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CONVERGENCE OF 2-DIMENSIONAL h-PROCESSES

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Abstract. Suppose that $D \subset C$ is a simply connected domain and p is a minimal Martin boundary point. Assume that there exists a curve in D which converges to p in the Martin topology and to $z \in C$ in the Euclidean topology. Then the same holds for almost all h-paths, where h is a minimal harmonic function represented by p. In such a case almost all h-paths have finite lifetime. This permits to define a Brownian excursion law in D starting from such a point p.

1. Introduction. The purpose of this note is to present probabilistic consequences of a theorem, proved by Jackson [6], which might have escaped the attention of non-specialists in potential theory.

Recently Cranston and McConnell [4] have proved that the expected lifetime of an h-process in a 2-dimensional domain with finite area is finite. This has been generalized by Cranston [3] to multidimensional domains with Lipschitz boundaries.

It will be shown that 2-dimensional h-paths converge a.s. in a simply connected domain under very mild assumptions. In a sense, one cannot assume less, see Remark 2.1 (ii) (a). It will be proved as a corollary that for a class of harmonic functions h, in simply connected domains (of possibly infinite area), almost all h-paths have finite lifetime. This implies in turn that for a suitable Martin boundary point p there exists a "standard" Brownian excursion law starting from p.

The inquiry resulting in the present note was stimulated by an unpublished example of Michael Cranston, who considered h-paths in a domain with infinite area.

The reader is referred to Doob [5] and Ohtsuka [8] for definitions of an h-process (i.e. "conditioned Brownian motion"), minimal-fine topology, prime ends and related concepts.

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2. Convergence of h-paths.

PROPOSITION 2.1. *Let $D \subset C$ be a simply connected domain. If a minimal harmonic function h is represented by a prime end p_h, then the cluster set of almost every h-path coincides with the set of all principal points of p_h.*

Proof. Almost every h-path converges to p_h in the Martin topology ([5], p. 691) and, therefore, its cluster set must contain all principal points of p_h (cf. [8], p. 270).

Moreover, almost all h-paths converge to p_h in the minimal-fine topology ([5], 3 III 3). By Theorem 3.2 (see also Remark 3.1) of Jackson [6] their cluster sets must be equal to the set of all principal points of p_h.

COROLLARY 2.2. *Let $D \subset C$ be a simply connected domain. If a minimal harmonic function h is represented by an accessible prime end p_h, then almost all h-paths converge (in the Euclidean topology) to the unique principal point of p_h.*

Remarks 2.1. (i) The corollary may be restated in the following form: *If at least one (non-random) path in D converges to p_h and terminates at $z \in \partial D$, then the same is true for almost all h-paths.*

(ii) Consider the following generalization of the problem. Let $D \subset R^n$, $n \geqslant 2$, be an open set and suppose that a curve $\Gamma = \{\Gamma(t), 0 < t < 1\} \subset D$ converges as $t \to 1$ to a minimal Martin boundary point p_h and it converges also to a point $z \in \partial D$ in the Euclidean topology. Does it follow that almost all h-paths converge to z?

(a) Yes, if $D \subset C$ is finitely connected.

(b) Not necessarily for general D. Almost all h-paths may converge to z or to $z_1 \in \partial D$, $z_1 \neq z$, or they may have no limit in the Euclidean topology. In dimensions $n \geqslant 3$ these kinds of behaviour may occur even if D is homeomorphic to a ball.

The observation (a) is a relatively easy corollary of Theorem 2.1. Examples illustrating (b) would require a lot of space (cf. [9]).

3. Applications.

THEOREM 3.1. *Let $D \subset C$ be a simply connected domain. If a minimal harmonic function h is represented by a prime end p_h and the set of its all principal points is bounded, then almost all h-paths have finite lifetime. In other words, p_h is an attainable minimal Martin boundary point.*

Proof. Fix the probability measure. Let A be the set of all principal points of p_h and

$$B = \{z \in C\colon |x-z| \geqslant 1 \text{ for all } x \in A\}.$$

Denote the last exit time from B as L_B. The lifetime of an h-path $X(\cdot)$ will be called R, i.e.

$$R = \inf\{t\colon \overline{\lim_{s \to t-}} X(s) \in \partial D\}.$$

Proposition 2.1 shows that $L_B < R$ a.s.; in particular, $L_B < \infty$ a.s.

The process $\{X(L_B+t), 0 < t < R-L_B\}$ is an h_1-process for some harmonic function h_1 on the bounded domain $D-B$ (see [7]). Theorem 1 of Cranston and McConnell [4], applied to this process, shows that its lifetime is finite a.s., that is $R-L_B < \infty$ a.s. and, therefore, $R < \infty$ a.s.

COROLLARY 3.1. *If a prime end is accessible, then it is attainable.*

Remarks 3.1. (i) One may assume in Theorem 3.1 that D is finitely connected (see Remark 2.1 (ii) (a)).

(ii) Theorem 3.1 extends Theorem 1 of Cranston and McConnell [4] to some h-processes and domains of infinite area. It does not guarantee however that the expected lifetime of h-paths is finite.

The next result is a generalization of Lemma 4.1 and Remark 4.2 (ii) of Burdzy [2]. In order to keep this note compact, the reader is referred to that paper for definitions and notation. Excursion laws below have Brownian transition probabilities.

PROPOSITION 3.1. *Let $D_1, D_2 \subset C$ be simply connected regions and f: $D_1 \to D_2$ be a conformal bijection. Suppose that p is a prime end of D_1, H^p is a standard excursion law in D_1, and $f(p)$ is a prime end such that the set of its all principal points is bounded. Then the f-mapping of H^p is well-defined excursion law in D_2.*

Proof. The proof of Lemma 4.1 in [2] may be repeated with the only change that Theorem 3.1 should be used instead of the result of Cranston and McConnell [4].

COROLLARY 3.2. *Let $D \subset C$ be a simply connected region and p be a prime end such that the set of its all principal points is bounded. Then there exists a (unique) standard non-null excursion law H^p in D. If p is accessible and $x \in C$ is its principal point, then there exists a (unique) standard non-null excursion law H^x in D.*

Proof. Let f: $\{\mathrm{Re}\, z > 0\} \to D$, $f(0) = p$, be a conformal bijection and H^0 be a standard non-null excursion law in $\{\mathrm{Re}\, z > 0\}$. Then H^p (H^x) may be defined as the f-mapping of H^0 by Proposition 3.1. It is easy to see, by Proposition 2.1, that H^p (H^x) has the desired properties.

An alternative proof is supplied by Theorem 4.1 of Burdzy [1] and the fact that p is attainable by Theorem 3.1.

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