty$ and we get

$$\lim_{n\to\infty} \mathrm{P}\{\|B_n x - B_0 x\| > \delta\} = 0,$$

which proves the Theorem.

3.3. THEOREM. *Let $\{A_n\}_{n\geqslant 0}$ be random operators from a separable Fréchet space X into a Banach space with the separable dual Y'. Suppose that A_n converge weakly to A_0 in distribution. Then there exist random operators B_n, $n \geqslant 0$, such that $A_n \sim B_n$ and B_n converge weakly to B_0.*

Proof. Let $Z = \{(x_i, y_i)\}_{i=1}^{\infty}$ be the countable set dense in $X \times Y'$. Because A_n converge weakly to A_0 in distribution, by using Skorokhod theorem and the same arguments as in the proof of the preceding theorem, we find random functions $U_n(x, y)$, $n \geqslant 0$, on $X \times Y'$ such that

(i) for each $n \geqslant 0$ the random function $U_n(x, y)$ is equivalent to the random function $(A_n x, y)$;

(ii) for each $n \geqslant 0$ and for each $(x, y) \in Z$, $U_n(x, y) \to U_0(x, y)$ in probability.

By (i) and arguments similar to those in the proof of Theorem 2.1 we find that there exist random operators B_n, $n = 0, 1, 2, \ldots$, such that, for each $n \geqslant 0$, $U_n(x, y) = (B_n x, y)$ P-a.s. for all $(x, y) \in X \times Y'$.

Clearly, $A_n \sim B_n$. Now we show that B_n converge weakly to B_0. Because $(B_n x, y)$ converges weakly for each $(x, y) \in X \times Y'$, we have

$$\lim_{t\to\infty}\sup_{n\geqslant 0} \mathrm{P}\{|(B_n x, y)| > t\} = 0.$$

Fix $y \in Y'$. Using the principle of the uniform boundedness for random linear functionals $(B_n x, y)$, we get

$$\lim_{t\to\infty}\sup_{\|x\|\leqslant 1}\sup_{n\geqslant 0}\mathrm{P}\{|(B_n x, y)|>t\}=0.$$

Again, using the principle of the uniform boundedness for the family $\{(B_n x, y),\ n\geqslant 0,\ \|x\|\leqslant 1\}$ of random functionals, we get

$$\lim_{t\to\infty}\sup_{\|y\|\leqslant 1}\sup_{\|x\|\leqslant 1}\sup_{n\geqslant 0}\mathrm{P}\{|(B_n x, y)|>t\}=0.$$

Given $(x, y)\in X\times Y'$, we choose a sequence $\{(x_k, y_k)\}\subset Z$ converging to (x, y). Then, for each $\delta>0$, we have

$$\mathrm{P}\{|(B_n x, y)-(B_n x_k, y_k)|>\delta\}$$
$$\leqslant \sup_{\|x\|\leqslant 1}\sup_{\|y\|\leqslant 1}\sup_{n\geqslant 0}\mathrm{P}\left\{|(B_n x, y)|>\frac{\delta}{2\|x_k-x\|\,\|y_k\|}\right\}+$$
$$+\sup_{\|x\|\leqslant 1}\sup_{\|y\|\leqslant 1}\sup_{n\geqslant 0}\mathrm{P}\left\{|(B_n x, y)|>\frac{\delta}{2\|x\|\,\|y_k-y\|}\right\}.$$

So

$$\mathrm{P}\{|(B_n x, y)-(B_0 x, y)|>\delta\}$$
$$\leqslant 2\sup_{\|x\|\leqslant 1}\sup_{\|y\|\leqslant 1}\sup_{n\geqslant 0}\mathrm{P}\left\{|(B_n x, y)|>\frac{\delta}{6\|x_k-x\|\,\|y_k\|}\right\}+$$
$$+2\sup_{\|x\|\leqslant 1}\sup_{\|y\|\leqslant 1}\sup_{n\geqslant 0}\mathrm{P}\left\{|(B_n x, y)|>\frac{\delta}{6\|x\|\,\|y_k-y\|}\right\}+$$
$$+\mathrm{P}\left\{|(B_n x_k, y_k)-(B_0 x_k, y_k)|>\frac{\delta}{3}\right\}.$$

Let $n\to\infty$. Then $k\to\infty$ and we get $\lim_{n\to\infty}\mathrm{P}\{|(B_n x, y)-(B_0 x, y)|>\delta\}=0$, as desired.

4. DECOMPOSABILITY OF RANDOM OPERATORS

4.1. Definition. Let X be a Banach space. A random operator A from X into Y is said to be *decomposable* if there exists an $L(X, Y)$-valued random variable B such that, for all $x\in X$, $\mathrm{P}\{\omega\colon Ax(\omega)=B(\omega)x\}=1$.

This definition is a natural extension of the notion of decomposability of random linear functionals to random operators. The decomposability of random linear functionals has been studied in many contexts (cf. for example [4] and [12]).

In this section we always assume that X and Y are separable.

4.2. PROPOSITION. *For each decomposable random operator A the decomposition of the random variable B is uniquely determined.*

Proof. Suppose that B_1 and B_2 are $L(X, Y)$-valued r.v.'s such that, for each $x \in X$, $Ax(\omega) = B_1(\omega)x$ and $Ax(\omega) = B_2(\omega)x$ P-a.s.

If Z is the countable linear subspace dense in X, then there exists a measurable set D, with $P(D) = 1$, such that $B_1(\omega)x = B_2(\omega)x$ for all $\omega \in D$ and for all $x \in Z$. Whence it follows that $B_1(\omega)x = B_2(\omega)x$ for all $\omega \in D$ and for all $x \in X$, i.e. $B_1 = B_2$ P-a.s.

Now we are going to find criteria which determine the decomposability of a random operator.

4.3. THEOREM. *A random operator A from X into Y' is decomposable if and only if, for every bounded sequence $\{x_n\}$ in X, we have $\sup_{n \geq 1} \|Ax_n\| < \infty$ P-a.s.*

Proof. Necessity. Suppose that A is decomposable. Then there exists an $L(X, Y)$-valued r.v. B such that, for all $x \in X$, $Ax(\omega) = B(\omega)x$ P-a.s.

Let $\{x_n\}$ be a sequence in X such that $\|x_n\| \leq 1$. Then there exists a measurable set D with $P(D) = 1$ such that $Ax_n(\omega) = B(\omega)x_n$ for all x_n and all $\omega \in D$. Therefore, for each $\omega \in D$, $\sup \|Ax_n(\omega)\| = \sup \|B(\omega)x_n\| \leq \|B(\omega)\| < \infty$, i.e. $\sup \|Ax_n\| < \infty$ P-a.s.

Sufficiency. Suppose that Q is a countable set, dense in X, and Z is a linear space spanned over the field of rational numbers of Q. Z is also countable.

Put $S_1 = \{z \in Z: \|z\| \leq 1\}$, $N(\omega) = \sup_{z \in Z} \|Az(\omega)\|$.

From the assumption it follows that there exists a measurable set D of probability 1 such that, for each $\omega \in D$, we have $N(\omega) < \infty$, $A(r_1 x + r_2 y)(\omega) = r_1 Ax(\omega) + r_2 Ay(\omega)$ for all x, y in Z and r_1, r_2 — rational numbers.

For each $\omega \in D$ define a mapping $B(\omega): Z \to Y$ by $B(\omega)z = Az(\omega)$.

The mapping $B(\omega)$ is linear and uniformly continuous on Z. Indeed, the linearity of $B(\omega)$ is obvious. Let now $x, y \in Z$ and r_n be a sequence of rational numbers such that $r_n \downarrow \|x-y\|$. Then $\|B(\omega)x - B(\omega)y\| = \|Ax(\omega) - Ay(\omega)\| = \|A(x-y)(\omega)\| = \|r_n A((x-y)/r_n)(\omega)\| \leq r_n N(\omega)$. Let $n \to \infty$. We get $\|B(\omega)x - B(\omega)y\| \leq N(\omega)\|x-y\|$, showing the uniform continuity of $B(\omega)$. Hence $B(\omega)$ can be extended to a linear continuous operator $B(\omega)$ on X.

To complete the proof of the Theorem, it remains to prove that, for each x, $Ax(\omega) = B(\omega)x$ P-a.s. Indeed, let $\{x_n\}$ be a sequence in Z converging to x. Then $Ax_n(\omega) = B(\omega)x_n$ for all x_n and for all $\omega \in D$. Since $B(\omega)x_n \to B(\omega)x$ for each $\omega \in D$, it follows that $Ax_n(\omega) \to B(\omega)x$ P-a.s. On the other hand, $Ax_n \to Ax$ in probability. Consequently, $B(\omega)x = Ax(\omega)$ P-a.s., as desired.

4.4. PROPOSITION. *Let X be a Banach space with the Schauder basis (e_n)*

and $A: X \to Y$ *be a random operator. Then* A *is decomposable if and only if there exists a measurable set* D *of probability* 1 *such that, for all* $\omega \in D$ *and for* $x \in X$, *the series* $\sum_{n=1}^{\infty} (x, e_n) Ae_n(\omega)$ *converges in* Y.

Proof. If A is decomposable by an $L(X, Y)$-valued r.v. B, then there exists a measurable set D of probability 1 such that $Ae_n(\omega) = B(\omega)e_n$ for all e_n and $\omega \in D$. Then, for each $x \in X$ and $\omega \in D$, we have $\sum (x, e_n) Ae_n(\omega) = \sum (x, e_n) B(\omega) e_n = B(\omega) \sum (x, e_n) e_n = B(\omega) x$.

Conversely, for each $\omega \in D$ we define a mapping $B(\omega): X \to Y$ by $B(\omega)x = \sum (x, e_n) Ae_n(\omega)$. By the Banach-Steinhaus theorem, $B(\omega) \in L(X, Y)$. Since $x = \sum (x, e_n) e_n$, we have $Ax = \sum (x, e_n) Ae_n(\omega)$ in probability. Hence $Ax(\omega) = B(\omega)x$ P-a.s.

4.5. THEOREM. *Let* $X = l_p$ $(1 < p < \infty)$ *with the standard Schauder basis* (e_n) *and* $A: l_p \to Y$ *be a random operator. Then the convergence a.s. of the series* $\sum \|Ae_n\|^q (1/p + 1/q = 1)$ *is a sufficient condition for* A *to be decomposable. This condition is necessary if and only if* Y *is finite-dimensional.*

Proof. Suppose that $\sum \|Ae_n\|^q < \infty$ P-a.s. Put

$$D = \{\omega: \sum \|Ae_n(\omega)\|^q < \infty\}.$$

Then, for each $\omega \in D$ and $x \in l_p$, $\sum \|(x, e_n) Ae_n(\omega)\| < \infty$, which implies the convergence of the series $\sum (x, e_n) Ae_n(\omega)$. By Proposition 4.4, A is decomposable.

Now suppose that A is decomposable. Consider first the case $Y = R$. There exists an l_q-valued r.v. φ such that, for all $x \in l_p$, $Ax(\omega) = \langle \varphi(\omega), x \rangle$ P-a.s. Hence there exists a set D of probability 1 such that $Ae_n(\omega) = \langle \varphi(\omega), e_n \rangle$ for all e_n and $\omega \in D$. Consequently, $\sum \|Ae_n(\omega)\|^q = \sum |\langle \varphi(\omega), e_n|^q < \infty$ for $\omega \in D$, i.e. $\sum \|Ae_n\|^q < \infty$ P-a.s.

Now let $Y = R^k$ and $f_1, f_2, \ldots, f_k$ be the standard basis in R^k. Then, for each $j = 1, 2, \ldots, k$, the random linear functional (Ax, f_j) is decomposable. Hence

$$\sum_{n=1}^{\infty} |(Ae_n, f_j)|^q < \infty \text{ P-a.s.}$$

So

$$\sum_{n=1}^{\infty} \|Ae_n\|^q \sim \sum_{n=1}^{\infty} \sum_{j=1}^{k} |(Ae_n, f_j)|^q = \sum_{j=1}^{k} \sum_{n=1}^{\infty} |(Ae_n, f_j)|^q < \infty \text{ P-a.s.}$$

To complete the proof of the Theorem, we give an example showing that in the case Y is infinite-dimensional the convergence a.s. of the series $\sum \|Ae_n\|^q$ is not necessary for A to be decomposable.

Let $\xi_1, \xi_2, \ldots$ be independent Gaussian real-valued r.v.'s with mean 0 and $\operatorname{Var}\xi_i = s_i^2$ such that $\sup s_i^2 < \infty$. We define a random operator $A\colon l_2 \to l_2$ by means of

$$Ax = \sum_{n=1}^{\infty} (x, e_n) e_n \xi_n.$$

It is not difficult to check that A is well-defined and that it is a random operator. We shall show that A is decomposable if and only if

$$\mathrm{P}\{\sup|\xi_r| < \infty\} = 1. \tag{4.1}$$

Indeed, if A is decomposable, then, by Theorem 4.3,

$$\mathrm{P}\{\sup\|Ae_n\| < \infty\} = \mathrm{P}\{\sup|\xi_n| < \infty\} = 1.$$

Conversely, if $\mathrm{P}\{\sup|\xi_n| < \infty\} = 1$, then put $N(\omega) = \sup|\xi_n(\omega)|$. Since the series $\sum (x, e_n) Ae_n(\omega) = \sum (x, e_n) e_n \xi_n$ converges in l_2 for all x in l_2 and $\omega \in D = \{\omega\colon N(\omega) < \infty\}$, A is, by Proposition 4.4, decomposable.

By Vakhania's theorem [16], condition (4.1) is equivalent to

$$\sum_{n=1}^{\infty} \exp\{-t/s_n^2\} < \infty \quad \text{for some } t > 0.$$

On the other hand, the series $\sum \|Ae_n\|^2 = \sum |\xi_n|^2$ converges a.s. if and only if $\sum s_n^2 < \infty$. So, if $\{s_n^2\}$ is a sequence such that $\sum s_n^2 = \infty$, but $\sum \exp\{-t/s_n^2\} < \infty$ for some $t > 0$, then A is decomposable but $\sum \|Ae_n\|^2 = \infty$ P-a.s.

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