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BAHADUR'S REPRESENTATION OF SAMPLE QUANTILES BASED ON SMOOTHED ESTIMATES OF A DISTRIBUTION FUNCTION*

BY

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Abstract. Suppose $\hat{F}_n$ is a convolution-smoother of the standard empirical distribution function based on a random sample from a distribution F with a positive density. Consider the smoothed sample quantile function $\hat{F}_n^{-1}(p) = \inf\{x\colon \hat{F}_n(x) \geqslant p\}$. Under appropriate conditions, we establish a pointwise Bahadur type representation theorem [1] from which local behavior can be inferred.

1. Introduction. Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. observations having common distribution function (d.f.) F with density $f > 0$. Let $F_n(\cdot)$ denote the empirical d.f. based on the X_j's and define the quantile function G^{-1} of any distribution function G by the left-continuous version

(1) $$G^{-1}(p) = \inf\{x\colon G(x) \geqslant p\}, \quad 0 < p < 1.$$

With this definition, the sample quantile function F_n^{-1} satisfies

(2) $$F_n^{-1}(p) = X_{n:k} \text{ if } k-1 < np \leqslant k, \quad k = 1, 2, \ldots, n,$$

where $X_{n:k}$ is the k^{th} order statistic from the X_j's. For notational convenience, also let

(3) $$Z_n(x) = \sqrt{n}\,[F_n(x) - F(x)], \quad -\infty < x < \infty,$$

and

(4) $$Q_n(p) = \sqrt{n}\,[F_n^{-1}(p) - F^{-1}(p)], \quad 0 < p < 1,$$

denote the empirical process and quantile process, respectively.

Consider

(5) $$R_n(p) = Z_n(F^{-1}(p)) + f(F^{-1}(p))\,Q_n(p).$$

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Bahadur [1] showed that if $f(F^{-1}(p)) > 0$ and f' exists and is bounded in a neighbourhood of $F^{-1}(p)$, then

$$R_n(p) = O[n^{-1/4}(\log n)^{1/2}(\log\log n)^{1/4}] \tag{6}$$

with probability 1 (w. p. 1) pointwise as $n \to \infty$. Then Kiefer [8] obtained the exact rate of strong uniform convergence of $R_n(p)$, $0 < p < 1$. Subsequent works of interest include Kiefer [9, 10], Sen [14], Csörgö and Révész [2], and Shorack [15].

While many of the above-mentioned investigations were probabilistically oriented, the main motivation of the present article is a statistical one. Needless to say, the use of quantiles in the context of the sample median and the interquartile range are statistical folklore. In recent years, Parzen [12, 13] has also suggested extensive use of quantiles and density-quantiles in data analysis. Of course when F has a density, it is perhaps more reasonable to use a smooth estimate $\hat{F}_n$ of F rather than the step function F_n. (For some discussion of $\hat{F}_n$ based on convolution, see [11].) Smoothing again turned out to be appropriate in generating bootstrap samples, as Efron [3] has indicated.

In this discussion, we first smooth the empirical d.f. F_n by convolution as in [11], i.e., let

$$\hat{F}_n(x) = \int W_n(x-t)\,dF_n(t) = \frac{1}{n}\sum_{j=1}^{n} W_n(x-X_j), \tag{7}$$

where $\{W_n\}$ is a *Heaviside sequence*, i.e., $\{W_n\}$ is a sequence of d. f.'s converging weakly to the d.f. corresponding to the unit mass at the origin. Then define

$$\hat{F}_n^{-1}(p) = \inf\{x:\ \hat{F}_n(x) \geqslant p\}. \tag{8}$$

Another type of quantile estimation is considered in [5] and [4] where the smoothing is applied to $F_n^{-1}(p)$ (i.e., take inverse first, then smooth). Still a third point of view, based on U-statistics, was employed in [7]. It would be interesting, although not for the present discussion, to compare the statistical behavior of all three versions.

For the remainder of this article, we will assume that W_n is differentiable with a positive derivative on its support, so that for n sufficiently large, $\hat{F}_n^{-1}(p)$ is uniquely defined and that $\hat{F}_n^{-1} \circ \hat{F}_n$ is the identity function.

The symbols "O" and "o" will be used with the understood qualification "as $n \to \infty$."

Our main result (a Bahadur type representation theorem for $\hat{F}_n^{-1}(p)$) together with its corollaries will be presented in the next section.

2. Main result. Let

$$\hat{R}_n(p) = \hat{Z}_n(F^{-1}(p)) + f(F^{-1}(p))\hat{Q}_n(p), \quad 0 < p < 1, \tag{10}$$

where $\hat{Z}_n$ and $\hat{Q}_n$ are defined in a similar way to (3) and (4) according to $\hat{F}_n$ and $\hat{F}_n^{-1}$, respectively. Our objective is to obtain a pointwise rate of almost sure convergence for $\hat{R}_n(p)$. We will proceed in the same way as Bahadur [1]. There are essentially three steps.

First show that

$$|\hat{F}_n^{-1}(p) - F^{-1}(p)| \leqslant a_n \text{ w.p. 1} \quad \text{as } n \to \infty, \tag{11}$$

where $a_n = c(n^{-1}\log n)^{1/2}$ for some constant $c > 0$ suitably chosen.

In the second step, we show that

$$\sup|[\hat{F}_n(x) - \hat{F}_n(F^{-1}(p))] - [F(x) - p]| = O[n^{-3/4}(\log n)^{3/4}] \text{ w. p. } 1, \tag{12}$$

with the supremum taken over all x such that $|x - F^{-1}(p)| \leqslant a_n$.

Thirdly, by Lagrange's form of Taylor's expansion w.r.t. $F^{-1}(p)$, if f' exists and is bounded in a neighborhood of $F^{-1}(p)$, we have

$$F(\hat{F}_n^{-1}(p)) = F(F^{-1}(p)) + f(F^{-1}(p))[\hat{F}_n^{-1}(p) - F^{-1}(p)] + O[(\hat{F}_n^{-1}(p) - F^{-1}(p))^2]. \tag{13}$$

The final result will be seen as a consequence of (11), (12) and (13). We begin with the following

LEMMA 1. *Suppose $f(F^{-1}(p))$ exists, and W_n satisfies*

$$\int t\, dW_n(t) = 0, \quad \int_{|t| > a_n} |t|\, dW_n(t) = o(a_n^2), \tag{W}$$

where $a_n = c(n^{-1}\log n)^{1/2}$. Then for $c > 0$ sufficiently large, (11) holds.

COROLLARY 1. *$\hat{F}_n^{-1}(p)$ is pointwise strong consistent.*

Proof of Lemma 1. Since $0 \leqslant W_n \leqslant 1$ for all n, by [6], for any $\varepsilon > 0$,

$$P(\hat{F}_n(x) - E\hat{F}_n(x) > \varepsilon) \leqslant \exp\{-2n\varepsilon^2\}. \tag{14}$$

Let $a_n' = c_1(n^{-1}\log n)^{1/2}$, where $c_1 > 0$ will be specified later. Now, for n large enough,

$$\begin{aligned} \text{(15)} \quad & P(|\hat{F}_n^{-1}(p) - F^{-1}(p)| > a_n') \\ &= P(\hat{F}_n^{-1}(p) > F^{-1}(p) + a_n') + P(\hat{F}_n^{-1}(p) < F^{-1}(p) - a_n') \\ &= P(\hat{F}_n(F^{-1}(p) + a_n') < p) + P(\hat{F}_n(F^{-1}(p) - a_n') > p). \end{aligned}$$

Next, write $\pi_n = F^{-1}(p) - a_n'$. We will show that

$$p - \hat{E}F_n(\pi_n) = f(F^{-1}(p))a_n' + o(a_n'). \tag{16}$$

By the first condition of (W),

(17) $$p - E\hat{F}_n(\pi_n) - f(F^{-1}(p))a'_n$$
$$= \int [F(F^{-1}(p)) - F(F^{-1}(p) - a'_n - t) - f(F^{-1}(p))(a'_n + t)]\, dW_n(t)$$
$$= \int_{|t| \leq a'_n} + \int_{|t| > a'_n} \equiv A + B, \text{ say.}$$

Now, for n sufficiently large

$$|A| \leq \int_{|t| \leq a'_n} \left| \frac{F(F^{-1}(p)) - F(F^{-1}(p) - a'_n - t)}{a'_n + t} - f(F^{-1}(p)) \right| |a'_n + t|\, dW_n(t),$$
$$= o(a'_n).$$

The absolute value of B is bounded by

$$[2 + f(F^{-1}(p))a'_n] \int_{|t| > a'_n} dW_n(t) + f(F^{-1}(p)) \int_{|t| > a'_n} |t|\, dW_n(t),$$

which, by the second condition in (W), is also $o(a'_n)$ for n large enough. Thus (16) is verified.

Consider the second term on the right of (15); by (14) (since $p - E\hat{F}_n(\pi_n) > 0$ for n large enough by (16))

(18) $$\mathrm{P}(\hat{F}_n(\pi_n) > p) = \mathrm{P}(\hat{F}_n(\pi_n) - E\hat{F}_n(\pi_n) > p - E\hat{F}_n(\pi_n))$$
$$\leq \exp\{-2n[p - E\hat{F}_n(\pi_n)]^2\},$$

whence from (17), as $n \to \infty$,

$$\mathrm{P}(\hat{F}_n(\pi_n) > p) \leq \exp\left\{-3nf^2(F^{-1}(p))c_1^2 \frac{\log n}{n}\right\}.$$

Choosing c_1 sufficiently large, we see that

(19) $$\sum_{n \geq N_1} \mathrm{P}(\hat{F}_n(\pi_n) > p) < \infty \quad \text{for some } 0 < N_1 < \infty.$$

Similarly, one can choose a $c_2 > 0$ sufficiently large such that for $a''_n = c_2(n^{-1}\log n)^{1/2}$,

(20) $$\sum_{n \geq N_2} \mathrm{P}(\hat{F}_n(F^{-1}(p) + a''_n) < p) < \infty \quad \text{for some } 0 < N_2 < \infty.$$

Taking $N = N_1 \vee N_2$, $c = c_1 \vee c_2$, and $a_n = c(n^{-1}\log n)^{1/2}$, we see that (11) follows by the Borel-Cantelli Lemma.

Next, we state a non-trivial result which allows us to bypass the detailed argument presented in Lemma 1 of [1]:

PROPOSITION ([16], Theorem 2.15). *Let* $\{a_n\}$ *be a bandsequence, that is,* $0 < a_n < 1$, *and that, as* $n \to \infty$,

(i) $na_n \uparrow \infty$,

(ii) $\log a_n^{-1} = o(na_n)$,

(iii) $\log a_n^{-1}/\log\log n \to \infty$.

Suppose J is an interval (possibly infinite) on which F has a (positive) uniformly continuous derivative f. Then

$$\lim_{n\to\infty} \sup_{\substack{|t-u|\leqslant a_n \\ t,u\in J}} \frac{|Z_n(t)-Z_n(u)|}{[2a_n \log a_n^{-1}]^{1/2}} = [\sup_{x\in J} f(x)]^{1/2} \quad w.p.1.$$

We now establish (12) by way of the following

LEMMA 2. *Suppose* a_n *is defined as in Lemma* 1 *and suppose f is uniformly continuous on the support J of F. Then* (12) *holds.*

Proof. Observe that

$$[\hat{F}_n(x)-\hat{F}_n(F^{-1}(p))]-[F(x)-p]$$
$$= n^{-1/2} \int [Z_n(x-u)-Z_n(F^{-1}(p)-u)]\, dW_n(u),$$

so that the left side of (12) is majorized by

$$\sup_{\substack{|x-F^{-1}(p)|\leqslant a_n \\ \text{all } u}} n^{-1/2} |[Z_n(x-u)-Z_n(F^{-1}(p)-u)]| \int dW_n(u)$$
$$\leqslant \sup_{\substack{|s-t|\leqslant a_n \\ s,t\in J}} n^{-1/2} |Z_n(s)-Z_n(t)| \cdot 1 = O[n^{-1/2}(a_n \log a_n^{-1})^{1/2}]$$
$$= O[n^{-3/4}(\log n)^{3/4}] \text{ w. p. 1}$$

by the Proposition above and by the fact that $\{a_n\}$ defined in Lemma 1 is a bandsequence.

We are now ready to state the main result.

THEOREM. *Suppose f is uniformly continuous,* $f'(F^{-1}(p))$ *exists and* f' *is bounded in a neighborhood of* $F^{-1}(p)$. *Suppose* $\{W_n\}$ *is a Heaviside sequence satisfying* (W). *Then* $\hat{R}_n(p)$ *defined in* (10) *satisfies*

(22) $$\hat{R}_n(p) = O[n^{-1/4}(\log n)^{3/4}] \ w.\ p.\ 1.$$

COROLLARY 2. *Under the same conditions as in Theorem, asymptotically,*

$$f(F^{-1}(p))\hat{Q}_n(p) \stackrel{\mathscr{L}}{=} -\hat{Z}_n(F^{-1}(p)).$$

Hence, by arguments analogous to Theorem 1 of Yamato [17],

(23) $$\hat{Q}_n(p)-\mu_n(p) \xrightarrow{\mathscr{L}} N(0, \sigma^2) \quad \text{as } n \to \infty,$$

where

$$\mu_n(p) = \sqrt{n}\,[p - E\hat{F}_n(F^{-1}(p))]/f(F^{-1}(p)), \qquad \sigma_p^2 = p(1-p)/f^2(F^{-1}(p)).$$

Proof of Theorem. By Lemmas 1 and 2 and (13), we have

$$\{\hat{F}_n[\hat{F}_n^{-1}(p)] - \hat{F}_n(F^{-1}(p))\} - \{f(F^{-1}(p))[\hat{F}_n^{-1}(p) - F^{-1}(p)] + O(a_n^2)\}$$
$$= O[n^{-3/4}(\log n)^{3/4}] \text{ w. p. 1.}$$

Simplification yields

$$\hat{F}_n^{-1}(p) = F^{-1}(p) + [p - \hat{F}_n(F^{-1}(p))]/f(F^{-1}(p)) + \tag{24}$$
$$+ O[n^{-3/4}(\log n)^{3/4}] \text{ w. p. 1,}$$

which is equivalent to (22).

COROLLARY 3. *Under the same conditions as in Theorem,*

$$\overline{\lim_{n\to\infty}}\, f(F^{-1}(p))\,|\hat{Q}_n(p)|/[2p(1-p)\log\log n]^{1/2} = 1 \text{ w. p. 1} \tag{25}$$

provided $\mu_n(p)/(\log\log n)^{1/2} = o(1)$.

Proof. This follows from the pointwise law of the iterated logarithm for $\hat{F}_n(x)$ (see [11]).

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