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ON LEVY'S AND DUDLEY'S TYPE ESTIMATES OF THE RATE CONVERGENCE IN THE CENTRAL LIMIT THEOREM FOR FUNCTIONS OF THE AVERAGE OF INDEPENDENT RANDOM VARIABLES

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Abstract. The Lévy and the Dudley metrics are used to give estimates of the rate convergence in the central limit theorem for some functions of the average of independent random variables.

1. Introduction and notation. Let $\{X_k, k \geqslant 1\}$ be a sequence of independent random variables and put $S_n = \sum X_k$, $k = 1, 2, \ldots, n$.

The asymptotical normality of $\{g(S_n/n), n \geqslant 1\}$, where g is a real function, was considered for instance in [1], [3], [4], [7], and [8]. We are interested in the rate of convergence in terms of the Lévy and Dudley metrics of the normalized sequence $\{g(S_n/n), n \geqslant 1\}$.

Throughout this paper we shall use the following notation:

$\mathscr{G}$ – the class of the real differentiable functions g such that g' satisfies the Lipschitz condition, i.e.

$$(1) \qquad |g'(x)-g'(y)| < L|x-y|^{\delta}, \quad 0 < \delta \leqslant 1;$$

$\mathscr{D}$ – the class of all sequences $\{d_n, n \geqslant 1\}$ of positive numbers such that $d_n \to \infty$, $n \to \infty$;

(D, L) – the metric space of all probability distributions on the real line, with the Lévy distance $L(\cdot, \cdot)$, i.e.

$$(2) \qquad L(F, G) = \inf\{\varepsilon: F(x-\varepsilon)-\varepsilon \leqslant G(x) \leqslant F(x+\varepsilon)+\varepsilon, \ x \in \boldsymbol{R}\}$$

(D is a complete space and convergence in the sense of the Lévy metric is equivalent to convergence in law [2]);

(D, d) – the metric space of all probability distributions on the real line, with the Dudley distance $d(\cdot, \cdot)$, i.e.

$$(3) \qquad d(F, G) = \sup\left|\int f(x)\, d(F-G)(x)\right|$$

where the supremum is taken over all such functions f for which

$$\sup_x |f(x)| \leqslant 1 \quad \text{and} \quad \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|} \leqslant 1$$

(the convergence in the sense of the Dudley metric is equivalent to convergence in law [2]);

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt, \qquad -\infty < x < \infty.$$

2. Convergence in the Lévy metric. First we shall give the estimates in the case where $\{X_k, k \geqslant 1\}$ is a sequence of independent identically distributed random variables (i.i.d.r.v.) with $EX_1 = \mu$, $0 < \sigma^2 X_1 = \sigma^2 < \infty$.

Let

$$Z_n = \frac{\sqrt{n}}{\sigma g'(\mu)} \left[g\left(\frac{S_n}{n}\right) - g(\mu) \right], \quad g \in \mathcal{G}. \tag{4}$$

In [7] there was given the estimate (with $\delta = 1$)

$$\sup_x |F_{Z_n}(x) - \Phi(x)| = O\left(\frac{d_n^{1+\delta}}{n^{\delta/2}} + \frac{1}{d_n} e^{-d_n^2/2}\right), \tag{5}$$

where $d_n \in \mathcal{D}$ and F_{Z_n} is the distribution function of Z_n. This note gives the rate of convergence of $\{F_{Z_n}\}$ to Φ in terms of the Lévy and Dudley metrics.

THEOREM 1. *Let $\{X_k, k \geqslant 1\}$ be a sequence of i.i.d.r.v. with* $EX_1 = \mu$, $\sigma^2 X_1 = \sigma^2$, $E|X_1|^3 < \infty$. *Then for every $g \in \mathcal{G}$ with $g'(\mu) \neq 0$, and any sequences $d_n \in \mathcal{D}$ and $\{\varepsilon_n, n \geqslant 1\}$, $\varepsilon_n \to 0$, $n \to \infty$,*

$$L(F_{Z_n}, \Phi) \leqslant C \max\left\{\varepsilon_n, \frac{1}{\sqrt{n}} + \frac{1}{d_n} e^{-d_n^2/2} + \frac{d_n^{1+\delta}}{A\sqrt{n}\,\varepsilon_n^{1/\delta}} e^{-\frac{A^2 n}{2}(\varepsilon_n/d_n)^{2/\delta}}\right\}, \tag{6}$$

where $A = \sigma^{-1}(L\theta)^{-1/\delta} |g'(\mu)|^{1/\delta}$.

Particularly, if $\varepsilon_n = A^{-\delta} d_n^{1+\delta} n^{-\delta/2}$,

$$L(F_{Z_n}, \Phi) \leqslant C \max\left\{\frac{d_n^{1+\delta}}{n^{\delta/2}}, \frac{1}{\sqrt{n}} + \frac{1}{d_n} e^{-d_n^2/2}\right\}. \tag{6'}$$

Proof. Note that, for any random variable X and Y, we have

$$F_{X+Y}(x) = P[X+Y < x] \leqslant P[|Y| \leqslant \varepsilon, X < x+\varepsilon] + P[|Y| > \varepsilon]$$
$$\leqslant F_X(x+\varepsilon) + P[|Y| > \varepsilon]$$

and by substituting $X \mapsto X+Y$, $Y \mapsto -Y$, $x \mapsto x-\varepsilon$ we get $F_X(x-\varepsilon) \leqslant F_{X+Y}(x)+\mathrm{P}[|Y|>\varepsilon]$. Hence

$$F_X(x-\varepsilon)-\mathrm{P}[|Y|>\varepsilon] \leqslant F_{X+Y}(x) \leqslant F_X(x+\varepsilon)+\mathrm{P}[|Y|>\varepsilon]. \tag{7}$$

Furthermore,

$$Z_n = \frac{S_n-n\mu}{\sigma\sqrt{n}} + \frac{S_n-n\mu}{\sigma\sqrt{n}}\left(h\left(\frac{S_n}{n}\right)-1\right), \tag{8}$$

where

$$h(x) = \begin{cases} \dfrac{g(x)-g(\mu)}{(x-\mu)g'(\mu)}, & x \neq \mu, \\ 1, & x = \mu, \end{cases}$$

and

$$L(F_{Z_n}, \Phi) \leqslant L(F_{Z_n}, F_{(S_n-n\mu)/\sigma\sqrt{n}}) + L(F_{(S_n-n\mu)/\sigma\sqrt{n}}, \Phi). \tag{9}$$

It is known [6] that

$$L(F_{(S_n-n\mu)/\sigma\sqrt{n}}, \Phi) = O(n^{-1/2}), \tag{10}$$

and, by (6) and (8),

$$F_{(S_n-n\mu)/\sigma\sqrt{n}}(x-\varepsilon_n)-\mathrm{P}[|Y_n|>\varepsilon_n] \leqslant F_{Z_n}(x) \leqslant F_{(S_n-n\mu)/\sigma\sqrt{n}}(x+\varepsilon_n)+\mathrm{P}[|Y_n|>\varepsilon_n], \tag{11}$$

where

$$Y_n = \frac{S_n-n\mu}{\sigma\sqrt{n}}\left(h\left(\frac{S_n}{n}\right)-1\right).$$

Now we shall use the following estimate:

$$\mathrm{P}[|Y_n|>\varepsilon_n] \leqslant C\frac{1}{\sqrt{n}} + 2(1-\Phi(d_n)) + 2\left(1-\Phi\left(A\sqrt{n}\left(\frac{\varepsilon_n}{d_n}\right)^{1/\delta}\right)\right) := \alpha_n. \tag{12}$$

To prove, we note that, for any $\{d_n, n \geqslant 1\} \in \mathscr{D}$,

$$\mathrm{P}[|Y_n|>\varepsilon_n] \leqslant \mathrm{P}\left[\left|\frac{S_n-n\mu}{\sigma\sqrt{n}}\right| \geqslant d_n\right] + \mathrm{P}\left[\left|h\left(\frac{S_n}{n}\right)-1\right| > \frac{\varepsilon_n}{d_n}\right] \tag{13}$$

$$\leqslant 2\sup_x \left|\mathrm{P}\left[\frac{S_n-n\mu}{\sigma\sqrt{n}} < x\right]-\Phi(x)\right| + 2(1-\Phi(d_n)) + \mathrm{P}\left[\left|h\left(\frac{S_n}{n}\right)-1\right| > \frac{\varepsilon_n}{d_n}\right].$$

But

$$\text{(14)}\quad P\left[\left|h\left(\frac{S_n}{n}\right)-1\right|>\frac{\varepsilon_n}{d_n}\right]=P\left[\left|\frac{g'(\mu+\theta(S_n/n-\mu))}{g'(\mu)}-1\right|>\frac{\varepsilon_n}{d_n}\right]$$

$$\leqslant P\left[\left|\frac{S_n-n\mu}{\sigma\sqrt{n}}\right|\geqslant\frac{(|g'(\mu)|/L\theta)^{1/\delta}}{\sigma}\left(\frac{\varepsilon_n}{d_n}\right)^{1/\delta}\sqrt{n}\right]$$

$$\leqslant 2\sup_x\left|P\left[\left|\frac{S_n-n\mu}{\sigma\sqrt{n}}\right|<x\right]-\Phi(x)\right|+2\left(1-\Phi\left(\frac{(|g'(\mu)|/L\theta)^{1/\delta}}{\sigma}\left(\frac{\varepsilon_n}{d_n}\right)^{1/\delta}\sqrt{n}\right)\right)$$

as $0<\theta<1$.

Putting $A=(|g'(\mu)|/\sigma^\delta L\theta)^{1/\delta}$ and combining (13), (14), we obtain (12). Now (11) can be written as follows:

$$F_{(S_n-n\mu)/\sigma\sqrt{n}}(x-\varepsilon_n)-\alpha_n\leqslant F_{Z_n}(x)\leqslant F_{(S_n-n\mu)/\sigma\sqrt{n}}(x+\varepsilon_n)+\alpha_n.$$

Hence, since the distribution function is nondecreasing, we have

$$\text{(15)}\quad F_{(S_n-n\mu)/\sigma\sqrt{n}}(x-\eta_n)-\eta_n\leqslant F_{Z_n}(x)\leqslant F_{(S_n-n\mu)/\sigma\sqrt{n}}(x+\eta_n)+\eta_n,$$

where $\eta_n=\max\{\varepsilon_n,\alpha_n\}$.

We observe that the infimum of numbers η_n for which (15) holds is the Lévy distance of F_{Z_n} and $F_{(S_n-n\mu)/\sigma\sqrt{n}}$. Thus $L(F_{Z_n}, F_{(S_n-n\mu)/\sigma\sqrt{n}})\leqslant\eta_n$, which, together with (9) and (10), completes the proof of Theorem 1.

Let now Φ be the class of all functions φ defined on R and satisfying the condition:

(16) φ is nonnegative, even, and nondecreasing on $[0,\infty]$, $x/\varphi(x)$ is defined for all x and nondecreasing on $[0,\infty]$.

THEOREM 2. *Let* $\{X_k, k\geqslant 1\}$ *be a sequence of i.i.d.r.v. with* $EX_1=\mu$, $\sigma^2X_1=\sigma^2$, $E(X_1-\mu)^2\varphi(X_1-\mu)<\infty$, *for some* $\varphi\in\Phi$. *Then, for every* $g\in\mathcal{G}$ *with* $g'(\mu)\neq 0$ *and any sequence* $\{d_n, n\geqslant 1\}\in\mathcal{D}$,

$$\text{(17)}\quad L(F_{Z_n},\Phi)\leqslant C\max\left\{\frac{d_n^{1+\delta}}{n^{\delta/2}},\frac{1}{\varphi(\sigma\sqrt{n})}+\frac{1}{d_n}e^{-d_n^2/2}\right\}.$$

Proof. It is enough to replace in the proof of Theorem 1 the estimate (10) by (cf. [5])

$$\text{(18)}\quad L(F_{(S_n-n\mu)/\sigma\sqrt{n}},\Phi)=O\left(\frac{1}{\varphi(\sigma\sqrt{n})}\right).$$

We now give the uniform estimates in the case where $\{X_k, k\geqslant 1\}$ is a sequence of nonidentically distributed random variables.

Put

(19) $$\mu_n = \frac{1}{n}\sum_{k=1}^{n} \mathrm{E}X_k, \quad s_n^2 = \sum_{k=1}^{n} \sigma^2 X_k, \quad X_k^0 = X_k - \mathrm{E}X_k,\ k \geqslant 1,$$

and

(20) $$V_n = \frac{n}{s_n g'(\mu_n)}\left[g\left(\frac{S_n}{n}\right) - g(\mu_n)\right].$$

THEOREM 3. *Let $\{X_k, k \geqslant 1\}$ be a sequence of independent random variables such that $\mathrm{E}(X_k^0)^2 \varphi(X_k^0) < \infty$, $k \geqslant 1$, for some $\varphi \in \Phi$. Then, for every $g \in \mathscr{G}$ with $g'(\mu_n) \neq 0$, $n \geqslant 1$, and any sequence $\{d_n, n \geqslant 1\} \in \mathscr{D}$,*

(21) $$L(F_{V_n}, \Phi) \leqslant C \max\left\{\frac{s_n^\delta d_n^{1+\delta}}{n^\delta |g'(\mu_n)|}, \frac{\sum_{k=1}^{n} \mathrm{E}(X_k^0)^2 \varphi(X_k^0)}{s_n^2 \varphi(s_n)} + \frac{1}{d_n} e^{-d_n^2/2}\right\}.$$

If $\varphi(x) \equiv |x|$, i.e. $\mathrm{E}|X_k^0|^3 < \infty$, then

(22) $$L(F_{V_n}, \Phi) \leqslant C \max\left\{\frac{s_n^\delta d_n^{1+\delta}}{n^\delta |g'(\mu_n)|}, \frac{\sum_{k=1}^{n} \mathrm{E}|X_k^0|^3}{s_n^3} + \frac{1}{d_n} e^{-d_n^2/2}\right\}.$$

Proof. Note that

$$V_n = \frac{S_n - n\mu_n}{s_n} + \frac{S_n - n\mu_n}{s_n}\left(\frac{n}{g'(\mu_n)} \frac{g(S_n/n) - g(\mu_n)}{S_n - n\mu_n} - 1\right)$$

and

(23) $$L(F_{V_n}, \Phi) \leqslant L(F_{V_n}, F_{(S_n - n\mu_n)/s_n}) + L(F_{(S_n - n\mu_n)/s_n}, \Phi).$$

It is known (see [5]) that

(24) $$L(F_{(S_n - n\mu_n)/s_n}, \Phi) = O\left(\frac{\sum_{k=1}^{n} \mathrm{E}|X_k^0|^3}{s_n^3}\right) \quad \text{if } \mathrm{E}|X_k^0|^3 < \infty,$$

and

(25) $$L(F_{(S_n - n\mu_n)/s_n}, \Phi) = O\left(\frac{\sum_{k=1}^{n} \mathrm{E}(X_k^0)^2 \varphi(X_k^0)}{s_n^2 \varphi(s_n)}\right) \quad \text{if } \mathrm{E}(X_k^0)^2 \varphi(X_k^0) < \infty.$$

Applying the method used in the proof of Theorem 1 we get

$$P\left[\left|\frac{S_n-n\mu_n}{s_n}\left(\frac{n}{g'(\mu_n)}\frac{g(S_n/n)-g(\mu_n)}{S_n-n\mu_n}-1\right)\right|>\varepsilon_n'\right]$$

$$\leqslant C\frac{\sum_{k=1}^{n}\mathrm{E}|X_k^0|^3}{s_n^3}+2\big(1-\Phi(d_n)\big)+2\left(1-\Phi\left(\frac{n}{s_n}\left(\frac{|g'(\mu_n)|\varepsilon_n}{Ld_n}\right)^{1/\delta}\right)\right):=\alpha_n'$$

and

$$F_{(S_n-n\mu_n)/s_n}(x-\varepsilon_n')-\alpha_n'\leqslant F_{V_n}(x)\leqslant F_{(S_n-n\mu_n)/s_n}(x+\varepsilon_n')+\alpha_n'.$$

Hence, for

$$\eta_n'=\max\{\varepsilon_n',\alpha_n'\}\quad\text{and}\quad\varepsilon_n'=\frac{Ld_n^{1+\delta}}{|g'(\mu_n)|}\left(\frac{s_n}{n}\right)^{\delta}$$

we have

$$F_{(S_n-n\mu_n)/s_n}(x-\eta_n')-\eta_n'\leqslant F_{V_n}(x)\leqslant F_{(S_n-n\mu_n)/s_n}(x+\eta_n')+\eta_n'.$$

Thus $L(F_{V_n},F_{(S_n-n\mu_n)/s_n})\leqslant\eta_n'$ which, together with (23)–(25), completes the proof of Theorem 3.

Note that, putting in (6'), $d_n=\sqrt{2\ln n}$, in (17), $d_n=\sqrt{2\ln[\varphi(\sigma\sqrt{n})+1]}$, in (22),

$$d_n=\left\{2\ln\left(\frac{s_n^3}{\sum_{k=1}^{n}\mathrm{E}|X_k^0|^3}+1\right)\right\}^{1/2},$$

and, in (21),

$$d_n=\left\{2\ln\left(\frac{s_n^2\,\varphi(s_n)}{\sum_{k=1}^{n}\mathrm{E}(X_k^0)^2\,\varphi(X_k^0)}+1\right)\right\}^{1/2},$$

we can give the following estimates:

COROLLARY 1. *Under the assumptions of Theorem* 1

$$L(F_{Z_n},\Phi)\leqslant C\frac{(\ln n)^{(1+\delta)/2}}{n^{\delta/2}}.\tag{26}$$

COROLLARY 2. *Under the assumptions of Theorem* 2

$$L(F_{Z_n},\Phi)\leqslant C\max\left\{\frac{1}{\varphi(\sigma\sqrt{n})},\ \frac{\{\ln[\varphi(\sigma\sqrt{n})+1]\}^{(1+\delta)/2}}{n^{\delta/2}}\right\}.\tag{27}$$

COROLLARY 3. *Under the assumptions of Theorem* 3

$$L(F_{V_n}, \Phi) \leqslant C \max\left\{\frac{\sum_{k=1}^{n} \mathrm{E}(X_k^0)^2 \varphi(X_k^0)}{s_n^2 \varphi(s_n)}, \frac{s_n^\delta (\ln \varphi(s_n))^{(1+\delta)/2}}{n^\delta |g'(\mu_n)|}\right\}$$

if $\mathrm{E}(X_k^0)^2 \varphi(X_k^0) < \infty$, *and*

$$L(F_{V_n}, \Phi) \leqslant C \max\left\{\frac{\sum_{k=1}^{n} \mathrm{E}(X_k^0)^3}{s_n^2}, \frac{s_n^\delta (\ln s_n)^{(1+\delta)/2}}{n^\delta |g'(\mu_n)|}\right\}$$

if $\mathrm{E}|X_k^0|^3 < \infty$.

If $g(x) = x^2$, $\delta = 1$, then we have

COROLLARY 4. *Under the assumptions of Theorem* 1, *we have*

$$\mathrm{P}\left[\frac{S_n^2 - (n\mu)^2}{n^{3/2}} < 2\sigma\mu x\right] \leqslant C \max\left\{\frac{d_n^2}{\sqrt{n}}, \frac{1}{\sqrt{n}} + \frac{1}{d_n} e^{-d_n^2/2}\right\}$$

and, for $d_n = \sqrt{2 \ln n}$,

$$\mathrm{P}\left[\frac{S_n^2 - (n\mu)^2}{n^{3/2}} < 2\sigma\mu x\right] \leqslant C \frac{\ln n}{\sqrt{n}}.$$

3. Convergence in the Dudley metric. We now estimate the rate of convergence in law of the sequences $\{Z_n, n \geqslant 1\}$ (see (4)) and $\{V_n, n \geqslant 1\}$ (see (20)) in the Dudley metric.

THEOREM 4. *Let* $\{X_k, k \geqslant 1\}$ *be a sequence of i.i.d.r.v. with* $\mathrm{E}X_1 = \mu$, $\sigma^2 X_1 = \sigma^2$, $\mathrm{E}|X_1|^3 < \infty$. *Then, for every* $g \in \mathcal{G}$ *with* $g'(\mu) \neq 0$,

$$d(F_{Z_n}, \Phi) = O\left(\frac{\mathrm{E}\left|\sum_{k=1}^{n} (X_k - \mu)\right|^{1+\delta}}{n^{1/2+\delta}}\right), \tag{28}$$

where $d(\cdot, \cdot)$ *is given by* (3).

Proof. It is known that

$$d(F_{(S_n - n\mu)/\sigma\sqrt{n}}, \Phi) = O(n^{-1/2}) \tag{29}$$

and

$$d(F_{Z_n}, \Phi) \leqslant d(F_{Z_n}, F_{(S_n - n\mu)/\sigma\sqrt{n}}) + d(F_{(S_n - n\mu)/\sigma\sqrt{n}}, \Phi). \tag{30}$$

By (3) and (1) we get

$$d(F_{Z_n}, F_{(S_n-n\mu)/\sigma\sqrt{n}}) = \sup_f \left|\mathrm{E}f(Z_n) - \mathrm{E}f\left(\frac{S_n-n\mu}{\sigma\sqrt{n}}\right)\right| \leqslant \left|\mathrm{E}Z_n - \mathrm{E}\left(\frac{S_n-n\mu}{\sigma\sqrt{n}}\right)\right|$$

$$\leqslant \mathrm{E}\left|Z_n - \frac{S_n-n\mu}{\sigma\sqrt{n}}\right| = \mathrm{E}\left|\frac{S_n-n\mu}{\sigma\sqrt{n}}\left(\frac{g'(\mu+\theta(S_n/n-\mu))}{g'(\mu)} - 1\right)\right|$$

$$\leqslant \frac{L\theta}{\sigma|g'(\mu)|}\frac{\mathrm{E}\left|\sum_{k=1}^{n}(X_k-\mu)\right|^{1+\delta}}{n^{1/2+\delta}},$$

which completes the proof of Theorem 4.

THEOREM 5. *Let* $\{X_k, k \geqslant 1\}$ *be a sequence of i.r.v. such that* (19) *holds and let* $g \in \mathscr{G}$, $g'(\mu_n) \neq 0$.

(i) *If* $\mathrm{E}(X_k^0)^2\varphi(X_k^0) < \infty$, $k \geqslant 1$ *for* $\varphi \in \Phi$ (*see* (16)), *then*

$$d(F_{V_n}, \Phi) = O\left(\frac{\sum_{k=1}^{n}\mathrm{E}(X_k^0)^2\varphi(X_k^0)}{s_n^2\varphi(s_n)} + \frac{\mathrm{E}\left|\sum_{k=1}^{n}X_k^0\right|^{1+\delta}}{n^\delta s_n|g'(\mu_n)|}\right). \tag{31}$$

(ii) *If* $\mathrm{E}|X_k^0|^3 < \infty$, $k \geqslant 1$, *then*

$$d(F_{V_n}, \Phi) = O\left(\frac{\sum_{k=1}^{n}\mathrm{E}|X_k^0|^3}{s_n^2} + \frac{\mathrm{E}\left|\sum_{k=1}^{n}X_k^0\right|^{1+\delta}}{n^\delta s_n|g'(\mu_n)|}\right). \tag{32}$$

Proof. It is known (cf. [5]) that

$$d(F_{(S_n-n\mu_n)/s_n}, \Phi) = O\left(\frac{\sum_{k=1}^{n}\mathrm{E}(X_k^0)^2\varphi(X_k^0)}{s_n^2\varphi(s_n)}\right) \tag{33}$$

if $\mathrm{E}(X_k^0)^2\varphi(X_k^0) < \infty$, $k \geqslant 1$, and

$$d(F_{(S_n-n\mu_n)/s_n}, \Phi) = O\left(\frac{\sum_{k=1}^{n}\mathrm{E}|X_k^0|^3}{s_n^3}\right) \tag{34}$$

if $\mathrm{E}|X_k^0|^3 < \infty$, $k \geqslant 1$, and

$$d(F_{V_n}, \Phi) \leqslant d(F_{V_n}, F_{(S_n-n\mu_n)/s_n}) + d(F_{(S_n-n\mu_n)/s_n}, \Phi). \tag{35}$$

But

$$d(F_{V_n}, F_{(S_n - n\mu_n)/s_n}) \leqslant \mathrm{E}\left|V_n - \frac{S_n - n\mu_n}{s_n}\right|$$

$$\leqslant \mathrm{E}\left|\frac{S_n - n\mu_n}{s_n}\left(\frac{n}{g'(\mu_n)}\,\frac{g(S_n/n) - g(\mu_n)}{S_n - n\mu_n} - 1\right)\right|$$

$$= \mathrm{E}\left|\frac{S_n - n\mu_n}{s_n}\left(\frac{g'(\mu_n + \theta(S_n/n - \mu_n))}{g'(\mu_n)} - 1\right)\right| \leqslant \frac{L\theta}{n^\delta s_n |g'(\mu_n)|}\,\mathrm{E}\,|S_n - n\mu_n|^{1+\delta},$$

which, together with (33)–(35), completes the proof of Theorem 5.

REFERENCES

[1] T. W. Anderson, *An introduction to multivariate analysis*, Wiley, New York 1958.

[2] R. M. Dudley, *Probabilities and metrics*, Lecture Notes Series No. 45, June 1976.

[3] K. W. Morris and D. Szynal, *On the limiting behaviour of some functions of the average of independent random variables*, Ann. Univ. M. Curie-Skłodowska, Sect. A, 31 (12) (1977), p. 85–95.

[4] — *On the convergence rate in the central limit theorem of some functions of the average of independent random variables*, PAMS 3 (1982), p. 85–95.

[5] V. V. Petrov, *An estimate of the deviation of the distribution of a sum of independent random variables from normal law*, Dok. Akad. Nauk USSR 160 (1965), p. 1013–1015.

[6] Yu. V. Prokhorov, *Convergence of random processes and limit theorems of the probability theory* (in Russian), Teor. Veroyatn. Primen. 1.2 (1956), p. 177–238.

[7] D. Szynal. *On the rate of convergence in the central limit theorem for functions of the average of independent random variables*, PAMS 7 (1986), p. 115–123.

[8] S. S. Wilks, *Mathematical statistics*, New York 1963.

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