PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 8 (1987), p. 177–181

SELF-DECOMPOSABLE PROBABILITY MEASURES ON LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

BY
CHU VAN DONG (HANOI)

Abstract. The paper presents a proof of the Levy-Khintchine representation for self-decomposable probability measures on complete locally convex topological vector spaces.

In recent years infinitely divisible (i.d.), stable and semi-stable probability measures (p.m.'s) on locally convex topological vector spaces have been extensively studied (see, e.g. [1–3]). However, another important class of self-decomposable p.m.'s, lying between i.d.p.m.'s and stable p.m.'s, has been considered only in Banach space. This fact motivates our study in this paper.

Let E be a real complete locally convex topological vector space (LCTVS) with the topological dual space E' separating points of E. Given $c > 0$ and a tight measure M on Borel subsets of R, let $T_0 M$ denote the image of M under the transformation $T_c x = cx$, $x \in E$.

A tight p.m. μ on E is said to be *self-decomposable* (s.d.) if for every $0 < c < 1$ there exists a p.m. μ_c such that

$$\mu = T_c \mu * \mu_c, \tag{1}$$

where the asterisk $*$ denotes the convolution operation.

In the same way as in the Banach space (cf. [5]) one can prove that if μ on E is s.d., then μ and its component μ_c in (1) are both i.d. Further, if μ is an i.d.p.m. on E, then its characteristic functional, denoted by $\hat{\mu}$, has (cf. [3]) the Levy-Khintchine representation

$$\hat{\mu}(y) = \exp\{i\langle y, a\rangle - \tfrac{1}{2}Q(y) + \int_E (e^{i\langle y,x\rangle} - 1 - i\langle y, x\rangle 1_K(x)) M(dx)\}, \tag{2}$$

where $y \in E'$, $a \in E$, Q is a positive definite quadric form on E and 1_K is the indicator of a convex balanced compact subset K of E such that $M(E \setminus K) < \infty$. The measure M, being a generalized Poisson exponent (Levy measure), has a finite mass outside every neighbourhood of zero in E and $M(\{0\}) = 0$.

It is clear from (1) and (2) that μ is an s.d. if and only if its generalized Poisson exponent M satisfies

$$M \geqslant T_c M \quad (0 < c < 1) \tag{3}$$

or, equivalently,

$$M = \sum_{j=0}^{\infty} T_{c^j} M_c \quad (0 < c < 1), \tag{4}$$

where $M_c = M - T_c M$ is the generalized Poisson exponent corresponding to μ_c in (1).

Thus the problem of representation of s.d.p.m.'s on E is reduced to that of solving inequality (2). In the Banach space this problem can be treated by the extreme points method (cf. [8]) and the polar coordinates method (cf. [4]). However, since there is no norm in the general LCTVS, another method would be more appropriate. In the sequel we apply the so-called "differentiation method" suggested by N. V. Thu to solve inequality (3). Namely, we shall first prove that there exists a limit

$$G = \lim_{c \to 1} \frac{M - T_c M}{-\log c} \tag{5}$$

which can be considered as a derivative of M (see Lemma 2). Then the measure M can be represented by an inverse operation (integration) – see Theorem 1.

Let K be a convex balance subset of E as in (2). Put $A = E \backslash K$ and define

$$Q_A(x) = \sup\{\lambda: \lambda \geqslant 1, x \in \lambda A\}, \qquad x \in E. \tag{6}$$

It is easy to prove the following properties of $Q_A(\cdot)$:

(i) $1_A(c^j x) = 1$ if and only if $0 \leqslant j \leqslant \log Q_A(x)/(-\log c)$, where $j = 0, 1, 2, \ldots, x \in E$ and $0 < c < 1$.

(ii) For every $\delta > 1$, $\delta_{\delta A}(x) = Q_A(x)/\delta$, whenever $Q_A(x) \geqslant \delta$.

Moreover, we get the following

LEMMA 1. *The family $G_t := M_c/t$, where (and in the sequel) $t = -\log c > 0$ and M_c is as in (4), is tight in the following sense: there exists a number $t_0 > 0$ such that for every $\varepsilon > 0$ there exists a convex balance compact subset W of E with the property*

$$\sup_{0 < t \leqslant t_0} G_t(E \backslash W) < \varepsilon. \tag{7}$$

Proof. Let K be a convex balance compact subset of E as in (2) and δ be a number from the interval (1,2). Setting $B = E \backslash K$ and taking into account that $M(E\backslash K) < \infty$, we get

$$M(B) < \infty. \tag{8}$$

Further, by (4) and (i),

$$M(B) = \sum_{j=0}^{\infty} T_{c^j} M_c(B) = \int_{E\backslash\{0\}} \Big(\sum_{j=0}^{\infty} 1_B(c^j x)\Big) M_c(dx) = \int_{E\backslash\{0\}} \left[\frac{1}{t}\log Q_B(x)\right] M_c(dx),$$

where $[\cdot]$ denotes the integer part and Q_B is defined by (6). Hence and by (ii) it follows that

$$\begin{aligned} M(B) &\geqslant \int_X \left[\frac{1}{t}\log Q_B(x)\right] M_c(dx) \geqslant \int_X \left[\frac{1}{t}\log \frac{Q_B(x)}{\delta}\right] M_c(dx) \\ &\geqslant \int_X \left[\frac{1}{t}\log Q_{\delta B}(x)\right] M_c(dx), \end{aligned} \tag{9}$$

where $X = \{x \in E \colon Q_B(x) \geqslant \delta\}$.

Now, observing that

$$\lim_{t\searrow 0} \frac{t^{-1}\log Q_{\delta B}(x)}{[t^{-1}\log Q_{\delta B}(x)]} = 1 \tag{10}$$

uniformly on the set X, one can choose a number $t_0 > 0$ such that, for any $0 < t \leqslant t_0$ and $Q_B(x) \geqslant \delta$,

$$\left[\frac{1}{t}\log Q_{\delta B}(x)\right] \geqslant \frac{1}{2t}\log Q_{\delta B}(x). \tag{11}$$

Therefore, by (9) and (i), it follows that, for $0 < t \leqslant t_0$,

$$M(B) \geqslant \frac{1}{2}\int_X \log Q_{\delta B}(x)\, G_t(dx) = \frac{1}{2}\int_X \int_0^{\infty} 1_{\delta B}(e^{-s}x)\, ds G_t(dx).$$

Hence and since $1_{\delta B}(e^{-s}x) = 0$ on the set $E\backslash X$, we get

$$\begin{aligned} M(B) &\geqslant \frac{1}{2}\int_E \int_0^{\infty} 1_{\delta B}(e^{-s}x)\, ds G_t(dx) = \frac{1}{2}\int_0^{\infty} G_t(\delta e^s B)\, ds \\ &\geqslant \frac{1}{2}\int_0^{b} G_t(\delta e^s B)\, ds \geqslant \frac{b}{2} G_t(\delta e^b B) \quad \text{for every } b > 0. \end{aligned}$$

Consequently,

$$2M(B)/b \geqslant G_t(\delta e^b B). \tag{12}$$

Since b can be arbitrarily chosen, the last inequality implies that if $\varepsilon > 0$

and $2M(B)/b < \varepsilon$, then $\sup G_t(E \setminus W) < \varepsilon$ for $0 < t \leqslant t_0$, where $W = \delta e^b K$ is a convex balanced compact set. Thus Lemma 1 is proved.

LEMMA 2. *There exists a measure G on E such that $G(\{0\}) = 0$, G is finite outside every neighbourhood of* 0 *in E and*

$$G_t \Rightarrow G \quad \text{as } t \searrow 0, \tag{13}$$

where the convergence is taken in the weak sense outside every neighbourhood of 0 *in E.*

Proof. By Theorem 3.1 in [7], it follows that Lemma 2 is true if $E = R^1$. Hence, for every functional $y \in E'$, the image of G_t under y converges to a limit on R^1 which, together with the tightness of the family G_t (see Lemma 1), implies that there exists a measure G such that $G(\{0\}) = 0$ and (14) holds. Moreover, G is finite outside every neighbourhood of 0 in E. Lemma 2 is thus proved.

LEMMA 3. *Measures M and G in* (3) *and* (13) *satisfy the relation*

$$M(\varepsilon) = \int_E \int_0^\infty 1_\varepsilon(e^{-t} x)\, dt G(dx) \tag{14}$$

for every Borel subset ε of E separated from 0. *Consequently, for every continuous seminorm P on E we have*

$$\int_E \log_+ P(x) G(dx) < \infty, \tag{15}$$

where $\log_+ a = \max(\log a, 0)$,

Proof. From Theorem 2.9 in [7] and Lemma 2 it follows that for every functional $y \in E'$ we have

$$yM(\varepsilon) = \int_{R^1} \int_0^\infty 1_\varepsilon(e^{-t} x)\, dt yG(dx), \qquad \varepsilon \subset R^1, \tag{16}$$

which, together with the fact that the image of the measure

$$\int_E \int_0^\infty 1_\varepsilon(e^{-t} x)\, dt G(dx), \qquad \varepsilon \subset E,$$

under y is equal to the right-hand side of (16), implies

$$yM(\cdot) = y\Big(\int_E \int_0^\infty 1\, (e^{-t} x)\, dt G(ds)\Big). \tag{17}$$

Consequently, formula (14) holds.

It remains to prove that (15) is satisfied. In fact, let U be a convex balanced neighbourhood of 0 in E and q the Minkowski functional defined by $p(x) = \inf\{\lambda : \lambda > 0,\ x \in \lambda U\}$.

By (14) and since $1_{E\setminus U}(e^{-t}x) = 1$ if and only if $0 \leqslant t \leqslant \log_+ p(x)$, we get

$$M(E\setminus U) = \int_E \int_0^\infty 1_{E\setminus U}(e^{-t}x)\,dt\,G(dx) = \int_E \log_+ p(x)\,G(dx).$$

Thus Lemma 3 is proved.

Lemma 3 together with formula (2) imply the following representation for s.d.p.m.'s on E:

THEOREM. *Let μ be an s.d.p.m. on E. Then its characteristic functional μ has the representation*

$$(18)\qquad \mu(y) = \exp\left\{i\langle y, a\rangle - \frac{1}{2}Q(y) + \int_X \left(\int_0^\infty h(y, e^{-t}x)\,dt\,G(dx)\right\}, \qquad y \in E',$$

where $h(y, x) = e^{i\langle y,x\rangle} - 1 - 1_K(x)\langle y, x\rangle$, a, Q and K have the same meaning as in (2), *and the measure G satisfies* (15) *with $G(\{0\}) = 0$.*

Acknowledgement. I would like to thank my adviser Professor Nguyen Van Thu for introducing me into the problem and for his great support.

REFERENCES

[1] A. de Acosta, *Stable measures and seminorms*, Ann. Prob. 3 (1975), p. 365–375.

[2] D. M. Chung, B. S. Rajput and A. Tortrat, *Semistable laws on TVS*, 1979.

[3] E. Dettweiler, *Grenzwertsätze für Wahrscheinlichkeitsmasse auf Badrixianschen Räumen*, Dissertation, Eberbard-Karls-Universität zu Tübingen, 1974.

[4] Z. J. Jurek, *How to solve the inequality $u_t m \leqslant m$ for every $0 < t < 1$?* Bull. Acad. Polon. 30.9–10 (1982), p. 477–482.

[5] A. Kumar and B. M. Schreiber, *Self-decomposable probability measures on Banach spaces*, Studia Math. 53 (1975), p. 55–71.

[6] A. P. Robertson and W. Robertson, *Topological vector spaces*, Cambridge 1964.

[7] N. V. Thu, *Universal multiply self-decomposable probability measures on Banach spaces*, Prob. Math. Stat. 3 (1982), p. 71–84.

[8] K. Urbanik, *Lévy's measures on Banach spaces*, Studia Math. 63 (1978), p. 284–308.

Institute of Mathematics, Hanoi
Vien Toan hoc
P. O. Box 631, Bo ho
Hanoi, Vietnam

Received on 23. 12. 1985