PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 9, Fasc. 2 (1988), p. 29–44

ADMISSIBILITY OF LIMITS OF THE UNIQUE LOCALLY BEST LINEAR ESTIMATORS WITH APPLICATION TO VARIANCE COMPONENTS MODELS

BY

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Abstract. The paper gives a sufficient condition for the limit of a sequence of the unique best linear estimators to be admissible. For commutative variance components models a complete characterization of limits of sequences fulfilling that condition is established. There are also presented some conditions imposed on the variance components model which guarantee that the described set of limits coincides with the minimal complete class.

1. Introduction. The work of LaMotte [6] has provided an algorithm for the characterization of admissible linear estimators among the linear estimators in the general linear model. In LaMotte's paper the general linear model is described as a random r-vector Y with mean vector μ and variance-covariance matrix V, with (μ, V) contained in an arbitrary subset of the Cartesian product of Euclidean r-space $\mathscr{R}^r$ and the set of $(n \times n)$-symmetric nonnegative definite matrices. Using that algorithm LaMotte characterized the class of all linear admissible estimators for any linear combination of the regression parameters for the linear regression model with nonsingular variance-covariance matrix of the error terms. Applications of LaMotte's method to other models in which there is a relationship between the mean vector and the variance-covariance matrix require to consider a number of cases too large to handle. Thus Klonecki and Zontek [4] investigated the possibility of characterizing the admissible linear estimators through the unique Bayes linear estimators and their limits. They defined a class of linear models, called regular, for which the class of the unique linear Bayes estimators and some of their limits form the minimal complete class. They also established a condition which guarantees that the limit of a sequence of unique linear Bayes estimators is admissible.

In this paper the assumptions of some of the results presented in [4] are weakened, so that they can be applied directly to the problem od characterization of the admissible invariant quadratic estimators for some mixed linear models. In fact, a number of examples of models which are regular for invariant quadratic estimation of any linear combination of the variance components are given. For those models we establish a complete characterization of the admissible invariant quadratic estimators for any linear combination of the variance components.

2. Minimal complete class for regular models. Let Y be a random vector as described in the introduction. It is desired to estimate $C'\mu$, where C is an $(r\times t)$-matrix $(r\geqslant t)$, while C' stands for the transposed matrix of C. The estimators considered are linear estimators $L'Y$, where L belongs to an affine set $\mathscr{L}=L_0+\mathscr{R}(N\otimes I)$. Here L_0 stands for any $(r\times t)$-matrix, N for any $(r\times r)$-matrix, I is the $t\times t$ identity matrix. As usual for any $(r\times r)$-matrix A and any $(t\times t)$-matrix B, the symbol $A\otimes B$ denotes the linear operator mapping the space $\mathscr{M}_{r\times t}$ of $(r\times t)$-matrices into itself and is defined for every C in $\mathscr{M}_{r\times t}$ by $(A\otimes B)C=ACB'$, whereas $\mathscr{R}(A\otimes B)$ stands for the range of the operator $A\otimes B$. For simplicity we refer to the estimator $L'Y$ of $C'\mu$ in terms "estimator $L\in\mathscr{L}$ of C".

To compare estimators we use the risk function $E(L'Y-C'\mu)'(L'Y-C'\mu)$. Writing this function as $[L,(V+\mu\mu')L]-2[L,\mu\mu'C]+[C,\mu\mu'C]$ shows that it depends on the distribution of Y through $V+\mu\mu'$ and $\mu\mu'C$ only. Here $[A,B]$ stands for the trace of the matrix AB'. Taking $(V+\mu\mu',\mu\mu'C)$ as the parameter, the new parameter space to be denoted by $\mathscr{T}$, becomes a subset of the Cartesian product $\mathscr{V}_r\times\mathscr{M}_{r\times t}$, where $\mathscr{V}_r$ is the set of all $r\times r$ n.n.d. matrices.

The relations "as good as" and "better than" on $\mathscr{T}$ are defined in the usual way. Estimator L is said to be *admissible for C among $\mathscr{L}$* if $L\in\mathscr{L}$ and if there exists no estimator in $\mathscr{L}$ better than L. The term "among $\mathscr{L}$" will be omitted when $\mathscr{L}$ coincides with $\mathscr{M}_{r\times t}$.

Now let τ be an a priori distribution on $\mathscr{T}$ such that $E_\tau\mu\mu'$ and $E_\tau V$ exist. The relevant Bayes risk becomes then

$$[L,E_\tau(V+\mu\mu')L]-2[L,E_\tau\mu\mu'C]+[C,E_\tau\mu\mu'C].$$

An estimator L in $\mathscr{L}$ is said to be a *linear Bayes estimator among $\mathscr{L}$* if it has the smallest Bayes risk among all estimators in $\mathscr{L}$. If we extend the risk function for each L in $\mathscr{L}$ from $\mathscr{T}$ to $\mathscr{W}=\operatorname{span}\mathscr{T}$ for each $W=(W_1,W_2)\in\mathscr{W}$ by

$$\varrho(W,L)=[L,W_1L]-2[L,W_2]+[C,W_2],$$

then every linear Bayes estimator among $\mathscr{L}$ may be viewed as an estimator minimizing the extended risk function at a point in conv $\mathscr{T}$ among $\mathscr{L}$ and, vice versa, every estimator minimizing the extended risk function among $\mathscr{L}$ at a point in conv$\mathscr{T}$ may be viewed as a linear Bayes estimator among $\mathscr{L}$ with respect to some a priori distribution τ on $\mathscr{T}$.

Similarly as in [7] and [6], we formulate the results in terms of (locally) best estimators instead in terms of linear Bayes estimators. The former terminology seems to be more convenient when dealing with linear Bayes estimators.

To begin with, we present the LaMotte theorem in a form most suitable for our considerations.

Since each L in $\mathscr{L}$ may be written as $L = L_0 + NZ$ for some Z in $\mathscr{M}_{r\times t}$, the risk function may be rewritten as

$$\varrho(W, L) = [Z, T(W)Z] + 2[Z, U(W)] + \varrho(W, L_0),$$

where $T(W) = N' W_1 N$ and $U(W) = N' W_1 L_0 + N' W_2$.

An estimator L in $\mathscr{L} = L_0 + \mathscr{R}(N \otimes I)$ is called *best among $\mathscr{L}$ at a point* $W \in \mathscr{W}$ if $\varrho(W, L) \leqslant \varrho(W, M)$ for all $M \in \mathscr{L}$. Let $\mathscr{B}(W|\mathscr{L})$ denote the subset of all those estimators in $\mathscr{L}$ which are best at W among $\mathscr{L}$. Notice that $\mathscr{B}(W|\mathscr{L})$ is not empty iff $T(W)$ is n.n.d. and $\mathscr{R}(U(W)) \subset \mathscr{R}(T(W))$. If $L \in \mathscr{B}(W|\mathscr{L})$, then $\mathscr{B}(W|\mathscr{L}) = L + (N \otimes I)(\mathscr{N}(T(W) \otimes I))$. The symbol $\mathscr{N}(T(W) \otimes I)$ denotes the null space of the operator $T(W) \otimes I$. Clearly, $\mathscr{B}(W|\mathscr{L}) = \{L\}$ iff $(N \otimes I)(\mathscr{N}(T(W) \otimes I)) = \{0\}$, i.e. iff $\mathscr{R}(N') = \mathscr{R}(T(W))$. In this case we say that L is the *unique best estimator* (UBE, for short) at W among $\mathscr{L}$.

To avoid some trivialities we assume throughout the paper that there exists a point $W = (W_1, W_2)$ in $\mathscr{W}$ such that W_1 is a p.d. matrix.

Following LaMotte [6], W in $\mathscr{W}$ is said to be a *trivial point* for $\mathscr{L}$ if $\mathscr{B}(W|\mathscr{L}) = \mathscr{L}$. The set of trivial points for $\mathscr{L}$ will be denoted by $\mathscr{S} = \mathscr{S}(\mathscr{L})$. Obviously, $\mathscr{S} = \{W \in \mathscr{W}\colon T(W) = 0, U(W) = 0\}$.

Since $T(W)$ is n.n.d. for each W in the closed convex cone containing $\mathscr{T} + \mathscr{S}$, to be denoted by $[\mathscr{T} + \mathscr{S}]$, it follows that $\mathscr{B}(W|\mathscr{L}) \neq \emptyset$ for W in $[\mathscr{T} + \mathscr{S}] \setminus \mathscr{S}$ iff $\mathscr{R}(U(W)) \subset \mathscr{R}(T(W))$.

Define

$$\mathscr{A}(\mathscr{L}) = \{W \in [\mathscr{T} + \mathscr{S}] \setminus \mathscr{S}\colon \mathscr{R}(U(W)) \subset \mathscr{R}(T(W))\}$$

and notice that $[\mathscr{T}] \setminus \mathscr{S} \subset \mathscr{A}(\mathscr{L})$.

THEOREM 2.1. *If L in $\mathscr{L}$ is an admissible estimator of C among $\mathscr{L}$, then there exists a point W in $\mathscr{A}(\mathscr{L})$ such that $L \in \mathscr{B}(W|\mathscr{L})$ unless $\mathscr{T} \subset \mathscr{S}$.*

This theorem, as indicated by LaMotte [6], gives a step-wise algorithm to characterize admissible linear estimators. To formalize this procedure let us introduce the following notation.

For $i = 1, \ldots, r$ define the following families of affine sets in $\mathcal{M}_{r\times t}$:

$$\mathcal{C}^{(0)} = \{\mathcal{M}_{r\times t}\},$$

$$\mathcal{C}^{(i)} = \{\mathcal{B}(W \mid \mathcal{L}): L \in \mathcal{C}^{(i-1)},\ W \in \mathcal{A}(\mathcal{L})\}.$$

Notice that if $\mathcal{L} = L_0 + \mathcal{R}(N \otimes I) \in \mathcal{C}^{(i)}$, then rank $N \leqslant r-i$.

To determine whether a linear estimator is admissible we may use the following corollary to LaMotte's theorem:

Corollary 2.2. *The set $\mathcal{D}$ defined by*

$$\mathcal{D} = \{L \in \mathcal{M}_{r\times t}: \{L\} \in \mathcal{C}^{(i)} \text{ for some } 0 \leqslant i \leqslant r\}$$

represents the class of all admissible estimators, i.e. $\mathcal{D}$ forms the minimal complete class.

We shall now recall the definition of a regular model.

W is said to be a *perfect trivial point* for the subspace $\mathcal{R}(N \otimes I)$ if $\mathcal{B}(W \mid L_0 + \mathcal{R}(N \otimes I)) = L_0 + \mathcal{R}(N \otimes I)$ for each L_0 in $\mathcal{M}_{r\times t}$. The set of all perfect trivial points for $\mathcal{R}(N \otimes I)$ is given by

$$\mathcal{S}_0 = \mathcal{S}_0(\mathcal{R}(N \otimes I)) = \{W = (W_1, W_2) \in \mathcal{W}: N'W_1 = N'W_2 = 0\}.$$

Clearly, W is a perfect trivial point for $\mathcal{R}(N \otimes I)$ iff W is a trivial point in the sense of LaMotte for each affine set $L_0 + \mathcal{R}(N \otimes I)$ whatever be L_0 in $\mathcal{M}_{r\times t}$.

W in $[\mathcal{T} + \mathcal{S}_0] \setminus \mathcal{S}$ is said to be a *perfect point* for a subspace $\mathcal{R}(N \otimes I)$ if $\mathcal{B}(W \mid L_0 + \mathcal{R}(N \otimes I)) \neq \emptyset$ for each L_0 in $\mathcal{M}_{r\times t}$. The set of the perfect points for $\mathcal{R}(N \otimes I)$, which are not trivial, is given by

$$\mathcal{A}_0(\mathcal{R}(N \otimes I)) = \{W \in [\mathcal{T} + \mathcal{S}_0] \setminus \mathcal{S}: \mathcal{R}(N'W_1) + \mathcal{R}(N'W_2) \subset \mathcal{R}(T(W))\},$$

Moreover, for $i = 1, \ldots, r$, define the following families of affine sets in $\mathcal{M}_{r\times t}$:

$$\mathcal{C}_0^{(0)} = \{\mathcal{M}_{r\times t}\},$$

$$\mathcal{C}_0^{(i)} = \{\mathcal{B}(W \mid \mathcal{L}): \mathcal{L} \in \mathcal{C}_0^{(i-1)},\ W \in \mathcal{A}_0(\mathcal{L})\}.$$

Definition 2.3. A model is said to be *regular for C* if $\mathcal{D} = \mathcal{D}_0$, where

$$\mathcal{D}_0 = \{L \in \mathcal{M}_{r\times t}: \{L\} \in \mathcal{C}_0^{(i)} \text{ for some } 0 \leqslant i \leqslant r\}.$$

The following result established in [4] gives a sufficient condition for a model to be regular:

Lemma 2.4. *If $\mathcal{A}_0(\mathcal{R}(N \otimes I)) + \mathcal{S}(\mathcal{L}) = \mathcal{A}(\mathcal{L})$ for every affine set $\mathcal{L} = L_0 + \mathcal{R}(N \otimes I)$ in $\bigcup \mathcal{C}^{(i)}$, $i = 1, 2, \ldots, r$, then the model is regular.*

As shown in [4], a mixed linear model is regular for the linear estimation of regression parameters and, under the additional assumption that the subspace spanned by the variance-covariance matrices is a quadratic commutative subspace of dimension less than or equal to 4, it is also regular for the invariant quadratic estimation of variance components.

Now we establish a new theorem, giving some conditions improved to those given in [4], which guarantee that a sequence of UBE's is convergent and that the limit is an admissible estimator.

THEOREM 2.5. (i) *Let* $\{L^{(n)}\}$ *be a sequence of estimators of* C *such that each* $L^{(n)}$ *is a UBE at a point* $W_0^{(n)} = (W_{01}^{(n)}, W_{02}^{(n)})$ *in* $[\mathscr{T}]$.

(ii) *Let* $W_i = (W_{i1}, W_{i2})$, $i = 1, \ldots, s$, *be points in* $\mathscr{W}$ *and let* N_i, $i = 1, \ldots, s$, *be* $(t \times t)$*-matrices such that* $N_1 = I$, $\mathscr{R}(N_{i+1}) = N_i(\mathscr{N}(N_i' W_{i1} N_i))$, $i = 1, \ldots, s-1$, *while* $\mathscr{R}(N_s') = \mathscr{R}(N_s' W_{s1} N_s) \neq \{0\}$.

If, for $i = 1, \ldots, s$,

$$\lim a_i^{(n)} N_i' W_0^{(n)} = N_i' W_i, \tag{2.1}$$

where $a_i^{(n)} = \|N_i' W_{01}^{(n)} N_i\|^{-1}$, *then the sequence* $\{L^{(n)}\}$ *converges and its limit is an admissible estimator of* C.

Proof. First note that, without loss of generality, we may assume that, for $i = 1, \ldots, s-1$,

$$N_{i+1} = N_i\big(I - (N_i W_{i1} N_i)^+ N_i W_{i1} N_i\big). \tag{2.2}$$

In view of (i)

$$a_i^{(n)} N_i W_{01}^{(n)} L^{(n)} = a_i^{(n)} N_i W_{02}^{\prime(n)}, \qquad i = 1, \ldots, s, \tag{2.3}$$

and summing over i yields

$$\sum_{i=1}^{s} a_i^{(n)} N_i W_{01}^{(n)} L^{(n)} = \sum_{i=1}^{s} a_i^{(n)} N_i^{(n)} W_{02}^{(n)}.$$

In order to show that $\{L^{(n)}\}$ converges we need in view of (2.1) to prove that

$$\lim_{n \to \infty} \sum_{i=1}^{s} a_i^{(n)} N_i W_{01}^{(n)} = \sum_{i=1}^{s} N_i W_{i1}$$

is invertable.

Suppose to the contrary that $\sum W_{i1} N_i x = 0$ $(i = 1, 2, \ldots, s)$ for some non-zero vector x in $\mathscr{R}^r$. This yields that

$$N_s W_{s1} N_s x + N_s \sum_{i=1}^{s-1} W_{i1} N_i x = 0.$$

From (2.2) it follows that the second sum must be equal to zero, so that $N_s W_{s1} N_s x = 0$. Since $\mathscr{R}(N_s) = \mathscr{R}(N_s W_{s1} N_s)$, we conclude that $N_s x = 0$. This and the formula $\mathscr{R}(N_{s-1} W_{s-1,1} N_{s-1}) = \mathscr{R}(N_{s-1} - N_s)$ show in an analogous way that $N_{s-1}x = 0$. Continuing this procedure we show that $N_1 x = x = 0$. This contradiction proves that $\{L^{(n)}\}$ converges, say, to L.

In view of (2.1) it follows from (2.3) that $N_i W_{i1} L \doteq N_i W_{i2}$ for $i = 1, \ldots, s$. Now let

$$\mathscr{L}_i = L + \mathscr{R}(N_i \otimes I), \qquad i = 1, \ldots, s,$$

and let $\mathscr{S}_{0i}$ be the set of perfect trivial points for $\mathscr{R}(N_i \otimes I)$. With this notation $\mathscr{L}_{i+1} = \mathscr{B}(W_i \mid \mathscr{L}_i)$ for $i = 1, \ldots, s$, and $\mathscr{L}_{s+1} = \{L\}$. Since $W_i \in [\mathscr{T} + \mathscr{S}_{0i}$ for $i = 1, \ldots, s$, it follows from Corollary 2.2 that L is an admissible estimator for C. In fact, lect $h_i\colon \mathscr{W} \to \mathscr{M}_{r\times r} \times \mathscr{M}_{r\times t}$ be defined by $h_i(W^*) = (N_i W_2^*, N_i W_2^*)$ for every point $W^* = (W_1^*, W_2^*)$ in $\mathscr{W}$ and for $i = 1, \ldots, s$. Then $\mathscr{N}(h_i) = \mathscr{S}_{0i}$, so that $h_i([\mathscr{T} + \mathscr{S}_{0i}]) = [h_i(\mathscr{T})]$ by Theorem 9.1 in [8]. Since $N_i W_i \in [h_i(\mathscr{T})]$, the point W_i must be then in $[\mathscr{T} + \mathscr{S}_{0i}]$ for $i = 1, \ldots, s$, as asserted.

Notice that sequence $\{a_i^{(n)}\}$, appearing in the assumption of the theorem, may be replaced by any sequence $\{b_i^{(n)}\}$ of positive numbers, provided $\{b_i^{(n)} N_i' W_0^{(n)}\}$ converges and the limit of $\{b_i^{(n)} N_i' W_{01}^{(n)} N_i\}$ is not equal to zero.

For regular models Klonecki and Zontek [4] described a method of constructing for every admissible estimator a sequence of UBE's which converges to a given admissible estimator. Now we shall show that the sequence constructed by this method fulfils the assumptions of Theorem 2.5.

THEOREM 2.6. *If a model is regular for C, then every admissible estimator of C is the limit of a sequence of estimators fulfilling the conditions of Theorem 2.5.*

Proof. Let L be an admissible estimator of C. If the model is regular, then there exists a sequence of affine sets $\mathscr{L}_1 = \mathscr{M}_{r\times t}, \mathscr{L}_2, \ldots, \mathscr{L}_{s+1}$ such that $\mathscr{L}_{i+1} = \mathscr{B}(W_i \mid \mathscr{L}_i) \in \mathscr{C}_0^{(i)}$ for $i = 1, \ldots, s$, and $\mathscr{L}_{s+1} = \{L\}$. Without loss of generality we may assume that each $\mathscr{L}_i$ admits the representation $\mathscr{L}_i = L + \mathscr{R}(N_i \otimes I)$, where N_i is idempotent and symmetric. Notice that $N_1, \ldots, N_s$ fulfil condition (ii) of Theorem 2.5.

The construction of a sequence of UBE's converging to a given admissible estimator, described in [4], is as follows.

Let $\mathscr{S}_{0i}$ denote the set of perfect trivial points for $\mathscr{R}(N_i \otimes I)$. For each $i = 1, \ldots, s$ there exist a sequence $\{W_i^{(n)}\} \subset [\mathscr{T}]$, $\{S_i^{(n)}\} \subset \mathscr{S}_{0i}$ and a sequence $\{\alpha_i^{(n)}\}$ of positive numbers such that

$$W_i^{(n)} + S_i^{(n)} \to W_i, \qquad 1 \leqslant i \leqslant s \tag{2.4}$$

and

$$(\alpha_i^{(n)}/\alpha_j^{(n)}) N_i W_j^{(n)} \to 0, \qquad 1 \leqslant i \neq j \leqslant s, \tag{2.5}$$

as $n \to \infty$. Since $N_i S^{(n)} = 0$ for $n = 1, 2, \ldots$, it follows from (2.4) that

(2.6) $$N_i W_i^{(n)} \to N_i W_i, \quad 1 \leqslant i \leqslant s.$$

Letting, for $n = 1, 2, \ldots$,

$$W_0^{(n)} = (W_{01}^{(n)}, W_{02}^{(n)}) = \sum_{i=1}^{s} (1/\alpha_i^{(n)}) W_i^{(n)},$$

notice that $\{W_0^{(n)}\} \subset [\mathcal{T}]$. Finally, let $\{L^{(n)}\}$ be a sequence of estimators of C defined by $W_{01}^{(n)} L^{(n)} = W_{02}^{(n)}$.

To see that $\{L^{(n)}\}$ satisfies condition (i) of Theorem 2.5, it is sufficient to establish that $W_{01}^{(n)}$ are invertible for sufficiently large n.

Suppose to the contrary that there exists a sequence $\{x^{(n)}\} \subset \mathscr{R}^r$ such that $x^{(n)} \to x \neq 0$ and $W_{01}^{(n)} x^{(n)} = 0$ for all $n > n_0$, say. From (2.5) and (2.6) we conclude that, as $n \to \infty$,

(2.7) $$\alpha_i^{(n)} N_i W_0^{(n)} \to N_i W_i, \quad 1 \leqslant i \leqslant s.$$

Hence

$$\lim_{n\to\infty} \alpha_1^{(n)} N_1 W_{01}^{(n)} x = N_1 W_{11} x = 0,$$

so that $(N_1 - N_2)x = 0$. Likewise, we may prove that $(N_i - N_{i+1}) = 0$ for $i = 2, \ldots, s-1$ and that $N_s x = 0$. This contradicts the assumption that $N_1 x = x \neq 0$ and proves that $L^{(n)}$ is a UBE for sufficiently large n.

Condition (2.1) follows from (2.7).

Since, as we have just shown, every sequence of estimators fulfilling the conditions of Theorem 2.5 converges to an admissible estimator and since for regular models every admissible estimator is the limit of a sequence of estimators also fulfilling the conditions of Theorem 1.5, the following result can be stated:

COROLLARY 2.7. *If a model is regular for C, then the limits of sequences fulfilling the conditions of Theorem 2.5. form the minimal complete class.*

Using the above results we may establish conditions under which limits of UBE's at points in $[\mathcal{T}]$ form the minimal complete class. The condition appearing in Theorem 2.8 implies that for every convergent sequence of UBE's at points in $[\mathcal{T}]$ there exists a subsequence to which Theorem 2.5 applies.

THEOREM 2.8. *If for every affine set $\mathscr{L}$ in $\bigcup \mathscr{C}^{(i)}$, $i = 1, \ldots, r$, the set of trivial points and the set of perfect trivial points coincide, then limits of UBE's at points in $[\mathcal{T}]$ form the minimal complete class.*

Proof. In view of Corollary 2.7 it is sufficient to show that (α) the model is regular, and (β) each limit of UBE's is admissible.

(α) Let $\mathscr{L} = L_0 + \mathscr{R}(N \otimes I)$ be as stated in the theorem and take any point $W = (W_1, W_2)$ in $\mathscr{A}(\mathscr{L})$. Since under the assumption of the theorem W is also a perfect trivial point for

$$\mathscr{B}(W \mid \mathscr{L}) \in \bigcup_{i=1}^{r} \mathscr{C}^{(i)},$$

it follows that $\mathscr{R}(N' W_1) + \mathscr{R}(N' W_2) \subset \mathscr{R}(N' W_1 N)$. Now this in turn implies that $\mathscr{A}(\mathscr{L}) \subset \mathscr{A}_0(\mathscr{R}(N \otimes I))$, which shows that the model is regular.

(β) For every $n = 1, 2, \ldots$ let $L^{(n)}$ be a UBE of C at a point $W_0^{(n)} = (W_{01}^{(n)}, W_{02}^{(n)})$ in $[\mathscr{T}]$, i.e. let

$$W_{01}^{(n)} L^{(n)} = W_{02}^{(n)}, \tag{2.8}$$

where $W_0^{(n)} \in [\mathscr{T}]$, while $W_{01}^{(n)}$ is non-singular. Assume that the sequence $\{L^{(n)}\}$ is convergent.

To show that there is a subsequence of $\{W_0^{(n)}|$ for which condition (2.1) is fulfilled, assume, to the contrary, that for some $N = N_j$, where $1 \leqslant j \leqslant s$, each subsequence of $\{a_j^{(n)} N' W_0^{(n)}\}$ is not convergent. Without loss of generality we may assume that $\|N' W_{01}^{(n)} N\| \equiv 1$. In this case there exists a subsequence $\{n_1\}$ of natural numbers and a sequence $\{\alpha^{(n_1)}\}$ of non-negative numbers such that $\alpha^{(n_1)} \to \infty$ and $\alpha^{(n_1)} N' W_0^{(n_1)} \to N' W^* \neq 0$ as $n_1 \to \infty$, where $W^* = (W_1^*, W_2^*) \in [\mathscr{T} + \mathscr{S}_0]$. Because $N' W_1^* L = N' W_2^*$ by (2.8) and because $N' W_1^* N = 0$, it follows that W^* is a trivial point for $L + \mathscr{R}(N \otimes I)$. But $N' W^* \neq 0$ implies that W^* is not a perfect trivial point for $\mathscr{R}(N \otimes I)$, which contradicts the assumption of the theorem. Thus the proof of the theorem is completed.

Remark. Stępniak [10] established that UBE's and their limits form a complete class. Using Stępniak's theorem it would be sufficient to prove part (α), only. In part (β) we have proved that the model is regular, which, as we have already shown, allows to construct a sequence of UBE's converging to an arbitrary admissible estimator.

3. Invariant estimation of variance components. Let X be a random m-vector with mean vector $\mu = A\beta$ and variance-covariance matrix

$$V = \operatorname{cov} X = \sum_{i=1}^{k} \sigma_i V_i,$$

where A is a known $(m \times p)$-matrix, $V_1, \ldots, V_k$ are known $(m \times m)$-matrices, while $\beta \in \mathscr{R}^p$ and $\sigma = (\sigma_1, \ldots, \sigma_k)' \geqslant 0$ are the unknown parameters. Here $\sigma \geqslant 0$ means that the coordinates of σ, called *variance components*, are all non-negative. To avoid some trivialities we assume that

$$\mathscr{R}(A) + \mathscr{R}\Big(\sum_{i=1}^{k} V_i\Big) = \mathscr{R}^m.$$

We are interested in estimation of $C'\sigma$, C being any fixed $k\times t$ matrix, by estimators of the form $(X'L_1X, \ldots, X'L_tX)'$, where L_i, $i=1, \ldots, t$, may be any $m\times m$ symmetric matrix satisfying the condition $L_i = ML_iM$, where $M = I - AA^+$. These estimators, which are translation invariant $X\to X+A\beta$, are called *invariant quadratic estimators* (IQE for short). Notice that $\mathrm{E}MX = 0$ and that $\operatorname{cov} MX = \sum \sigma_i MV_iM$ $(i=1, \ldots, k)$. We assume throughout the paper that $MV_1M, \ldots, MV_kM$ are non-zero commuting n.n.d. matrices and that the covariance operator of $MXX'M$ is given by

$$\operatorname{cov} MXX'M = 2\Big(\sum_{i=1}^{k} \sigma_i MV_iM\Big) \otimes \Big(\sum_{i=1}^{k} \sigma_i MV_iM\Big).$$

The function $C'\sigma$ is said to be *invariantly estimable* if there exists an unbiased IQE of $C'\sigma$.

If a quadratic loss function is assumed, then the invariant quadratic estimation reduces to a linear estimation within a general linear model.

In fact, let $Y = (X'D_1X, \ldots, X'D_rX)'$, where $\{D_1, \ldots, D_r\}$ is a basis of the quadratic subspace spanned by $\{MV_1M, \ldots, MV_kM\}$ consisting of $m\times m$ idempotent symmetric matrices such that $D_iD_j = 0$ for all $i\neq j$ (see [9]). In terms of this new basis the covariance operator of MX becomes

$$\operatorname{cov} MX = \sum_{i=1}^{r} \theta_i D_i, \qquad \text{where } \theta = (\theta_1, \ldots, \theta_r)' = H'\sigma,$$

while H is a $(k\times r)$-matrix. The i-th row of the matrix H consists of coordinates of MV_iM in the basis $\{D_1, \ldots, D_r\}$.

Letting $R = \operatorname{diag}(\operatorname{rank} D_1, \ldots, \operatorname{rank} D_r)$ and introducing for every $(r\times r)$-matrix $A = (a_{ij})$ the notation $A_d = \operatorname{diag}(a_{11}, \ldots, a_{rr})$, we easily find that

$$\mathrm{E}Y = R\theta, \qquad \operatorname{cov} Y = 2R(\theta\theta')_d, \tag{3.1}$$

where θ ranges over the set $\{H'\sigma\colon \sigma\geqslant 0\}$.

This model is a particular case of the general linear model as defined by LaMotte [6]. If a function $C'\sigma$ is invariantly estimable in the original model, then there exists a matrix F such that $C'\sigma = F'\theta$. Moreover, for every IQE of $C'\sigma$ there exists a linear estimator based on Y which is as good (see [2] or [4]). Consequently, the problem of invariant quadratic estimation of $C'\sigma$ in then there exists a matrix F such that $C'\sigma = F'\theta$. Moreover, for every IQE corresponding function $F'\theta$ in the model defined by (3.1).

Now we find implicitly the class of admissible linear estimators of $F'\theta$ in the model (3.1), which can be presented as limits of the sequences of estimators fulfilling the conditions of Theorem 2.5.

Let $\Omega = \{H'\sigma\sigma'H\colon \sigma\geqslant 0\}$ and let $\bar{\mathscr{G}}$ be the closure of

$$\mathscr{G} = \{\omega_d^{-1}\omega\colon \omega\in[\Omega],\ \omega_d \text{ is p.d.}\}.$$

THEOREM 3.1. *The limits of all sequences satisfying the assumptions of Theorem 2.5 form the set*

$$\mathscr{D}^* = \{(I-2(2I+GR)^{-1})R^{-1}F \colon G \in \mathscr{G}\}. \tag{3.2}$$

Proof. For the model (3.1) the relevant $\mathscr{T}$ becomes

$$\mathscr{T} = \{(2R\omega_d + R\omega R,\ R\omega F) \colon \omega \in \Omega\}.$$

For any $r \times r$ symmetric matrix A define $W(A)$ in $\mathscr{W} = \operatorname{span} \mathscr{T}$ by

$$W(A) = (W_1(A),\ W_2(A)) = (2RA_d + RAR,\ RAF).$$

In the sequel we use the fact that $\mathscr{R}(W_1(A)) = \mathscr{R}((W_1(A))_d)$ if $A \geqslant 0$.

For $n = 1, 2, \ldots$ let $L^{(n)}$ be UBE at point $W(\omega^{(n)})$ in $[\mathscr{T}]$, i.e. let $\omega^{(n)}$ be a point in $[\Omega]$ such that $\omega_d^{(n)}$ is p.d. and let $W_1(\omega^{(n)})L^{(n)} = W_2(\omega^{(n)})$.

Letting $G^{(n)} = (\omega_d^{(n)})^{-1}\omega^{(n)}$, we obtain $L^{(n)} = (2I + G^{(n)}R)^{-1}G^{(n)}F$ or $L^{(n)} = (I - 2(2I + G^{(n)}R)^{-1})R^{-1}F$.

If $\lim_{n\to\infty} G^{(n)} = G$, then $\lim_{n\to\infty} L^{(n)} = (I-2(2I+GR)^{-1})R^{-1}F$ since the sequence of the determinants $\{|2I + G^{(n)}R|\}$ is bounded from below by 2^r.

Now we show that if $\{G^{(n)}\}$ converges, then there exists a subsequence of $\{L^{(n)}\}$ to which Theorem 2.5 is applicable.

Let $a_1^{(n)} = \|W_1(\omega^{(n)})\|^{-1}$. Since

$$a_1^{(n)} W(\omega^{(n)}) = a_1^{(n)} \omega_d^{(n)} R(2I + G^{(n)}R,\ G^{(n)}F),$$

and since entries of $a_1^{(n)}\omega_d^{(n)}$ are in the closed interval $[0, 1]$, it follows that there exists a sequence of natural numbers $\{n_1\}$ such that $\{a_1^{(n_1)} W(\omega^{(n_1)})\}$ converges to, say, $W_1 = (W_{11}, W_{12})$ in $\mathscr{W}$ with $W_{11} \neq 0$. If W_{11} is p.d., then $\{L^{(n_1)}\}$ fulfils the assumptions of Theorem 2.5 with $s = 1$.

Otherwise, define $N_2 = I - (W_{11})^+ W_{11} = I - (W_{11})_d^+ (W_{11})_d$ and let $a_2^{(n)} = \|N_2 W_1(\omega^{(n)}) N_2\|^{-1}$. As above, since all entries of $a_2^{(n)} N_2 \omega_d^{(n)}$ are in the closed interval $[0, 1]$, it follows that there exists a subsequence $\{n_2\}$ of $\{n_1\}$ such that $\{a_2^{(n_2)} N_2 W(\omega^{(n_2)})\}$ converges to, say, $N_2 W_2 = N_2(W_{21}, W_{22})$ with $N_2 W_{21} N_2 \neq 0$. If $\mathscr{R}(N_2) = \mathscr{R}(N_2 W_{21} N_2)$, then $\{L^{(n_2)}\}$ fulfils the assumptions of Theorem 2.5 with $s = 2$. Otherwise the above argumentation is continued.

To end the proof we need yet to show that the assumptions of Theorem 2.5 imply the convergence of $\{G^{(n)}\}$.

Thus suppose that

$$W_1 = (W_{11}, W_{12}) = \lim_{n\to\infty} a_1^{(n)} W(\omega^{(n)}), \quad \text{where } a_1^{(n)} = \|W_1(\omega^{(n)})\|^{-1}.$$

Since $\{a_1^{(n)}\omega_d^{(n)}\}$ converges to $\omega_* \neq 0$, say, and since $\mathscr{R}(\omega_*) = \mathscr{R}(W_{11}) = \mathscr{R}((W_{11})_d)$, we may conclude that the limit

$$\lim_{n\to\infty} (I-N_2)\, G^{(n)} \tag{3.3}$$

exists, where $N_2 = I-(W_{11})_d^+ (W_{11})_d$.

If $s = 1$, then $N_2 = 0$ and, consequently, $\{G^{(n)}\}$ is a convergent sequence. In the converse case, suppose that

$$W_2 = (W_{21}, W_{22}) = \lim_{n\to\infty} a_2^{(n)} N_2 W(\omega^{(n)}),$$

where $a_2^{(n)} = \|N_2 W(\omega^{(n)}) N_2\|^{-1}$ for $n = 1, 2, \ldots$

Since $\{a_2^{(n)} N_2 \omega_d^{(n)}\}$ converges, say, to $\omega_{**} \neq 0$, $\mathscr{R}(\omega_{**}) = \mathscr{R}(N_2 W_{21} N_2) = \mathscr{R}(N_2 (W_{21})_d)$ implies that the limit

$$\lim_{n\to\infty} (N_2-N_3)\, G^{(n)} \tag{3.4}$$

exists, where $N_3 = N_2\big(I-(W_{21})_d^+ (W_{21})_d\big)$.

If $s = 2$, i.e. if $N_3 = 0$, then $\{G^{(n)}\}$ is a convergent sequence by (3.3) and (3.4). If $s \neq 2$, then the above argumentation may be continued.

Clearly, if the model is regular, then the set $\mathscr{D}^*$, defined by (3.2), coincides with the class of all admissible estimators of F.

Klonecki and Zontek [4] showed that the variance component model with $k \leqslant 4$ is regular (under assumption that the covariance matrices commute). The theorem below gives a sufficient condition for the model (3.1) to be regular. For completeness we also present a proof of regularity of (3.1) for $k \leqslant 4$. In Section 4 there are given some examples of regular variance components models with k greater than 4.

THEOREM 3.2. *If* (i) *for every* $\mathscr{L} = L_0 + \mathscr{R}(N \otimes I) \in \bigcup_{i=1}^{k} \mathscr{C}^{(i)}$, *all non-zero columns of* $N'H'$ *are linearly independent or if* (ii) $k \leqslant 4$, *then model* (3.1) *is regular for* F.

Proof. In view of Lemma 2.4 it is sufficient to show that

$$\mathscr{A}(\mathscr{L}) \subset \mathscr{A}_0(\mathscr{L}) + \mathscr{S}(\mathscr{L}) \tag{3.5}$$

for each affine set $\mathscr{L}$ in $\bigcup \mathscr{C}^{(i)}$ $(i = 1, \ldots, k)$.

First we show that (3.5) holds for all $\mathscr{L}$ in $\mathscr{C}^{(1)}$. If $\mathscr{L} \in \mathscr{C}^{(1)}$, then there exists a point $W(\omega)$ in $[\mathscr{T}] \backslash \{0\}$ such that $\mathscr{L} = L_0 + \mathscr{R}(N_1 \otimes I)$, where $L_0 = \big(W_1(\omega)\big)^+ W_2(\omega)$. Since $\mathscr{R}\big(W_1(\omega)\big) \subset \mathscr{R}(\omega_d)$, we may assume that $N_1 = I - \omega_d^+ \omega_d$.

Now we find the closure of $\mathscr{T} + \mathscr{S}_0$ and $\mathscr{T} + \mathscr{S}$, where $\mathscr{S}_0$ and $\mathscr{S}$ are the set of trivial and the set of perfect trivial points for $\mathscr{L}$, respectively.

Let $\{W(H'\sigma_n\sigma_n' H+S^{(n)})\}$ be a convergent sequence with limit $W(\omega_1)$, say, where $\{H'\sigma_n\sigma_n'H\}\subset\Omega$, while $\{W(S^{(n)})\}\subset\mathscr{S}_0$.

Define $\mathscr{I}_1$ as the set of numbers of these columns of $N_1 H'$ which are equal to the zero vector and let P_1 denote the diagonal matrix in $\mathscr{M}_{k\times k}$ with the i-th diagonal element equal to 0 or 1 if $i\in\mathscr{I}_1$ or $i\in\{1,\ldots,k\}\setminus\mathscr{I}_1$, respectively.

Because the i-th column of $N_1 H'$ has non-negative elements and is non-zero for $i\in\mathscr{I}_1$, we conclude that, as $n\to\infty$,

$$W\big(H'(I-P_1)\sigma_n\sigma_n'(I-P_1)H+S^{(n)}\big)\to W(S)\in\mathscr{S}_0, \tag{3.6}$$

say, and that there exists a convergent subsequence of $\{P_1\sigma_n\}$. To avoid double indexed sequences assume that $\{P_1\sigma_n\}$ is convergent. Thus, as $n\to\infty$,

$$W(H'P_1\sigma_n\sigma_n'P_1H)\to W(H'\sigma_*\sigma_*'H)\in\mathscr{T},\qquad\text{where } \sigma_*=\lim_{n\to\infty}P_1\sigma_n. \tag{3.7}$$

Likewise, we may infer from (3.6) and (3.7) that

$$W\big(H'(P_1\sigma_n\sigma_n'(I-P_1)+(I-P_1)\sigma_n\sigma_n'P_1)H\big)\to W(H'MH)\in\mathscr{M},$$

say, where $\mathscr{M}=\{W\big(H'(ab'+ba')H\big)\colon a\in\mathscr{R}(P_1),\ b\in\mathscr{N}(P_1)\}$. Thus $W(\omega_1)\in\mathscr{T}+\mathscr{M}+\mathscr{S}_0$. From this it follows that the closure of $\mathscr{T}+\mathscr{S}_0$ is contained in the cone $[\mathscr{T}]+[\mathscr{M}]+\mathscr{S}_0$. Hence $[\mathscr{T}+\mathscr{S}_0]\subset[\mathscr{T}]+[\mathscr{M}]+\mathscr{S}_0$. But $\mathscr{M}\subset[\mathscr{T}+\mathscr{S}_0]$, so

$$[\mathscr{T}+\mathscr{S}_0]=[\mathscr{T}]+[\mathscr{M}]+\mathscr{S}_0. \tag{3.8}$$

Similarly, we can show that

$$[\mathscr{T}+\mathscr{S}]=[\mathscr{T}]+[\mathscr{M}]+\mathscr{S}. \tag{3.9}$$

Now let $W(\omega_*)=W(H'\Delta_*H)\in\mathscr{A}(\mathscr{L})$. In view of (2.9) we may assume without loss of generality that $W(\omega_*)\in[\mathscr{T}]+[\mathscr{M}]$.

To show that $W(\omega_*)\in\mathscr{A}_0(\mathscr{L})+\mathscr{S}$, we need the following notation.

Let $N_2=N_1\big(I_r-(\omega_*)_d^+(\omega_*)_d\big)$ and let P_2 denote the diagonal matrix in $\mathscr{M}_{k\times k}$ with the i-th diagonal element equal to 0 or 1 when the i-th column of $N_1 H'$ is zero or a non-zero vector, respectively.

The defined matrices P_1 and P_2 have the property

$$N_i W(A)=N_i W(H'P_i\Delta H),\qquad i=1,2, \tag{3.10}$$

where $A=H'\Delta H$. Moreover, since $P_1\Delta P_1$ is n.n.d. by the assumption that $W(\omega_*)\in[\mathscr{T}]+[\mathscr{M}]$ and since $P_2\Delta_*P_2=0$, we get

$$P_1\Delta_*P_2=0. \tag{3.11}$$

Put $\omega_{**}=H'(P_2\Delta_*+\Delta_*P_2)H$ and decompose $W(\omega_*)$ as $W(\omega_*)$

$= W(\omega_* - \omega_{**}) + W(\omega_{**})$. Now we show that $W(\omega_* - \omega_{**}) \in \mathscr{A}_0(\mathscr{L})$ and that $W(\omega_{**}) \in \mathscr{S}$.

To prove the former inclusion first note that the entries of the matrix $\Delta_* - P_2 \Delta_* - \Delta_* P_2$ are non-negative. Hence $W(\omega_* - \omega_{**}) \in [\mathscr{T}] + [\mathscr{M}] \subset [\mathscr{T} + \mathscr{S}_0]$. From this and from the evident equality $N_1 W(\omega_* - \omega_{**}) = (N_1 - N_2) W(\omega_* - \omega_{**})$ it is clear that the desired inclusion holds.

To show that $W(\omega_{**}) \in \mathscr{S}$ assume first that a model satisfies condition (a) of the theorem.

Because $N_2 H'$ has exactly $\operatorname{rank}(N_2 H')$ non-zero columns, therefore $P_2 = (N_2 H')^+ N_2 H'$. This, (3.10) and (3.11) imply that

$$N_1 W_1(\omega_{**}) L_0 - N_1 W_2(\omega_{**}) = N_1 R H' P_2 \Delta_* H(RL_0 - F)$$
$$= N_1 RH'(N_2 H')^+ (N_2 \omega_* RL_0 - N_2 \omega_* F)$$
$$= N_1 RH'(N_2 H')^+ R^{-1} (N_2 W_1(\omega_*) L_0 - N_2 W_2(\omega_*)) = 0$$

and that $N_1 W_1(\omega_{**}) N_1 = 0$, which proves that $W(\omega_{**}) \in \mathscr{S}$. Now assume that $k \leqslant 4$. If $\operatorname{rank}(I - P_1) > 1$, then $\operatorname{rank} P_2 \leqslant 1$. Hence $N_2 H'$ has exactly $\operatorname{rank} P_2$ non-zero columns. Thus we can conclude as above that $W(\omega_{**}) \in \mathscr{S}_0$.

It remains to verify the case where $\operatorname{rank}(I - P_1) = 1$. Using (3.10) and (3.11) simple algebra shows that

$$N_2 RH' P_2 \Delta_* (I - P_1) H(RL_0 - F) = N_2 W_1(\omega_{**}) L_0 - N_2 W_2(\omega_{**}) = 0.$$

Because $(I - P_1) \Delta_* P_2 HRN_2$ is non-zero when $(I - P_1) \Delta_* P_2 \neq 0$, therefore $\mathscr{R}(I - P_1) \subset \mathscr{N}((RL_0 - F)' H')$ by the assumption that $\operatorname{rank}(I - P_1) = 1$. This implies that

$$N_1 W_1(\omega_{**}) L_0 - N_1 W_2(\omega_{**}) = N_1 RH' P_2 \Delta_* (I - P_1) H(RL_0 - F) = 0.$$

Since $N_1 W_1(\omega_{**}) N_1 = 0$, we obtain that $W(\omega_{**}) \in \mathscr{S}$.

The proof of (3.5) for $\mathscr{L} \in \mathscr{C}^{(i)}$, where $i = 2, \ldots, k$, is similar as for $i = 1$, and is, therefore, omitted.

Remark. From the proof of the theorem it is clear that the matrix N, appearing in condition (i), may be assumed to be idempotent and diagonal.

If a model does not fulfil assumptions (i) and (ii) of Theorem 3.2, then this does not imply yet that the model is not regular for F. For instance, as it was shown by Klonecki and Zontek [5], each model defined by (3.1) is regular for $F = I$. In the case where the assumptions of this theorem are not satisfied, condition (3.5) may not hold for some F. But this condition does not characterize regularity of a model.

COROLLARY 3.3. *If the model* (3.1) *is regular for* F, *then the set* $\mathscr{D}^*$ *coincides with the minimal complete class.*

For the two variance components model Gnot and Kleffe [1] showed that UBE's and some of their limits form the minimal complete class. The sequences of UBE's, considered by them, fulfil the assumptions of Theorem 2.5.

It is an open problem whether $\mathscr{D}^*$ forms the minimal complete class in the general case.

4. Examples of regular variance components models. In view of Theorem 3.2 each model with $k \leqslant 4$ variance components is regular for any estimable function $C'\sigma$ under the assumptions specified at the beginning of Section 3.

Now we give examples of models which fulfil assumption (i) of Theorem 3.2.

Example 4.1. Assume that

$$\mathscr{R}(MV_1 M) \subsetneq \mathscr{R}(MV_2 M) \subsetneq \ldots \subsetneq \mathscr{R}(MV_k M). \tag{4.1}$$

Letting $H = (h_{ij})$, we notice that in this case there exist natural numbers $1 \leqslant n_1 < n_2 < \ldots < n_k = r$ such that, for $i = 1, \ldots, k$,

$$\begin{aligned} h_{ij} &\neq 0 \quad \text{for } j = 1, \ldots, n_i, \\ h_{ij} &= 0 \quad \text{for } j = n_i+1, \ldots, r. \end{aligned} \tag{4.2}$$

Let N be a non-zero idempotent diagonal matrix in $\mathscr{M}_{r\times r}$. If

$$L_0 + \mathscr{R}(N \otimes I) \in \bigcup_{i=1}^{k} \mathscr{C}^{(i)},$$

then there exists a u, $1 \leqslant u < k$, such that

$$N = diag(0, \ldots, \underset{(n_u)}{0}, 1, \ldots, 1)$$

by (4.1). It is clear now that each of the n_u first columns of NH' is the zero-vector and that the remaining columns are linearly independent. This shows that a model satisfying (4.2) is regular for each $F \in \mathscr{M}_{r\times t}$.

For instance, condition (4.1) is fulfilled by the random balanced nested classification model given by (see [3])

$$\begin{gathered} X = \mathbf{1}_{p_1} \otimes \ldots \otimes \mathbf{1}_{p_k}, \\ V_i = I_{p_1} \otimes \ldots \otimes I_{p_i} \otimes J_{p_{i+1}} \otimes \ldots \otimes J_{p_k}, \quad i = 1, \ldots, k-1, \\ V_k = I_{p_1} \otimes \ldots \otimes I_{p_k}, \end{gathered}$$

where $\mathbf{1}_p$ is the p-vector of one's, while $J_p = \mathbf{1}_p \mathbf{1}_p'$.

Example 4.2. Assume that H is $(r \times r)$-matrix of the form

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \dots & h_{1r} \\ 0 & h_{22} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & h_{rr} \end{bmatrix}, \tag{4.3}$$

where $h_{1i} \geqslant 0$ and $h_{ii} > 0$ for $i = 1, \dots, r$.

It is easy to see that condition (a) of Theorem 3.2 is fulfilled for each diagonal idempotent matrix N such that the first diagonal element of N is equal to 1.

Now assume that the first diagonal element of N is equal to 0. Then

$$L_0 + \mathscr{R}(N \otimes I) \in \bigcup_{i=1}^{k} \mathscr{C}^{(i)}$$

for an $L_0 \in \mathscr{M}_{r \times t}$ iff $N = N_0 N$, where N_0 is the diagonal matrix with the i-th diagonal element equal to 0 or 1 depending on whether $h_{1i} > 0$ or $h_{1i} = 0$, respectively, for $i = 1, \dots, r$. For such N the matrix NH' is diagonal. This shows that N satisfies assumption (i) of Theorem 3.2 and proves that the model is regular.

The matrix H has structure (4.3) for the random balanced multi-way ANOVA model given by

$$X = \mathbf{1}_{p_1} \otimes \dots \otimes \mathbf{1}_{p_{k+1}},$$

$$V_1 = I_{p_1} \otimes J_{p_2} \otimes \dots \otimes J_{p_{k+1}},$$

$$V_i = J_{p_1} \otimes \dots \otimes J_{p_{i-1}} \otimes I_{p_i} \otimes J_{p_{i+1}} \otimes \dots \otimes J_{p_{k+1}}, \quad i = 2, \dots, k,$$

$$V_{k+1} = I_{p_1} \otimes \dots \otimes I_{p_{k+1}}.$$

Another example of a model for which H has structure (4.3) is a model fulfilling the following assumption:

$$\mathscr{R}(MV_i M) \cap \mathscr{R}(MV_j M) = \{0\} \quad \text{for } i \neq j = 1, \dots, k.$$

For this model the matrix H is diagonal.

Acknowledgement. I am greatly indebted to Professor Witold Klonecki for his helpful suggestions during the preparation of the manuscript.

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Received on 2. 4. 1987