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CONDITIONED RANDOM WALKS WITH RANDOM INDICES

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Abstract. Let $\{X_k, k \geqslant 1\}$ be a sequence of i.i.d. random variables with $EX_1 = 0$, $EX_1^2 = \sigma^2 < \infty$, and let $\{N_m, m \geqslant 0\}$, $N_0 = 0$ a.s., be a sequence of positive integer-valued random variables. Let $\{S_n, n \geqslant 0\}$ and $\{S_{N_m}, m \geqslant 0\}$ be defined by $S_0 = 0$ a.s., $S_n = X_1 + \ldots + X_n$, $n \geqslant 1$, $S_{N_0} = 0$ a.s., $S_{N_m} = X_1 + X_2 + \ldots + X_{N_m}$, $m \geqslant 1$. Put

$$N = \inf\{n\colon S_n < 0\}, \quad M = \max\{S_n\colon n \leqslant N\}.$$

In this note, under additional conditions on sequences $\{X_k, k \geqslant 1\}$ and $\{N_m, m \geqslant 0\}$, we investigate the limit behaviour of $P[M/\sigma\sqrt{N_m} \leqslant v | N > N_m]$, $P[\max_{0 \leqslant k \leqslant N_m} S_k/\sigma\sqrt{N_m} \leqslant v | N > N_m]$, and $P[N > N_m | M > v\sigma\sqrt{N_m}]$, where $v > 0$.

1. Introduction. Let $\{X_k, k \geqslant 1\}$ be a sequence of independent identically distributed random variables with $EX_1 = 0$, $0 < EX_1^2 = \sigma^2 < \infty$, and let $\{S_n, n \geqslant 0\}$ with $S_0 = 0$ and $S_n = X_1 + \ldots + X_n$, $n \geqslant 1$, denote the random walk. Put $N = \inf\{n\colon S_n < 0\}$, $M = \max\{S_n\colon n \leqslant N\}$, and write $\mu = -ES_N$. The following result is known:

THEOREM 1. *Under the above assumptions*:

(i) $$\lim_{x \to \infty} xP[M > x] = \mu;$$

(ii) $$\lim_{x \to \infty} P[\sigma^2 N/2x^2 \leqslant u | M > x] = (2/\sqrt{\pi u}) \sum_{k=1}^{\infty} \exp(-k^2/u),\ 0 < u < \infty;$$

(iii) $$\lim_{n \to \infty} P[M/\sqrt{n}\sigma \leqslant v | N > n] = 1 - v^{-1}\sqrt{\pi/2} + 2\sum_{k=1}^{\infty} \exp(-2k^2 v^2);$$

(iv) $$\lim_{n \to \infty} P[N > n | M > \sigma\sqrt{n}v] = 2\sqrt{2/\pi}v \sum_{k=1}^{\infty} \exp(-2k^2 v^2),\ 0 < v < \infty.$$

Proof. Statements (i)–(iii) have been proved in [2]. Putting now, in (ii), $n = [2ux^2/\sigma^2]$, $u = 1/(2v^2)$, we have

$$(1)\quad \lim_{n\to\infty} (1-\mathrm{P}[N > n \mid M > \sigma\sqrt{nv}])$$

$$= \lim_{n\to\infty} (1-\mathrm{P}[S_1 \geqslant 0, \ldots, S_n \geqslant 0 \mid M > \sigma\sqrt{nv}])$$

$$= 1-2\sqrt{2/\pi}\,v \sum_{k=1}^{\infty} \exp(-2k^2 v^2), \quad 0 < v < \infty,$$

which proves (iv).

Moreover, we have

THEOREM 2. *Under the above assumptions, for* $x > 0$,

$$(2)\quad \lim_{n\to\infty} \mathrm{P}[\max_{1\leqslant k\leqslant n} S_k/\sigma\sqrt{n} \leqslant x \mid N > n] = \sum_{k=-\infty}^{+\infty} (-1)^k \exp(-k^2 x^2/2).$$

Proof. It is known that

$$(S_{[n\cdot]}/\sigma\sqrt{n} \mid N > n) \Rightarrow W^+, \quad n\to\infty,$$

where W^+ is a Brownian meander (see [1] and [4]). In [3] (Corollary (2.2)) it has been proved that, for $x > 0$,

$$\mathrm{P}[\sup_{0\leqslant s\leqslant 1} W^+(s) \leqslant x] = \sum_{k=-\infty}^{+\infty} (-1)^k \exp(-k^2 x^2/2).$$

Hence, we conclude that, for $x > 0$,

$$\lim_{n\to\infty} \mathrm{P}[\max_{1\leqslant k\leqslant n} S_k/\sigma\sqrt{n} \leqslant x \mid N > n] = \sum_{k=-\infty}^{+\infty} (-1)^k \exp(-k^2 v^2/2),$$

which completes the proof of (2).

Let now $\{N_m, m \geqslant 1\}$ be a sequence of positive integer-valued random variables and put

$$S_{N_m} = X_1 + X_2 + \ldots + X_{N_m}, \quad m \geqslant 1.$$

We are interested in the asymptotic behaviour of

$$\mathrm{P}[M/\sigma\sqrt{N_m} \leqslant v \mid N > N_m], \quad 0 < v < \infty,$$

$$\mathrm{P}[\max_{1\leqslant k\leqslant N_m} S_k/\sigma\sqrt{N_m} \leqslant v \mid N > N_m],$$

and

$$\mathrm{P}[S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0 \mid M > v\sigma\sqrt{N_m}] = \mathrm{P}[N > N_m \mid M > v\sigma\sqrt{N_m}].$$

2. Results. We now prove the following

THEOREM 3. *Let $\{X_k, k \geqslant 1\}$ be a sequence of i.i.d. random variables with* $EX_1 = 0$, $0 < EX_1^2 = \sigma^2 < \infty$. *Suppose that $\{N_m, m \geqslant 0\}$, $N_0 = 0$ a.s., is a sequence of positive integer-valued random variables independent of $\{X_k, k \geqslant 1\}$ and $\{\alpha_m, m \geqslant 1\}$ is a sequence of positive integers with $\alpha_m \to \infty$, $m \to \infty$, such that*

$$\inf_m P[N_m \leqslant a\alpha_m] = d \text{ for some positive } a, \quad \text{where } d > 0, \tag{3}$$

and

$$\sqrt{\alpha_m} P[N_m = n] \to 0, \quad m \to \infty,\ n \geqslant 1. \tag{4}$$

Then for $0 < v < \infty$

$$\lim_{m\to\infty} P[M/\sigma\sqrt{N_m} \leqslant v | S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0] = 1 - v^{-1}\sqrt{\pi/2} + 2\sum_{k=1}^{\infty} \exp(-2k^2 v^2), \tag{5}$$

$$\lim_{m\to\infty} P[\max_{1\leqslant k\leqslant N_m} S_k/\sigma\sqrt{N_m} \leqslant v | S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0] = \sum_{k=-\infty}^{+\infty} (-1)^k \exp(-k^2 v^2/2), \tag{6}$$

and

$$\lim_{m\to\infty} P[S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0 | M > \sigma\sqrt{N_m}\, v] = 2\sqrt{2/\pi}\, v \sum_{k=1}^{\infty} \exp(-2k^2 v^2). \tag{7}$$

Proof. Assuming that, in (3), $a = 1$, we prove now (5) and (6). Note that

$$P[M/\sigma\sqrt{N_m} \leqslant v | S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0] = \sum_{n=1}^{\infty} t_{m,n} P[M/\sigma\sqrt{n} \leqslant v | S_1 \geqslant 0, \ldots, S_n \geqslant 0],$$

where

$$t_{m,n} = P[S_1 \geqslant 0, \ldots, S_n \geqslant 0] P[N_m = n]/P[S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0], \quad n \geqslant 1,\ m \geqslant 1,$$

and

$$P[\max_{1\leqslant k\leqslant N_m} S_k/\sigma\sqrt{N_m} \leqslant v | S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0] = \sum_{n=1}^{\infty} t_{m,n} P[\max_{1\leqslant k\leqslant n} S_k/\sigma\sqrt{n} \leqslant v | S_1 \geqslant 0, \ldots, S_n \geqslant 0].$$

Now taking into account that (Theorem 3.5, [7])

(8) $$P[S_1 \geq 0, \ldots, S_n \geq 0] \sim c/\sqrt{n}, \quad n \to \infty,$$

where

$$c = \exp\{\sum_{k=1}^{\infty} (1/k)(1/2 - P[S_k > 0])\},$$

we see, by assumption (3), that

(9) $$P[S_1 \geq 0, \ldots, S_{N_m} \geq 0] \geq P[S_1 \geq 0, \ldots, S_{N_m} \geq 0, N_m \leq \alpha_m]$$
$$\geq P[S_1 \geq 0, \ldots, S_{\alpha_m} \geq 0] P[N_m \leq \sigma_m] \geq (d \cdot c)/\sqrt{\alpha_m}, \quad m \to \infty.$$

Hence, using assumption (4), we have

$$0 \leq t_{m,n} \leq \sqrt{\alpha_m} P[N_m = n]/(dc) \to 0, \quad m \to \infty.$$

Since

$$\sum_{n=1}^{\infty} t_{m,n} = 1, \quad m \geq 1,$$

$\{t_{m,n}\}$, $m \geq 1$, $n \geq 1$, is a Toeplitz matrix ([6], p. 472). Therefore, by (iii) and (2), we get (5) and (6), respectively.

Similarly arguing, we get

(10) $$P[S_1 \geq 0, \ldots, S_{N_m} \geq 0 | M > \sigma\sqrt{N_m} v]$$
$$= \sum_{n=1}^{\infty} c_{m,n} P[S_1 \geq 0, \ldots, S_n \geq 0 | M > \sigma\sqrt{n} v],$$

where

$$c_{m,n} = P[M > \sigma\sqrt{n} v] P[N_m = n]/P[M > \sqrt{N_m}\sigma v], \quad m \geq 1,\ n \geq 1,$$
$$\sum_{n=1}^{\infty} c_{m,n} = 1, \quad m \geq 1.$$

Using assumption (3), we get

$$P[M > \sigma\sqrt{N_m} v] \geq \sum_{\{n: N_m \leq \alpha_m\}} P[M > \sigma\sqrt{N_m} v, N_m = n]$$
$$\geq P[M > \sigma\sqrt{\alpha_m} v] P[N_m \leq \alpha_m] \geq P[M > \sigma\sqrt{\alpha_m} v] \cdot d$$

for sufficiently large m. Hence, by (4) and (i), we have

$$0 \leq c_{m,n} \leq P[N_m = n]/(P[M > \sigma\sqrt{\alpha_m} v] d)$$
$$= \sigma v\sqrt{\alpha_m} P[N_m = n]/(d\sigma\sqrt{\alpha_m} P[M > \sqrt{\alpha_m} v\sigma]) \to 0, \quad m \to \infty.$$

These facts together with (iv) imply (7).

COROLLARY 1. *Suppose that a sequence $\{X_k, k \geqslant 1\}$ of i.i.d. random variables satisfies the assumptions of Theorem 3. If $\{N_m, m \geqslant 0\}$, $N_0 = 0$ a.s., is a sequence of positive integer-valued random variables independent of $\{X_k, k \geqslant 1\}$, and $\{\alpha_m, m \geqslant 1\}$ is a sequence of positive integers with $\alpha_m \to \infty$, $m \to \infty$, such that, for any given $\varepsilon > 0$,*

$$\mathrm{P}[|N_m/\alpha_m - \lambda| \geqslant \varepsilon] = o(1/\sqrt{\alpha_m}), \quad m \to \infty, \tag{11}$$

where λ is a random variable such that there exists an $a > 0$ such that

$$\mathrm{P}[\lambda > a] = 1, \tag{12}$$

then (5), (6) and (7) hold true.

Proof. It is obvious that we can find $\{\alpha_m, m \geqslant 1\}$ such that (3) holds. For any fixed $n \in N$ assumptions (11) and (12) imply that for any given ε, $0 < \varepsilon < a$, and sufficiently large m

$$\sqrt{\alpha_m}\,\mathrm{P}[N_m = n] \leqslant \sqrt{\alpha_m}\,\mathrm{P}[N_m \leqslant \alpha_m(a-\varepsilon)] \leqslant \sqrt{\alpha_m}\,\mathrm{P}[N_m \leqslant \alpha_m(\lambda-\varepsilon)] \times$$
$$\times \sqrt{\alpha_m}\,\mathrm{P}[|N_m/\alpha_m - \lambda| \geqslant \varepsilon].$$

Hence, we have (4), which completes the proof of Corollary 1.

The following example shows that assumption (11), in general, cannot be weakened in that sense that o can not be replaced by O, as it is in the random central limit theorem.

Example 1. Let $\{N_m, m \geqslant 5\}$ be a sequence of positive integer valued random variables such that $\mathrm{P}[N_m = 1] = 1/\sqrt{m}$, $\mathrm{P}[N_m = 2] = 1/\sqrt{m}$, $\mathrm{P}[N_m = m] = 1 - 2/\sqrt{m}$, $m \geqslant 5$.

Suppose that $\{X_k, k \geqslant 1\}$ is a sequence of Theorem 3 independent of $\{N_m, m \geqslant 5\}$. We see that $N_m/m \xrightarrow{\mathrm{P}} 1$, $m \to \infty$, and, for ε, $0 < \varepsilon < 1/4$, $\mathrm{P}[|N_m/m - 1| \geqslant \varepsilon] = 2/\sqrt{m}$, $m \geqslant 5$.

Moreover, by (2) and (8), one can verify that, for $v > 0$,

$$\mathrm{P}[\max_{1 \leqslant k \leqslant N_m} S_k/\sigma\sqrt{N_m} \leqslant v \mid S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0]$$
$$\to (\mathrm{P}[S_1 \leqslant \sigma v, S_1 \geqslant 0] + \mathrm{P}[\max(S_1, S_2) \leqslant \sigma\sqrt{2}v, S_1 \geqslant 0, S_2 \geqslant 0] +$$
$$+ c\sum_{k=-\infty}^{+\infty} (-1)^k \exp(-k^2 v^2/2))/(\mathrm{P}[S_1 \geqslant 0] + \mathrm{P}[S_1 \geqslant 0, S_2 \geqslant 0] + c)$$
$$\neq \sum_{k=-\infty}^{+\infty} (-1)^k \exp(-k^2 v^2/2).$$

Similarly, one can verify that (5) and (7) do not hold with the considered sequence $\{N_m, m \geqslant 5\}$.

Consider now the case where $\{X_k, k \geqslant 1\}$ and $\{N_m, m \geqslant 0\}$ are not independent. We are able to prove the following

THEOREM 4. *Let $\{X_k, k \geqslant 1\}$ be a sequence of i.i.d. random variables with $EX_1 = 0$, $0 < EX_1^2 = \sigma^2 < \infty$. Suppose that $\{N_m, m \geqslant 0\}$, $N_0 = 0$ a.s., is a sequence of positive integer-valued random variables, and $\{\alpha_m, m \geqslant 1\}$ is a sequence of positive numbers with $\alpha_m \to \infty$, $m \to \infty$, such that, for any given $\varepsilon > 0$,*

$$\mathrm{P}[a-\varepsilon \leqslant N_m/\alpha_m \leqslant b+\varepsilon] = o(1/\sqrt{\alpha_m}), \tag{13}$$

where $0 < a \leqslant b < \infty$ are constant.

Then, for $v > 0$,

$$\begin{aligned}(14)\quad & \sqrt{a/b}\,\{1-(va/b)^{-1}\sqrt{\pi/2}+2\sum_{k=1}^{\infty}\exp(-2k^2(va/b)^2)\} \\ & \leqslant \liminf_{m\to\infty} \mathrm{P}[M \leqslant v\sigma\sqrt{N_m} \,|\, S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0] \\ & \leqslant \limsup_{m\to\infty} \mathrm{P}[M \leqslant v\sigma\sqrt{N_m} \,|\, S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0] \\ & \leqslant \sqrt{b/a}\,\{1-(vb/a)^{-1}\sqrt{\pi/2}+2\sum_{k=1}^{\infty}\exp(-2k^2(vb/a)^2)\},\end{aligned}$$

$$\begin{aligned}(15)\quad & \sqrt{a/b}\sum_{k=-\infty}^{+\infty}\{\exp(-k^2(va/b)^2/2)\}(-1)^k \\ & \leqslant \liminf_{m\to\infty} \mathrm{P}[\max_{1\leqslant k\leqslant N_m} S_k/\sigma\sqrt{N_m} \leqslant v \,|\, S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0] \\ & \leqslant \limsup_{m\to\infty} \mathrm{P}[\max_{1\leqslant k\leqslant N_m} S_k/\sigma\sqrt{N_m} \leqslant v \,|\, S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0] \\ & \leqslant \sqrt{b/a}\sum_{k=-\infty}^{+\infty}\{\exp(-k^2(vb/a)^2/2)\}(-1)^k,\end{aligned}$$

and

$$\begin{aligned}(16)\quad & \sqrt{a/b}\,2\sqrt{2/\pi}\,v\sum_{k=1}^{\infty}\exp(-2k^2v^2) \\ & \leqslant \liminf_{m\to\infty} \mathrm{P}[S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0 \,|\, M > v\sigma\sqrt{N_m}] \\ & \leqslant \limsup_{m\to\infty} \mathrm{P}[S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0 \,|\, M > v\sigma\sqrt{N_m}] \\ & \leqslant \sqrt{b/a}\,2\sqrt{2/\pi}\,v\sum_{k=1}^{\infty}\exp(-2k^2v^2).\end{aligned}$$

Proof. Put $A_m = \{n, (a-\varepsilon)\alpha_m \leqslant n \leqslant (b+\varepsilon)\alpha_m\}$. Then, by (13) and (8), we have

$$\begin{aligned} r_m &= \mathrm{P}[S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0] \\ &\leqslant \mathrm{P}[S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0, N_m \in A_m] + \mathrm{P}[N_m \in A_m^c] \\ &\leqslant c/\sqrt{[(a-\varepsilon)\alpha_m]} + o(1/\sqrt{\alpha_m}). \end{aligned}$$

Similarly, we get

$$\begin{aligned} r_m &\geqslant \mathrm{P}[S_1 \geqslant 0, \ldots, S_{[(b+\varepsilon)\alpha_m]} \geqslant 0] - \mathrm{P}[N_m \in A_m^c] \\ &\geqslant c/\sqrt{[(b+\varepsilon)\alpha_m]} - o(1/\sqrt{\alpha_m}). \end{aligned}$$

Hence, we obtain

$$-o(1/\sqrt{\alpha_m}) + c/\sqrt{[(b+\varepsilon)\alpha_m]} \leqslant r_m \leqslant c/\sqrt{[(a-\varepsilon)\alpha_m]} + o(1/\sqrt{\alpha_m}). \tag{17}$$

Thus

$$\lim_{m\to\infty} \mathrm{P}[N_m \in A_m^c]/r_m = 0. \tag{18}$$

Hence, to prove (14), it is enough to consider

$$\mathrm{P}[M \leqslant v\sigma\sqrt{N_m}, S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0, N_m \in A_m]/r_m.$$

Note that

$$\mathrm{P}[M \leqslant v\sigma\sqrt{[(b+\varepsilon)\alpha_m]}, S_1 \geqslant 0, \ldots, S_{[(a-\varepsilon)\alpha_m]} \geqslant 0, N_m \in A_m]$$

$$\leqslant \mathrm{P}[M \leqslant v\sigma\sqrt{[(b+\varepsilon)\alpha_m]/[(a-\varepsilon)\alpha_m]}\sqrt{[(a-\varepsilon)\alpha_m]}, S_1 \geqslant 0, \ldots, S_{[(a-\varepsilon)\alpha_m]} \geqslant 0].$$

Moreover, we see that for any given $\eta > 0$ and sufficiently large m,

$$\sqrt{[(b+\varepsilon)\alpha_m]/[(a-\varepsilon)\alpha_m]} \leqslant \sqrt{(b+\varepsilon)/(a-\varepsilon)} + \eta.$$

Hence, we get

$$\begin{aligned} \mathrm{P}[M \leqslant v\sigma\sqrt{N_m}, S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0]/r_m \\ \leqslant (\mathrm{P}[S_1 \geqslant 0, \ldots, S_{[(a-\varepsilon)\alpha_m]} \geqslant 0]/r_m) \times \end{aligned}$$

$$\times \mathrm{P}[M \leqslant v\sigma(\sqrt{(b+\varepsilon)/(a-\varepsilon)} + \eta)\sqrt{[(a-\varepsilon)\alpha_m]} \mid S_1 \geqslant 0, \ldots, S_{[(a-\varepsilon)\alpha_m]} \geqslant 0].$$

Note that, for any given $\delta > 0$ and sufficiently large m,

$$\mathrm{P}[S_1 \geqslant 0, \ldots, S_{[(a-\varepsilon)\alpha_m]} \geqslant 0]/r_m \leqslant \sqrt{(b+\varepsilon)/(a-\varepsilon)} + \delta.$$

Therefore, for sufficiently large m,

$$(19)\quad P[M \leqslant v\sigma\sqrt{N_m},\ S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0,\ N_m \in A_m]/r_m$$
$$\leqslant P[M \leqslant v\sigma(\sqrt{(b+\varepsilon)/(c-\varepsilon)}+\eta)\sqrt{[(a-\varepsilon)\alpha_m]} \mid S_1 \geqslant 0,$$
$$\ldots, S_{[(a-\varepsilon)\alpha_m]} \geqslant 0](\sqrt{(b+\varepsilon)/(a-\varepsilon)}+\delta).$$

Similarly we get, for any given $\Delta > 0$ and sufficiently large m,

$$(20)\quad P[M \leqslant v\sigma(\sqrt{(a-\varepsilon)/(b+\varepsilon)}-\Delta)\sqrt{[(b+\varepsilon)\alpha_m]} \mid S_1 \geqslant 0, \ldots, S_{[(b+\varepsilon)\alpha_m]} \geqslant 0]\times$$
$$\times(\sqrt{(a-\varepsilon)/(b+\varepsilon)}-\Delta)-o(1/\sqrt{\alpha_m})$$
$$\leqslant P[M \leqslant v\sigma\sqrt{N_m},\ S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0,\ N_m \in A_m]/r_m.$$

Letting now $m \to \infty$, $\eta \to 0$, $\delta \to 0$, $\Delta \to 0$, $\varepsilon \to 0$, we get (14). In the similar way one can get (15).

We now prove (16). Note that, for any given $\varepsilon > 0$,

$$P[M > v\sigma\sqrt{N_m}] \geqslant P[M > v\sigma\sqrt{[(b+\varepsilon)\alpha_m]}]-P[N_m \in A_m^c],$$

and

$$P[M > v\sigma\sqrt{N_m}] \leqslant P[M > v\sigma\sqrt{[(a-\varepsilon)\alpha_m]}]+P[N_m \in A_m^c].$$

Moreover, we see that

$$P[S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0 \mid M > v\sigma\sqrt{N_m}]$$
$$= P[S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0,\ M > v\sigma\sqrt{N_m}]/P[M > v\sigma\sqrt{N_m}]$$
$$\leqslant \frac{P[S_1 \geqslant 0, \ldots, S_{[(a-\varepsilon)\alpha_m]} \geqslant 0,\ M > \sigma v\sqrt{[(a-\varepsilon)\alpha_m]}]+P[N_m \in A_m^c]}{P[M > v\sigma\sqrt{N_m}]}$$
$$\leqslant \frac{P[N_m \in A_m^c]+P[S_1 \geqslant 0, \ldots, S_{[(a-\varepsilon)\alpha_m]} \geqslant 0 \mid M > v\sigma\sqrt{[(a-\varepsilon)\alpha_m]}]}{P[M > v\sigma\sqrt{[(b+\varepsilon)\alpha_m]}]-P[N_m \in A_m^c]}\times$$
$$\times P[M > v\sigma\sqrt{[(a-\varepsilon)]\alpha_m}]$$

and

$$P[S_1 \geqslant 0, \ldots, S_{N_m} \geqslant 0 \mid M > v\sigma\sqrt{N_m}]$$
$$\geqslant \frac{-P[N_m \in A_m^c]+P[M > \sigma v\sqrt{[(b+\varepsilon)\alpha_m]}]}{P[M > v\sqrt{[(a-\varepsilon)\alpha_m]}]+P[N_m \in A_m^c]}\times$$
$$\times P[S_1 \geqslant 0, \ldots, S_{[(b+\varepsilon)\alpha_m]} \geqslant 0 \mid M > v\sigma\sqrt{[(b+\varepsilon)\alpha_m]}].$$

Hence, by the assumptions and (i), we obtain (16).

COROLLARY 2. *If* (13) *holds with* $a = b$, *then* (5), (6), *and* (7) *hold true.*

Remark 1. Assumption (13) holds true if we assume, for example, that, for any given $\varepsilon > 0$,

(13) $$P[|N_m/\alpha_m - \lambda| \geqslant \varepsilon] = o(1/\sqrt{\alpha_m}), \qquad P[a \leqslant \lambda \leqslant b] = 1,$$

where λ is a random variable, and $0 < a \leqslant b < \infty$ are constants.

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