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FREE INFINITE DIVISIBILITY FOR GENERALIZED POWER DISTRIBUTIONS WITH FREE POISSON TERM*

BY

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Abstract. We study free infinite divisibility (FID) for a class of generalized power distributions with free Poisson term by using complex analytic methods and free cumulants. In particular, we prove that (i) if X follows the free generalized inverse Gaussian distribution, then the distribution of X^r is FID when $|r| \geqslant 1$; (ii) if S follows the standard semicircle law and $u > 2$, then the distribution of $(S+u)^r$ is FID when $r \leqslant -1$; (iii) if B_p follows the beta distribution with parameters p and $3/2$, then (iii-a) the distribution of B_p^r is FID when $|r| \geqslant 1$ and $0 < p \leqslant 1/2$; (iii-b) the distribution of B_p^r is FID when $r \leqslant -1$ and $p > 1/2$.

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1. INTRODUCTION

In classical probability theory, many people studied infinitely divisible distributions as laws of Lévy processes (e.g. Brownian motions, Poisson processes, Cauchy processes, etc.). A probability measure μ on $\mathbb{R}$ is said to be *infinitely divisible* if for each $n \in \mathbb{N}$ there exists a probability measure ρ_n on $\mathbb{R}$ such that $\mu = \rho_n^{*n}$ where $*$ is the classical convolution. We recall that a probability measure μ on $\mathbb{R}$ is infinitely divisible if and only if its characteristic function $\hat{\mu}$ has the Lévy–Khinchin representation

$$\hat{\mu}(u) = \exp\left(i\eta u - \frac{1}{2}au^2 + \int_{\mathbb{R}} \left(e^{iut} - 1 - iut1_{[-1,1]}(t)\right) d\nu(t)\right), \quad u \in \mathbb{R},$$

where $\eta \in \mathbb{R}$, $a \geqslant 0$ (the *Gaussian component*) and ν is the *Lévy measure* on $\mathbb{R}$, that is, $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge t^2)\, d\nu(t) < \infty$. The triplet (η, a, ν) is uniquely de-

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termined and is called the *characteristic triplet*. For example, the normal distribution $N(m, \sigma^2)$ (with mean $m \in \mathbb{R}$ and variance $\sigma > 0$) has the characteristic triplet $(m, \sigma^2, 0)$ (so that it is infinitely divisible). In general, it is very difficult to find the characteristic triplet of probability measures. Many people studied subclasses of probability measures as criteria for infinite divisibility. In particular, these subclasses were studied to understand infinitely divisible distributions which are preserved by powers, products and quotients of independent random variables. We say that the distribution of X belongs to the ME (*mixture of exponential distributions*) class if X is of the form EZ where the random variables E, Z are independent, E follows an exponential distribution and Z is positive. By the Goldie–Steutel theorem [9], [16], the ME class is a subclass of infinitely divisible distributions. This class is important to understand infinitely divisible distributions because is preserved by powers and products of independent random variables: if the distribution of X belongs to the ME class then so does the distribution of X^r when $r \geqslant 1$, and if X, Y are independent and their distributions belong to the ME class then so does the distribution of XY. As one of the most important subclasses of infinitely divisible distributions, the GGC (*generalized gamma convolution*) class was introduced by Thorin [18], [19]. Bondesson [6] proved that the HCM (*hyperbolically completely monotone*) class which is a subclass of GGC is preserved by powers (if the distribution of X belongs to HCM then so does the distribution of X^r if $|r| \geqslant 1$), products and quotients of independent random variables. Moreover Bondesson [7] proved that if the distributions of X, Y belong to GGC and are independent then the distribution of XY belongs to GGC. However, we do not know whether the GGC class is preserved by powers of random variables.

In free probability theory, we have the corresponding problems for *freely infinitely divisible* (for short, FID) distributions which will be defined in Section 2.2. There are two important subclasses of freely infinitely divisible distributions: the FR (*free regular*) class and the UI (*univalent inverse Cauchy transform*) class. Firstly, Pérez-Abreu and Sakuma [15] introduced the FR class in terms of the Bercovici–Pata bijection. In [2], the FR class was developed as a characterization of nonnegative free Lévy processes. Next, the UI class was introduced by Arizmendi and Hasebe [1] as a subclass of freely infinitely divisible distributions. We will give its precise definition in Section 2.3. We do not know whether the classes FR and UI are closed with respect to powers, products and quotients of free independent random variables. Hasebe [11] studied the powers and products for the classes FR and UI by using complex analytic methods.

In this paper, we study free infinite divisibility for *generalized power distributions with free Poisson term* (for short, GPFP distributions). A probability measure on $\mathbb{R}$ is said to be a GPFP distribution if its PDF (probability density function) is given by

$$\frac{\sqrt{(b-x)(x-a)}}{x} \sum_{k=1}^{N} \frac{\alpha_k}{x^{l_k}} 1_{(a,b)}(x) \tag{1.1}$$

for some $0 < a < b$, $N \in \mathbb{N}$, $\alpha = (\alpha_1, \ldots, \alpha_N) \in (0, \infty)^N$ such that (1.1) is a PDF and $l \in \mathbb{R}^N_< := \{(l_1, \ldots, l_N) \in \mathbb{R}^N : l_1 < \cdots < l_N\}$. We write GPFP$(a, b, N, \alpha, l)$ for the distribution whose PDF is given by (1.1). We prove that if a random variable X follows the GPFP distribution then

(i) the distribution of X^r belongs to the UI class if $r \geqslant 1$ when $l \in [t, t+1]^N_< := \{(l_1, \ldots, l_N) \in \mathbb{R}^N : t \leqslant l_1 < \cdots < l_N \leqslant t+1\}$ for some $t \geqslant 0$;

(ii) the distribution of X^r belongs to UI if $|r| \geqslant 1$ when $l \in [0, 1]^N_<$.

The weak closure of the GPFP class contains the free Poisson distributions which have no atoms at 0, the free positive stable laws with index $1/2$, the free Generalized Inverse Gaussian (fGIG) distributions, the shifted semicircle laws and some beta distributions. From the above results (i) and (ii), we obtain free infinite divisibility for powers of random variables which follow the fGIG distributions, the shifted semicircle laws and the beta distributions.

There are examples of GPFP distributions which are not FID. In more detail, we prove that

(iii) there is a distribution $X \sim$ GPFP(a, b, N, α, l) with $l \notin [0, 1]^N_<$ such that the distribution of X^{-1} is not FID;

(iv) there is a GPFP(a, b, N, α, l) distribution with $l \notin [t, t+1]^N_<$ for any $t \geqslant 0$ that is not FID.

This paper consists of five sections. In Section 2, we introduce freely infinitely divisible distributions and the UI class. In Section 3, we prove that probability measures whose PDF satisfied some assumptions are UI by using complex analytic methods (see Theorem 3.1). In Section 4, we first give examples of GPFP distributions and distributions contained in the weak closure of the GPFP class (see Example 4.1). Next we prove the above results (i) and (ii) (see Theorem 4.1 and Corollary 4.1, respectively). Moreover we give applications of (i) and (ii) (see Example 4.2). In Section 5, we introduce the concept of free cumulants and give the moment-cumulant formula. Finally, we prove (iii) and (iv) by using free cumulants (see Examples 5.1 and 5.2, respectively).

2. PRELIMINARIES

2.1. Notations. In this paper, we use the following notations.

- Let X be a (non-commutative) random variable and μ a probability measure on $\mathbb{R}$. The notation $X \sim \mu$ means that X follows μ.
- $\mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $\mathbb{C}^- := \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$
- $x \pm i0 = \lim_{y \to 0\pm}(x + iy)$ and $f(x \pm i0) = \lim_{y \to 0\pm} f(x + iy)$.

- $I \pm i0 := \{x \pm i0 : x \in I\}$ (I an interval in $\mathbb{R}$).
- The complex function z^p ($p \in \mathbb{R}$) means the principal value defined on $\mathbb{C} \setminus (-\infty, 0]$. The complex function $\arg(z)$ means the argument of z defined on $\mathbb{C} \setminus (-\infty, 0]$ taking values in $(-\pi, \pi)$.
- $\mathbb{R}^N_< := \{(l_1, \dots, l_N) \in \mathbb{R}^N : l_1 < \cdots < l_N\}$ ($N \in \mathbb{N}$).
- $[t, t+1]^N_< := \{(l_1, \dots, l_N) \in \mathbb{R}^N : t \leqslant l_1 < \cdots < l_N \leqslant t+1\}$ ($n \in \mathbb{N}, t \in \mathbb{R}$).

2.2. Analytic tools in free probability theory. Let μ be a probability measure on $\mathbb{R}$. The *Cauchy transform* of μ is defined by

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z - x}\, d\mu(x), \qquad z \in \mathbb{C} \setminus \mathrm{supp}(\mu). \tag{2.1}$$

In particular, if $X \sim \mu$ then we sometimes write G_X for the Cauchy transform of μ. The function G_μ is analytic on the complex upper half-plane $\mathbb{C}^+$. Bercovici and Voiculescu [5, Proposition 5.4] proved that for any $\gamma > 0$ there exist $\lambda, M, \delta > 0$ such that G_μ is univalent in the truncated cone

$$\Gamma_{\lambda,M} := \{z \in \mathbb{C}^+ : \lambda|\mathrm{Re}(z)| < \mathrm{Im}(z),\ \mathrm{Im}(z) > M\},$$

and the image $G_\mu(\Gamma_{\lambda,M})$ contains the triangular domain

$$\Lambda_{\gamma,\delta} := \{z \in \mathbb{C}^- : \gamma|\mathrm{Re}(z)| < -\,\mathrm{Im}(z),\ \mathrm{Im}(z) > -\delta\}.$$

Therefore the right inverse function G_μ^{-1} exists on $\Lambda_{\gamma,\delta}$. We define the *Voiculescu transform* of μ by

$$\phi_\mu(z) := G_\mu^{-1}(1/z) - z, \qquad 1/z \in \Lambda_{\gamma,\delta}. \tag{2.2}$$

It was proved in [5] (historically, see also [13], [20]) that for any probability measures μ and ν on $\mathbb{R}$, there exists a unique probability measure λ on $\mathbb{R}$ such that $\phi_\lambda(z) = \phi_\mu(z) + \phi_\nu(z)$ for all z in the intersection of the domains of the three transforms. We write $\lambda := \mu \boxplus \nu$ and call it the *additive free convolution* (for short, *free convolution*) of μ and ν. The operation $\boxplus$ is the free analogue of the classical convolution $*$. The *free cumulant transform* of μ is defined by

$$C_\mu(z) := z\phi_\mu(1/z), \qquad z \in \Lambda_{\gamma,\delta}.$$

In particular, if $X \sim \mu$ then we sometimes write C_X for the free cumulant transform of μ. By the definition, $C_{\mu\boxplus\nu}(z) = C_\mu(z) + C_\nu(z)$ for all z in the intersection of the domains of the three transforms. The transform C_μ is the free analogue of $\log \hat{\mu}$.

A probability measure μ on $\mathbb{R}$ is said to be *freely infinitely divisible* if for each $n \in \mathbb{N}$ there exists a probability measure ρ_n on $\mathbb{R}$ such that $\mu = \rho_n^{\boxplus n}$. Bercovici and Voiculescu [5, Theorem 5.10] proved that μ is freely infinitely divisible if and only if the Voiculescu transform ϕ_μ has an analytic continuation to a map from $\mathbb{C}^+$ to $\mathbb{C}^- \cup \mathbb{R}$, and therefore it has the *Pick–Nevanlinna representation*

$$\phi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+xz}{z-x}\, d\sigma(x), \qquad z \in \mathbb{C}^+,$$

for some $\gamma \in \mathbb{R}$ and a nonnegative finite measure σ on $\mathbb{R}$. The pair (γ, σ) is uniquely determined and called the *free generating pair*. In this case, we can rewrite the free cumulant transform C_μ as

$$C_\mu(z) = \eta z + a z^2 + \int_{\mathbb{R}} \left(\frac{1}{1-tz} - 1 - tz 1_{[-1,1]}(t) \right) d\nu(t), \qquad z \in \mathbb{C}^-, \tag{2.3}$$

where $\eta \in \mathbb{R}$, $a \geqslant 0$ (called the *semicircular component*) and ν is a nonnegative measure on $\mathbb{R}$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge t^2)\, d\nu(t) < \infty$ (see [3, Proposition 5.2]). The measure ν is called the *free Lévy measure* of μ. The representation (2.3) is called the *free Lévy–Khinchin representation* and it is the free analogue of the classical Lévy–Khinchin representation of the Fourier transform $\hat{\mu}$. The triplet (η, a, ν) is uniquely determined by μ and called the *free characteristic triplet*. Moreover the relation between the free generating pair (γ, σ) and the free characteristic triplet (η, a, ν) is given by

$$\begin{aligned} a &= \sigma(\{0\}), \qquad d\nu(t) = \frac{1+t^2}{t^2} \cdot 1_{\mathbb{R}\setminus\{0\}}(t) d\sigma(t), \\ \eta &= \gamma + \int_{\mathbb{R}} t \left(1_{[-1,1]}(t) - \frac{1}{1+t^2} \right) d\nu(t). \end{aligned}$$

EXAMPLE 2.1. (1) The *semicircle distribution* $S(m, \sigma^2)$ is the probability measure with PDF

$$\frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - (x-m)^2}\, 1_{(m-2\sigma, m+2\sigma)}(x), \qquad m \in \mathbb{R},\ \sigma > 0.$$

It is known (for example, see [14]) that $C_{S(m,\sigma^2)}(z) = mz + \sigma^2 z^2$ for $z \in \mathbb{C}^-$. Therefore $S(m, \sigma^2)$ is freely infinitely divisible and has free characteristic triplet $(m, \sigma^2, 0)$. This distribution appears as the limit of the eigenvalue distributions of Wigner matrices as the size of the random matrices goes to infinity.

(2) The *free Poisson distribution* (or Marchenko–Pastur distribution) $\mathbf{fp}(p, \theta)$ is the probability measure given by

$$\begin{aligned} &\max\{1-p, 0\}\delta_0 \\ &\qquad + \frac{\sqrt{(\theta(\sqrt{p}+1)^2 - x)(x - \theta(\sqrt{p}-1)^2)}}{2\pi x} 1_{(\theta(\sqrt{p}-1)^2, \theta(\sqrt{p}+1)^2)}(x) dx \end{aligned}$$

for $p, \theta > 0$. In particular, we write $\mathbf{fp}(p) := \mathbf{fp}(p, 1)$. We have $C_{\mathbf{fp}(p,\theta)}(z) = \frac{p\theta z}{1-\theta z}$ for $z \in \mathbb{C}^-$. Therefore $\mathbf{fp}(p, \theta)$ is freely infinitely divisible. This distribution appears as the limit of the eigenvalue distributions of Wishart matrices as the size of the random matrices goes to infinity.

2.3. Univalent inverse Cauchy transform. In Section 2.2, we introduced freely infinitely divisible distributions. Recall that a probability measure is freely infinitely divisible if and only if its free cumulant transform has the free Lévy–Khinchin representation (2.3) (or its Voiculescu transform has the Pick–Nevanlinna representation). However, in general, it is very difficult to find the free characteristic triplet (or free generating pair) of probability measures. Many people studied conditions implying free infinite divisibility.

In particular, Arizmendi and Hasebe [1, Definition 5.1] defined the UI class as a criterion for free infinite divisibility.

DEFINITION 2.1. A probability measure μ on $\mathbb{R}$ is said to be in the UI (*univalent inverse Cauchy transform*) class if the right inverse function G_μ^{-1}, originally defined in a triangular domain $\Lambda_{\gamma,\delta}$, has a univalent analytic continuation to $\mathbb{C}^-$. In particular, if $X \sim \mu$ and $\mu \in$ UI, then we write $X \sim$ UI.

By [1, Proposition 5.2 and p. 2763], the UI class has the following property.

LEMMA 2.1. *$\mu \in$ UI implies that μ is freely infinitely divisible. The UI class is closed with respect to weak convergence.*

EXAMPLE 2.2. (1) The semicircle distribution $S(m, \sigma^2)$ is in UI. The free Poisson distribution $\mathbf{fp}(p, \theta)$ is also in UI (see [11, Section 2.3 (1), (2)]).

(2) The *beta distribution* $\beta_{p,q}$ is the probability measure with PDF

$$\frac{1}{B(p,q)} x^{p-1}(1-x)^{q-1} 1_{(0,1)}(x), \quad p, q > 0.$$

The beta distribution $\beta_{p,q}$ is in UI if (i) $p, q \geqslant 3/2$, (ii) $0 < p \leqslant 1/2$, $q \geqslant 3/2$ or (iii) $0 < q \leqslant 1/2$, $p \geqslant 3/2$ (see [10, Theorem 1.2] and [11, Theorem 3.4]).

3. UI PROPERTY

In this section, $\tilde{G}_\mu(z)$ denotes the Cauchy transform of a probability measure μ on $\mathbb{R}$ defined in (2.1). In [11, Lemma 3.11], Hasebe mentioned that the Cauchy transform of μ has an analytic continuation to $\mathbb{C}^+ \cup \operatorname{supp}(\mu) \cup \mathbb{C}^-$ when the PDF of μ has some analytic properties. We prove an assertion almost the same as [11, Lemma 3.11].

LEMMA 3.1. *Let $0 < a < b$. Let σ be a probability measure on (a, b) whose PDF is $f(x)$ with respect to Lebesgue measure. Assume that f is analytic in a neighborhood of $(a, b) \cup \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$ for some $0 < \theta \leqslant \pi$. Then* *the Cauchy transform* $G_\sigma := \tilde{G}_\sigma|_{\mathbb{C}^+}$ *defined on* $\mathbb{C}^+$ *has an analytic continuation to* $\mathbb{C}^+ \cup (a,b) \cup \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$ (*still denoted by* G_σ), *and*

$$G_\sigma(z) = \tilde{G}_\sigma(z) - 2\pi i f(z) \quad \textit{in } \{z \in \mathbb{C}^- : \arg(z) \in (-\theta,0)\}. \tag{3.1}$$

Proof. The proof is very similar to that of [10, Proposition 4.1], but for the reader's convenience we include it. Since f is analytic in a neighborhood of $(a,b) \cup \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$ for some $0 < \theta \leqslant \pi$, we have

$$G_\sigma(z) = \int_\gamma \frac{f(w)}{z-w}\, dw, \qquad z \in \mathbb{C}^+,$$

where γ is an arbitrary simple arc contained in $\{z \in \mathbb{C}^- : \arg(z) \in (-\theta,0)\}$ except for its endpoints a, b. Therefore G_σ has an analytic continuation to the domain containing $\mathbb{C}^+$, surrounded by γ and (a,b). Since γ is arbitrary, G_σ has an analytic continuation to $\mathbb{C}^+ \cup (a,b) \cup \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$.

We now prove (3.1). For $z \in \mathbb{C}^-$ with $\arg(z) \in (-\theta, 0)$, take a simple arc γ contained in $\{z \in \mathbb{C}^- : \arg(z) \in (-\theta,0)\}$ with endpoints a, b such that the closed curve $\tilde{\gamma} := \gamma \cup [a,b]$ surrounds z. By the residue theorem, we have

$$\int_{\tilde{\gamma}} \frac{f(w)}{z-w}\, dw = 2\pi i \operatorname{Res}_{w=z}\left(\frac{f(w)}{z-w}\right) = -2\pi i f(z). \tag{3.2}$$

Since the LHS of (3.2) equals $G_\sigma(z) - \tilde{G}_\sigma(z)$, the formula (3.1) holds for all $z \in \mathbb{C}^-$ with $\arg(z) \in (-\theta, 0)$. ∎

We now prove that a probability measure whose PDF satisfies the following properties is in the UI class and therefore the measure is freely infinitely divisible.

THEOREM 3.1. *Let* $0 < a < b$. *Let* μ *be a probability measure on* (a,b) *which has a PDF* $f(x)$ *that is real analytic and positive on* (a,b) *and there exists* $\theta \in (0, \pi]$ *such that*

(A1) f *analytically extends to a function* (*still denoted by* f) *defined in* $\{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in (-\theta, 0)\} \cup (a,b)$, *and further to a continuous function on* $\{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in [-\theta, 0]\}$;

(A2) $\mathrm{Re}(f(x - i0)) = 0$ *for all* $0 < x < a$ *and* $x > b$;

(A3) *there exist* $\alpha > 0$ *and* $0 < l < (2\pi - \theta)/\theta$ *such that*

$$f(z) = -\frac{i\alpha}{z^{1+l}}(1 + o(1)) \quad \textit{as } z \to 0,\ \arg(z) \in (-\theta, 0);$$

(A4) $\mathrm{Re}(f(ue^{-i\theta})) \leqslant 0$ *for all* $u > 0$;

(A5) $\lim_{|z|\to\infty,\, \arg(z)\in(-\theta,0)} f(z) = 0$.

(A6) $G(z) := G_\mu(z)$ *has an analytic continuation to* $\mathbb{C}^+ \cup (a,b) \cup \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$ (*still denoted by* G), *and further to a continuous function on* $\mathbb{C}^+ \cup [a,b] \cup \{z \in \mathbb{C}\setminus\{0\} : \arg(z) \in [-\theta, 0)\} \cup ((-\infty, a] \cup [b, \infty) + i0) \cup ([0, a] \cup [b, \infty) - i0)$.

Then $\mu \in UI$.

REMARK 3.1. Before the proof let us remark that the existence of an analytic continuation of G follows from (A1) and Lemma 3.1, so the assumption (A6) is only about the continuous extension.

Proof of Theorem 3.1. We define the following eight lines and curves (see Figure 1). In the following, $\eta > 0$ is supposed to be large and $\delta > 0$ small.

c_1: the real line segment from $-\eta + i0$ to $a + i0$;

c_2: the real line segment from $a - i0$ to $\delta - i0$;

c_3: the clockwise circle $\delta e^{i\psi}$ where ψ starts from 0 and ends with $-\theta$;

c_4: the line segment from $\delta e^{-i\theta} + 0$ to $\eta e^{-i\theta} + 0$ if $\theta \in (0, \pi)$ or the line segment from $-\delta - i0$ to $-\eta - i0$ if $\theta = \pi$;

c_5: the counterclockwise circle $\eta e^{i\psi}$ where ψ starts from $-\theta$ and ends with 0;

c_6: the real line segment from $\eta - i0$ to $b - i0$;

c_7: the real line segment form $b + i0$ to $\eta + i0$;

c_8: the counterclockwise circle $\eta e^{i\psi}$ where ψ starts from 0 and ends with π.

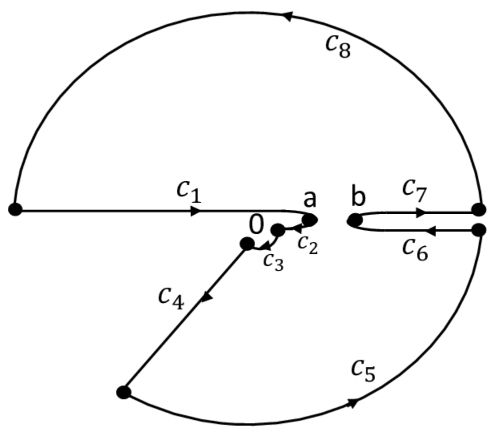

FIGURE 1. The curves c_k

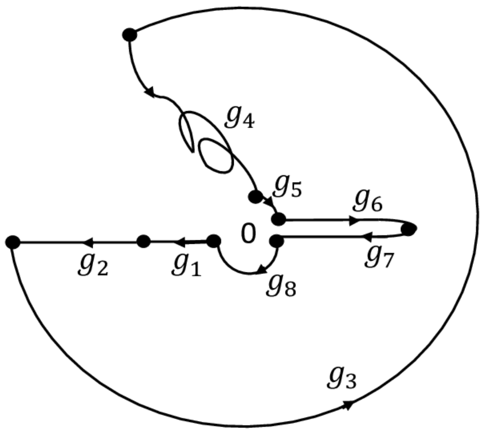

FIGURE 2. The curves g_k

Define the curves $g_k := G(c_k)$ for all $k = 1, \dots, 8$ (see Figure 2). It is easy to prove that g_1 is contained in the negative real line and g_7 is contained in the positive real line. Since

$$G(z) = \tilde{G}(z) = \frac{1}{z}(1 + o(1)) \quad \text{as } |z| \to \infty,\ z \in \mathbb{C}^+,$$

the image g_8 is contained in a ball of some small radius and it is a clockwise curve whose argument starts from 0 and ends with $-\pi$ when η is sufficiently large. By (A6), the curves $g_1 \cup g_2$ and $g_6 \cup g_7$ are continuous since so are $c_1 \cup c_2$ and $c_6 \cup c_7$. By (A2), we have $\mathrm{Re}(f(x - i0)) = 0$ for all $0 < x < a$ and $x > b$. Hence $\mathrm{Im}(G(x-i0)) = 0$ for all $0 < x < a$, $x > b$, and therefore g_2 and g_6 are contained in the real line. By (A1) and (A4), when $\theta \in (0, \pi)$,

$$\mathrm{Im}(G(ue^{-i\theta} + 0)) = \mathrm{Im}(\tilde{G}(ue^{-i\theta})) - 2\pi \mathrm{Re}(f(ue^{-i\theta})) > 0$$

for all $\delta < u < \eta$, and when $\theta = \pi$,

$$\mathrm{Im}(G(-u - i0)) = \mathrm{Im}(\tilde{G}(-u)) - 2\pi \mathrm{Re}(f(-u)) = -2\pi \mathrm{Re}(f(-u)) \geqslant 0$$

for all $\delta < u < \eta$. Therefore g_4 is contained in $\mathbb{C}^+ \cup \mathbb{R}$. Note that $\tilde{G}(z) = o(1/z)$ as $z \to 0$, $z \in \mathbb{C}^-$, since $\lim_{z\to 0,\, z\in\mathbb{C}^-} |\tilde{G}(z)| = \int_a^b \frac{f(t)}{t}\, dt < \infty$. On account of (A3) and (3.1), we have

$$G(z) = -\frac{2\pi\alpha}{z^{1+l}}(1 + \varphi(z)), \tag{3.3}$$

where $\varphi(z) = o(1)$ as $z \to 0$, $\arg(z) \in (-\theta, 0)$.

Let $\epsilon > 0$ be small. By the asymptotics of (3.3), we can take $\delta \in (0, (\pi\alpha\epsilon)^{1/(1+l)})$ small enough such that $|\varphi(\delta e^{i\psi})| < 1/2$ uniformly in $\psi \in (-\theta, 0)$, and therefore

$$|G(\delta e^{i\psi}) + 2\pi\alpha(\delta e^{i\psi})^{-(1+l)}| = 2\pi\alpha\delta^{-(1+l)}|\varphi(\delta e^{i\psi})| < \pi\alpha\delta^{-(1+l)}$$

uniformly in $\psi \in (-\theta, 0)$. Hence by the triangle inequality we have

$$\begin{aligned}|G(\delta e^{i\psi})| &\geqslant |-2\pi\alpha(\delta e^{i\psi})^{-(1+l)}| - |G(\delta e^{i\psi}) + 2\pi\alpha(\delta e^{i\psi})^{-(1+l)}| \\ &> 2\pi\alpha\delta^{-(1+l)} - \pi\alpha\delta^{-(1+l)} = \pi\alpha\delta^{-(1+l)} > 1/\epsilon,\end{aligned}$$

and therefore the distance between the curve g_3 and 0 is larger than $1/\epsilon$. Since $G(\delta e^{i\psi}) = \frac{2\pi\alpha}{\delta^{1+l}} e^{(-\pi-\psi(1+l))i}$ as $\delta \to 0$ and the argument of ψ starts from 0 and ends with $-\theta$, the image g_3 is a large counterclockwise curve whose radius is larger than $1/\epsilon$ and the argument of g_3 starts from $-\pi$ and ends with $-\pi + \theta(1+l)$ ($< \pi$ by (A3)) when δ is sufficiently small.

Note that $\tilde{G}(z) = \frac{1}{z}(1 + o(1))$ as $|z| \to \infty$, $z \in \mathbb{C}^-$. By (A5), we have $|f(z)| \to 0$ as $|z| \to \infty$, $\arg(z) \in (-\theta, 0)$, and therefore

$$|G(z)| \leqslant |\tilde{G}(z)| + 2\pi|f(z)| \to 0$$

as $|z| \to \infty$, $\arg(z) \in (-\theta, 0)$. Hence there exists $\eta > 0$ large enough such that $|z| \geqslant \eta$ implies $|G(z)| < \epsilon$, that is, g_5 is contained in a small circle with radius $\epsilon > 0$.

Therefore every point of $D_\epsilon := \{z \in \mathbb{C}^- : \epsilon < |z| < 1/\epsilon\}$ is surrounded by the closed curve $g_1 \cup \cdots \cup g_8$ exactly once. By the argument principle, for any $w \in D_\epsilon$ there exists only one element z in the bounded domain surrounded by the closed curve $c_1 \cup \cdots \cup c_8$ such that $G(z) = w$. Therefore we can define a right inverse function G^{-1} on D_ϵ. Since $G^{-1}(D_\epsilon)$ is connected and G is univalent on it, the inverse function G^{-1} is univalent on D_ϵ. We can define a univalent right inverse function (with the same symbol G^{-1}) in $\mathbb{C}^-$ by letting $\epsilon \to 0$. By the identity theorem, the right inverse function, originally defined in some triangular domain $\Lambda_{\gamma,\delta}$, has a univalent analytic continuation to our function G^{-1} on $\mathbb{C}^-$. Hence $\mu \in$ UI. ∎

REMARK 3.2. From Figure 2, it may seem that the curves g_1 and g_2 (resp. g_6 and g_7) do not intersect, but this is not necessarily so. However, it does not affect the proof of Theorem 3.1 since every point of D_ϵ is surrounded by the closed curve $g_1 \cup \cdots \cup g_8$ exactly once even if $g_1 \cap g_2 \neq \emptyset$ (resp. $g_6 \cap g_7 \neq \emptyset$).

4. FREE INFINITE DIVISIBILITY FOR GENERALIZED POWER DISTRIBUTIONS WITH A FREE POISSON TERM

Consider a random variable X which follows the free Poisson distribution $\mathbf{fp}(p)$ (see Example 2.1(2)). By [11, Theorem 3.5], if either $p \geqslant 1$ and $r \in (-\infty, 0] \cup [1, \infty)$, or $0 < p < 1$ and $r \geqslant 1$, then the distribution of X^r is FID (UI and FR).

In this section, we study free infinite divisibility for the distribution of X^r when X follows the generalized power distribution with free Poisson term (a GPFP distribution). We give a precise definition of such distributions.

DEFINITION 4.1. A probability measure μ on $\mathbb{R}$ is said to be a *generalized power distribution with free Poisson term* (for short, *GPFP distribution*) if its PDF is given by

$$f_{a,b,N,\alpha,l}(x) := \frac{\sqrt{(b-x)(x-a)}}{x}\sum_{k=1}^{N}\frac{\alpha_k}{x^{l_k}}1_{(a,b)}(x)$$

for some $0 < a < b$, $N \in \mathbb{N}$, $\alpha = (\alpha_1, \dots, \alpha_N) \in (0,\infty)^N$ with $\int_a^b f_{a,b,N,\alpha,l}(x)\,dx = 1$ and $l = (l_1, \dots, l_N) \in \mathbb{R}^N_<$. In this case, the probability measure μ is denoted by GPFP(a, b, N, α, l).

REMARK 4.1. The GPFP class is closed under taking inverses of random variables: if $X \sim$ GPFP(a, b, N, α, l) with $\alpha = (\alpha_1, \dots, \alpha_N) \in (0,\infty)^N$ and $l = (l_1, \dots, l_N) \in \mathbb{R}^N_<$, then

$$X^{-1} \sim \text{GPFP}\bigg(\frac{1}{b}, \frac{1}{a}, N, (\alpha_N\sqrt{ab}, \dots, \alpha_1\sqrt{ab}), (1-l_N, \dots, 1-l_1)\bigg).$$

The GPFP class contains the free Poisson distributions and the free generalized inverse Gaussian (fGIG) distributions. Moreover the weak closure of the GPFP class contains a lot of examples. We give some examples before the main result (Theorem 4.1).

EXAMPLE 4.1. (1) Suppose that $p > 1$, $a = (\sqrt{p}-1)^2$, $b = (\sqrt{p}+1)^2$, $N = 1$, $\alpha = \frac{1}{2\pi}$ and $l = 0$. Then

$$\begin{aligned}\text{GPFP}\bigg(a = (\sqrt{p}-1)^2,\ b = (\sqrt{p}+1)^2,\ N = 1,\ \alpha = \frac{1}{2\pi},\ l = 0\bigg)& \\ = \frac{\sqrt{((\sqrt{p}+1)^2 - x)(x - (\sqrt{p}-1)^2)}}{2\pi x} \cdot 1_{((\sqrt{p}-1)^2, (\sqrt{p}+1)^2)}(x)dx& \\ = \mathbf{fp}(p).&\end{aligned}$$

In this case, the GPFP distribution corresponds to the free Poisson distribution (Marchenko–Pastur distribution) which has no atom at 0.

(2) Consider $\lambda \in \mathbb{R}$ and $\alpha_1, \alpha_2 > 0$. Let $N = 2$, $l = (0,1)$ and $\alpha = \left(\frac{\alpha_1}{2\pi}, \frac{\alpha_2}{2\pi\sqrt{ab}}\right)$ where $0 < a < b$ are the unique solution of

$$\begin{cases} 1 - \lambda + \alpha_1\sqrt{ab} - \alpha_2\frac{a+b}{2ab} = 0, \\ 1 + \lambda + \frac{\alpha_2}{\sqrt{ab}} - \alpha_1\frac{a+b}{2} = 0. \end{cases} \tag{4.1}$$

Then

$$\mathrm{GPFP}\biggl(a,\, b,\, N=2,\, \alpha=\Bigl(\frac{\alpha_1}{2\pi}, \frac{\alpha_2}{2\pi\sqrt{ab}}\Bigr),\, l=(0,1)\biggr)$$
$$= \frac{\sqrt{(x-a)(b-x)}}{2\pi x}\Bigl(\alpha_1 + \frac{\alpha_2}{\sqrt{ab}x}\Bigr)\cdot 1_{(a,b)}(x)dx =: \mathrm{fGIG}(\alpha_1, \alpha_2, \lambda).$$

In this case, the GPFP distribution corresponds to the fGIG distribution. This distribution was studied in [12] (the free version of the generalized inverse Gaussian distributions, free selfdecomposability, free regularity and unimodality for the fGIG distributions) and [17] (the free version of Matsumoto-Yor property, the R-transform of the fGIG distributions).

Next we give examples of distributions in the weak closure of the GPFP class.

(3) Let $n \in \mathbb{N}$ and $b > 0$. Define

$$c(n,b) := \Biggl(\int_{1/n}^{b} \frac{\sqrt{(x-1/n)(b-x)}}{2\pi x}\,dx\Biggr)^{-1}. \tag{4.2}$$

Note that $\lim_{n\to\infty} c(n,b) = 4/b$ for each $b > 0$. We define the following GPFP distribution:

$$\mu_{n,b}(dx) := \mathrm{GPFP}\biggl(\frac{1}{b},\, n,\, N=1,\, \alpha=\frac{c(n,b)}{2\pi}\sqrt{\frac{b}{n}},\, l=1\biggr) \tag{4.3}$$
$$= c(n,b)\sqrt{\frac{b}{n}}\cdot\frac{\sqrt{(x-1/b)(n-x)}}{2\pi x^2}\cdot 1_{(1/b,n)}(x)dx.$$

We can show that $\mu_{n,b} \to \mu_b$ weakly as $n \to \infty$ for each $b > 0$, where

$$\mu_b(dx) := \frac{4}{b}\cdot\frac{\sqrt{bx-1}}{2\pi x^2}\cdot 1_{(1/b,\infty)}(x)dx. \tag{4.4}$$

In particular, if $b = 4$, then the probability measure μ_b corresponds to the free positive stable law with index $1/2$. The free stable laws were introduced in [4].

(4) Let $n \in \mathbb{N}$, $b > 0$. We define the following GPFP distribution:

$$\nu_{n,b}(dx) := \mathrm{GPFP}\biggl(\frac{1}{n}, b, N=1, \alpha=\frac{c(n,b)}{2\pi}, l=0\biggr), \tag{4.5}$$

where $c(n,b)$ is the constant defined in (4.2). We can show that $\nu_{n,b} \to \nu_b$ weakly as $n \to \infty$ for each $b > 0$, where

$$\nu_b(dx) := \frac{2\sqrt{x(b-x)}}{\pi b x} 1_{(0,b)}(x)dx. \tag{4.6}$$

In particular, if $b = 4$, then the probability measure ν_b corresponds to the free Poisson distribution $\mathbf{fp}(1)$. Note that $X \sim \mu_{n,b}$ if and only if $X^{-1} \sim \nu_{n,b}$ for each $n \in \mathbb{N}$ and $b > 0$ by Remark 4.1.

(5) Let $n \in \mathbb{N}$ and $l < 1/2$. Define

$$\alpha(n,l) := \left(\frac{1}{B(1/2-l,3/2)} \int_{1/n}^{1} \frac{\sqrt{(1-x)(x-1/n)}}{x^{1+l}}\, dx\right)^{-1},$$

where $B(p,q)$ is the beta distribution. Note that $\lim_{n\to\infty} \alpha(n,l) = 1$ for each $l < 1/2$. For all $n \in \mathbb{N}$ and $l < 1/2$, we consider the following GPFP distribution:

$$\rho_{n,l}(dx) := \mathrm{GPFP}\left(\frac{1}{n},\ 1,\ N=1,\ \frac{\alpha(n,l)}{B(1/2-l,3/2)},\ l\right). \tag{4.7}$$

We can show that $\rho_{n,l} \to \beta_{1/2-l,3/2}$ weakly as $n \to \infty$ for each $l < 1/2$. Recall that $\beta_{1/2-l,3/2}$ is the beta distribution, defined in Example 2.2(2).

We start to discuss free infinite divisibility for the distribution of X^r where X follows the GPFP distribution.

THEOREM 4.1. *Suppose that* $X \sim \mathrm{GPFP}(a,b,N,\alpha,l)$ *with* $l = (l_1,\ldots,l_N) \in [t,t+1]^N_<$ *for some* $t \geqslant 0$. *Then for* $r \geqslant 1$, $X^r \sim$ *UI.*

Proof. Fix $N \in \mathbb{N}$. Suppose first that $r > 1$ and $l = (l_1,\ldots,l_N) \in [t,t+1]^N_<$ satisfies $t < l_1 < \cdots < l_N < t+1$ for some $t \geqslant 0$. Assume that $[t] = n-1$ for some $n \in \mathbb{N}$. Define $\tilde{l}_k := l_k - [t] = l_k - (n-1) \in (0,1)$ for all $k = 1,\ldots,N$. Then

$$X^r \sim \frac{s\sqrt{(B^s - x^s)(x^s - A^s)}}{x} \sum_{k=1}^{N} \frac{\alpha_k}{x^{s(n-1+\tilde{l}_k)}} \cdot 1_{(A,B)}(x)dx =: h(x)dx,$$

where $s = 1/r \in (0,1)$, $A = a^r$ and $B = b^r$. We define $h_k(x)$ to be the kth term of $h(x)$. Set $n_0 := s(n-1)+1 \geqslant 1$. Note that $s(n-1+\tilde{l}_k) = n_0 - 1 + \tilde{l}_k s$ for each $k = 1,\ldots,N$. Consider $\theta = \pi/n_0$. Let $G = G_{X^r}$ be the Cauchy transform of X^r. We prove that θ, h and G satisfy assumptions (A1)–(A6) of Theorem 3.1.

(A1) Suppose that $z \in \mathbb{C} \setminus \{0\}$ and $\arg(z) \in (-\theta, 0)$. An argument similar to the proof of [11, Theorem 3.5] shows that $\arg((B^s - z^s)(z^s - A^s)) \in (-\pi,\pi)$. Moreover, $\arg(z^{s(n-1+\tilde{l}_k)}) \in (-\theta s(n-1+\tilde{l}_k), 0) \subset (-\pi, 0)$. Thus every h_k analytically extends to a function (still denoted by h_k) defined in $\{z \in \mathbb{C}\setminus\{0\} : \arg(z) \in (-\theta,0)\} \cup (A,B)$, so that assumption (A1) holds.

(A2) For $0 < x < A$ we have

$$h_k(x - i0) = -\frac{s\alpha_k\sqrt{(A^s - x^s)(B^s - x^s)}}{x^{1+s(n-1+\tilde{l}_k)}}\, i,$$

and for $x > B$ we have

$$h_k(x - i0) = \frac{s\alpha_k \sqrt{(x^s - A^s)(x^s - B^s)}}{x^{1+s(n-1+\tilde{l}_k)}} i.$$

Therefore $\mathrm{Re}(h_k(x - i0)) = 0$ for $0 < x < A$ and for $x > B$, so that assumption (A2) holds.

(A3) For each $k = 1, \ldots, N$, we have

$$h_k(z) = -\frac{is\alpha_k \sqrt{(AB)^s}}{z^{1+s(n-1+\tilde{l}_k)}}(1 + o(1)) \quad \text{as } z \to 0,\ \arg(z) \in (-\theta, 0).$$

Moreover, $0 < s(n - 1 + \tilde{l}_k) < (2\pi - \theta)/\theta$. Therefore assumption (A3) holds.

(A4) Let $0 < \theta < \pi$ (equivalently, $n > 1$). For all $k = 1, \ldots, N$, $u > 0$ we have

$$\begin{aligned}
\arg(h_k(ue^{-i\theta})) &= \arg\left(\frac{s\alpha_k \sqrt{(B^s - u^s e^{-is\theta})(u^s e^{-is\theta} - A^s)}}{u^{1+s(n-1+\tilde{l}_k)} e^{-i\theta(1+s(n-1+\tilde{l}_k))}} \right) \\
&\in \left(-\frac{\pi}{2} + \theta(1 + s(n-1+\tilde{l}_k)), \frac{\pi - 2s\theta}{2} + \theta(1 + s(n-1+\tilde{l}_k)) \right) \\
&= \left(-\frac{\pi}{2} + \theta(n_0 + s\tilde{l}_k), \frac{\pi - 2s\theta}{2} + \theta(n_0 + s\tilde{l}_k) \right) \\
&= \left(\frac{\pi}{2} + \theta s\tilde{l}_k, \frac{3}{2}\pi - \theta s(1 - \tilde{l}_k) \right) \subset \left(\frac{\pi}{2}, \frac{3}{2}\pi \right),
\end{aligned}$$

since

$$\arg\left(\sqrt{(B^s - u^s e^{-is\theta})(u^s e^{-is\theta} - A^s)}\right) \in \left(-\frac{\pi}{2}, \frac{\pi - 2s\theta}{2} \right).$$

Therefore $\mathrm{Re}(h_k(ue^{-i\theta})) \leqslant 0$ for all $u > 0$. When $\theta = \pi$ (equivalently, $n = 1$), for all $k = 1, \ldots, N$, $u < 0$ we have

$$\begin{aligned}
\arg(h_k(u)) &= \arg\left(\frac{s\alpha_k \sqrt{(B^s - (u - i0)^s)((u - i0)^s - A^s)}}{u^{1+s(n-1+\tilde{l}_k)}} \right) \\
&= \arg\left(\frac{s\alpha_k \sqrt{(B^s - (u - i0)^s)((u - i0)^s - A^s)}}{|u|^{1+s\tilde{l}_k} e^{-i\pi(1+s\tilde{l}_k)}} \right) \\
&\in \left(-\frac{\pi}{2} + \pi(1 + s\tilde{l}_k), \frac{1 - 2s}{2}\pi + \pi(1 + s\tilde{l}_k) \right) \\
&= \left(\frac{\pi}{2} + \pi s\tilde{l}_k, \frac{3}{2}\pi - \pi s(1 - \tilde{l}_k) \right) \subset \left(\frac{\pi}{2}, \frac{3}{2}\pi \right),
\end{aligned}$$

since

$$\arg\left(\sqrt{(B^s - (u - i0)^s)((u - i0)^s - A^s)}\right) \in \left(-\frac{\pi}{2}, \frac{1 - 2s}{2}\pi \right).$$

Therefore $\mathrm{Re}(h_k(u)) \leqslant 0$ for all $u < 0$. Hence assumption (A4) holds.

(A5) We have

$$h_k(z) = \frac{-is\alpha_k}{z^{1+s(n-2+\tilde{l}_k)}}(1+o(1)) \qquad \text{as } |z| \to \infty,\ \arg(z) \in (-\theta, 0).$$

Hence $\lim_{|z|\to\infty,\, \arg(z)\in(-\theta,0)} h_k(z) = 0$, so that assumption (A5) holds.

Since $h(x) = \sum_{k=1}^N h_k(x)$ for all $x \in (A, B)$ and every function h_k satisfies assumption (A1), the function h also satisfies (A1) and $h(z) = \sum_{k=1}^N h_k(z)$ for all $z \in \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\} \cup (A, B)$. Finally, we conclude that h satisfies (A1) to (A5) since so does every h_k $(k = 1, \dots, N)$.

(A6) Since h satisfies (A1), by applying Lemma 3.1, the Cauchy transform G has an analytic continuation to $\mathbb{C}^+ \cup (A, B) \cup \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$ (still denoted by G) and

$$G(z) = \int_A^B \frac{h(x)}{z-x}\, dx - 2\pi i h(z) = \sum_{k=1}^N \left(\int_A^B \frac{h_k(x)}{z-x}\, dx - 2\pi i h_k(z) \right)$$

for $z \in \mathbb{C}^-$ with $\arg(z) \in (-\theta, 0)$. We set $\tilde{G}_k(z) = \int_A^B \frac{h_k(x)}{z-x}\, dx$. To prove that G can be extended to a continuous function on $\mathbb{C}^+ \cup [A, B] \cup \{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in [-\theta, 0)\} \cup ((-\infty, A] \cup [B, \infty) + i0) \cup ([0, A] \cup [B, \infty) - i0)$, we find the asymptotic behavior of G at $z = A$ and $z = B$. We define

$$G_k(z) := \tilde{G}_k(z) - 2\pi i h_k(z)$$

for $z \in \mathbb{C}^-$ with $\arg(z) \in (-\theta, 0)$. By the Taylor expansion at $x = A$, there exist $c_k > 0$ and a real analytic function $\psi_k(x)$ on a neighborhood of A which satisfies $\psi_k(x) = o(1)$ as $x \to A + 0$ and has an analytic continuation to $\{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\} \cup (A, B)$ (still denoted by ψ_k) such that

$$h_k(x) = c_k(x-A)^{1/2}(1 + \psi_k(x)) \qquad \text{as } x \to A + 0.$$

Therefore

$$G_k(z) = \beta_k + \gamma_k(-(z-A))^{1/2} + o(|z-A|^{1/2})$$

as $|z - A| \to 0$ for $z \in \mathbb{C}^-$ with $\arg(z) \in (-\theta, 0)$, where $\beta_k = \lim_{z\to A,\, z\in\mathbb{C}^-} \tilde{G}_k(z) < 0$ (is finite) and $\gamma_k = 2\pi c_k$. Finally, if $\beta := \sum_{k=1}^N \beta_k < 0$ and $\gamma = \sum_{k=1}^N \gamma_k > 0$, then

$$G(z) = \sum_{k=1}^N G_k(z) = \beta + \gamma(-(z-A))^{1/2} + o(|z-A|^{1/2})$$

as $|z - A| \to 0$ for $z \in \mathbb{C}^-$ with $\arg(z) \in (-\theta, 0)$. Moreover $G(A + i0) = \beta = G(A - i0)$. Similarly, we can find the asymptotic behavior of G at $z = B$. Hence

G can be extended to a continuous function on $\mathbb{C}^+ \cup [A,B] \cup \{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in [-\theta, 0)\} \cup ((-\infty, A] \cup [B,\infty) + i0) \cup ([0,A] \cup [B,\infty) - i0)$. Thus G satisfies (A6).

By Theorem 3.1, we have $X^r \sim$ UI. By the weak closedness of the UI class (see Lemma 2.1), we have $X^r \sim$ UI even if $r \geqslant 1$ and $t \leqslant l_1 < \cdots < l_N \leqslant t+1$. ∎

COROLLARY 4.1. *If $X \sim$ GPFP(a,b,N,α,l) and $l \in [0,1]^N_<$ then $X^r \sim$ UI when $r \leqslant -1$.*

Proof. Let $r \leqslant -1$. We have

$$X^r \sim \frac{t\sqrt{(AB)^{-t}}\sqrt{(B^t - x^t)(x^t - A^t)}}{x} \sum_{k=1}^{N} \frac{\alpha_k}{x^{t(1-l_k)}} \cdot 1_{(A,B)}(x)dx := p(x)dx,$$

where $t := -1/r \in (0,1)$, $A := b^{-1/t}$ and $B := a^{-1/t}$. We set $\theta = \pi$ and let $p_k(x)$ be the kth term of $p(x)$. A similar argument to the proof of Theorem 4.1 shows that θ, p_k, p, and $G = G_{X^r}$ satisfy assumptions (A1)–(A6) of Theorem 3.1. Therefore $X^r \sim$ UI if $r \leqslant -1$. ∎

Finally, if $X \sim$ GPFP(a,b,N,α,l) with $l \in [0,1]^N_<$ then $X^r \sim$ UI for all $|r| \geqslant 1$ by Theorem 4.1 and Corollary 4.1.

To end this section, we give applications of Theorem 4.1 and Corollary 4.1.

EXAMPLE 4.2. (1) Let $p > 1$. If

$$X \sim \mathbf{fp}(p) = \text{GPFP}\left(a = (\sqrt{p}-1)^2,\ b = (\sqrt{p}+1)^2,\ N = 1,\ \alpha = \frac{1}{2\pi},\ l = 0\right),$$

then $X^r \sim$ UI for all $|r| \geqslant 1$ by Theorem 4.1 and Corollary 4.1 (but the result in [11] is stronger).

(2) Let $\lambda \in \mathbb{R}$ and $\alpha_1, \alpha_2 > 0$. Suppose that $0 < a < b$ are the unique solution of (4.1). If

$$X \sim \text{fGIG}(\alpha_1, \alpha_2, \lambda) = \text{GPFP}\left(a,\ b,\ N = 2,\ \alpha = \left(\frac{\alpha_1}{2\pi}, \frac{\alpha_2}{2\pi\sqrt{ab}}\right),\ l = (0,1)\right),$$

then $X^r \sim$ UI for all $|r| \geqslant 1$ by applying Theorem 4.1 and Corollary 4.1.

(3) Assume $S \sim S(0,1)$ and $u \neq 0$, where $S(0,1)$ is the semicircle law with mean 0 and variance 1 (see Example 2.1(1)). Then the distribution of $S+u$ is FID. However, the distribution of $(S+u)^2$ is not FID (see [8]).

If $u > 2$, then

$$\begin{aligned} S + u &\sim \text{GPFP}\left(a = u-2,\ b = u+2,\ N = 1,\ \alpha = \frac{1}{2\pi},\ l = -1\right) \\ &= \frac{1}{2\pi}\sqrt{4-(x-u)^2} \cdot 1_{(u-2,u+2)}(x)dx. \end{aligned}$$

By Remark 4.1,

$$(S+u)^{-1} \sim \mathrm{GPFP}\Big(a = \frac{1}{u+2},\, b = \frac{1}{u-2},\, N = 1,\, \alpha = \frac{1}{2\pi}\sqrt{u^2-4},\, l = 2\Big).$$

By Theorem 4.1, we have $(S+u)^{-r} = ((S+u)^{-1})^r \sim \mathrm{UI}$ for $r \geqslant 1$. Finally, we conclude that $(S+u)^s \sim \mathrm{UI}$ for all $u > 2$ and $s \leqslant -1$.

(4) Consider a random variable $S_{n,b} \sim \mu_{n,b}$, where $\mu_{n,b}$ is the GPFP distribution given by (4.3) in Example 4.1(3) for $n \in \mathbb{N}$ and $b > 0$. Then $S_{n,b}^r \sim \mathrm{UI}$ for all $|r| \geqslant 1$ by Theorem 4.1 and Corollary 4.1.

Recall that $\mu_{n,b} \to \mu_b$ weakly as $n \to \infty$, where μ_b is the probability measure given by (4.4) in Example 4.1(3) for each $b > 0$. If $S_b \sim \mu_b$, then the weak closedness of the UI class implies that $S_b^r \sim \mathrm{UI}$ for all $|r| \geqslant 1$.

(5) Consider a random variable $B_{n,l} \sim \rho_{n,l}$, where $\rho_{n,l}$ is the GPFP distribution given by (4.7) in Example 4.1(5) for $n \in \mathbb{N}$ and $l < 1/2$.

If $0 \leqslant l < 1/2$ then $B_{n,l}^r \sim \mathrm{UI}$ for all $|r| \geqslant 1$ by Theorem 4.1 and Corollary 4.1.

By Remark 4.1, we have

$$B_{n,l}^{-1} \sim \mathrm{GPFP}\Big(1,\, n,\, 1,\, \sqrt{\frac{1}{n}} \cdot \frac{\alpha(n,l)}{B(1/2-l,3/2)},\, 1-l\Big). \tag{4.8}$$

If $l < 0$ then $B_{n,l}^{-r} = (B_{n,r}^{-1})^r \sim \mathrm{UI}$ for all $r \geqslant 1$ by Theorem 4.1 and condition (4.8). Finally we conclude that if $l < 0$ then $B_{n,l}^s \sim \mathrm{UI}$ for all $s \leqslant -1$.

Recall that $\rho_{n,l} \to \beta_{1/2-l,3/2}$ weakly as $n \to \infty$ for each $l < 1/2$. If $B_l \sim \beta_{1/2-l,3/2}$, then the weak closedness of the UI class implies that (i) if $0 \leqslant l < 1/2$ then $B_l^r \sim \mathrm{UI}$ for all $|r| \geqslant 1$, and (ii) if $l < 0$ then $B_l^s \sim \mathrm{UI}$ for all $s \leqslant -1$.

5. NON-FREE-INFINITE DIVISIBILITY FOR GPFP DISTRIBUTIONS

In this section, we give a few of examples of the GPFP distributions which fail to satisfy the conclusions of Theorem 4.1 or Corollary 4.1.

Let $(\mathcal{A}, \phi)$ be a C^*-probability space and $X \in \mathcal{A}$ a random variable. The *nth moment* $m_n = m_n(X)$ of X is defined to be $\phi(X^n)$. In particular, if $X \sim \mu$, we write $m_n(\mu)$ for the nth moment of X (or μ). The *nth free cumulant* $\kappa_n = \kappa_n(X)$ of X is defined as the nth coefficient of the power series of the free cumulant transform $C_X(z)$. If $X \sim \mu$, we write $\kappa_n(\mu)$ for the nth free cumulant of X (or μ). We have the *moment-cumulant formula*

$$\kappa_n = \sum_{\pi \in \mathrm{NC}(n)} \Big(\prod_{V \in \pi} m_{|V|}\Big)\mu(\pi, 1_n), \qquad n \in \mathbb{N},$$

where $\mathrm{NC}(n)$ is the set of non-crossing partitions of the set $\{1, \ldots, n\}$, the symbol 1_n is the non-crossing partition which has the block $\{1, \ldots, n\}$, $\mu(\pi, \sigma)$ (for

$\pi \leqslant \sigma$ in $\mathrm{NC}(n)$) is the Möbius function of $\mathrm{NC}(n)$, and $|V|$ is the number of elements in a block V of π (see [14] for details). By the moment-cumulant formula, we can compute the free cumulants. For example, we have

$$\begin{aligned} &\kappa_1 = m_1, \quad \kappa_2 = m_2 - m_1^2, \quad \kappa_3 = m_3 - 3m_1m_2 + 2m_1^3, \\ &\kappa_4 = m_4 - 4m_1m_3 - 2m_2^2 + 10m_1^2m_2 - 5m_1^4, \\ &\kappa_5 = m_5 - 5m_1m_4 - 5m_2m_3 + 15m_1^2m_3 + 15m_1m_2^2 - 35m_1^3m_2 + 14m_1^5. \end{aligned} \tag{5.1}$$

The nth free cumulant of the free Poisson distribution $\mathbf{fp}(p)$ is $\kappa_n(\mathbf{fp}(p)) = p$ for all $n \in \mathbb{N}$ (from Example 2.1(2)). Then (5.1) yields

$$\begin{aligned} &m_1(\mathbf{fp}(p)) = p, \quad m_2(\mathbf{fp}(p)) = p(p+1), \\ &m_3(\mathbf{fp}(p)) = p(p^2+3p+1), \\ &m_4(\mathbf{fp}(p)) = p(p^3+6p^2+6p+1), \\ &m_5(\mathbf{fp}(p)) = p(p^4+10p^3+20p^2+10p+1). \end{aligned} \tag{5.2}$$

In particular, we need to compute the moments of $\mathbf{fp}(p)$ of degree $s \in \mathbb{C}$:

$$m_s(\mathbf{fp}(p)) = \int_0^\infty x^s \mathbf{fp}(p)(dx), \quad s \in \mathbb{C},\ p > 1.$$

This is an analytic function of s in $\mathbb{C}$.

LEMMA 5.1. *For $s \in \mathbb{C}$ and $p > 1$, we have*

$$m_s(\mathbf{fp}(p)) = \frac{m_{-s-1}(\mathbf{fp}(p))}{(p-1)^{-1-2s}}.$$

Proof. Let $s \in \mathbb{C}$ and $p > 1$. Then

$$\begin{aligned} m_s(\mathbf{fp}(p)) &= \int_{(\sqrt{p}-1)^2}^{(\sqrt{p}+1)^2} x^{s-1} \frac{\sqrt{((\sqrt{p}+1)^2 - x)(x - (\sqrt{p}-1)^2)}}{2\pi}\, dx \\ &= (p-1) \int_{\frac{1}{(\sqrt{p}+1)^2}}^{\frac{1}{(\sqrt{p}-1)^2}} x^{-s-1} \frac{\sqrt{\left(\frac{1}{(\sqrt{p}-1)^2} - x\right)\left(x - \frac{1}{(\sqrt{p}+1)^2}\right)}}{2\pi x}\, dx \\ &= \frac{1}{(p-1)^{-1-2s}} \int_{(\sqrt{p}-1)^2}^{(\sqrt{p}+1)^2} x^{-s-1} \frac{\sqrt{((\sqrt{p}+1)^2 - x)(x - (\sqrt{p}-1)^2)}}{2\pi}\, dx \\ &= \frac{m_{-s-1}(\mathbf{fp}(p))}{(p-1)^{-1-2s}}, \end{aligned}$$

where the second equality holds by changing variables from x to x^{-1} and the third equality holds by changing variables from x to $x/(p-1)^2$. ∎

By Lemma 5.1, for all $p > 1$,

$$m_{-1}(\mathbf{fp}(p)) = \frac{1}{p-1}, \quad m_{-2}(\mathbf{fp}(p)) = \frac{p}{(p-1)^3}. \tag{5.3}$$

LEMMA 5.2. *Let $p > 1$ and $\alpha_1, \alpha_2 > 0$.*

(1) *The measure*

$$\frac{\sqrt{\left(x - \frac{1}{(\sqrt{p}+1)^2}\right)\left(\frac{1}{(\sqrt{p}-1)^2} - x\right)}}{2\pi x^2}\left(\alpha_1 + \frac{\alpha_2}{x}\right)1_{\left(\frac{1}{(\sqrt{p}+1)^2}, \frac{1}{(\sqrt{p}-1)^2}\right)}(x)dx \tag{5.4}$$

is a probability measure if and only if $\alpha_1 + \alpha_2 p = p - 1$.

(2) *The measure*

$$\frac{\sqrt{((\sqrt{p}+1)^2 - x)(x - (\sqrt{p}-1)^2)}}{2\pi x}\left(\alpha_1 + \frac{\alpha_2}{x^2}\right)1_{((\sqrt{p}-1)^2, (\sqrt{p}+1)^2)}(x)dx \tag{5.5}$$

is a probability measure if and only if $\alpha_1 + \frac{p}{(p-1)^3}\alpha_2 = 1$.

Proof. (1) Assume that (5.4) is a probability measure. Suppose that X is a random variable which follows the measure (5.4). Then

(5.6)
$$X^{-1} \sim \frac{\sqrt{(x - (\sqrt{p}-1)^2)((\sqrt{p}+1)^2 - x)}}{2\pi(p-1)x}(\alpha_1 + \alpha_2 x)1_{((\sqrt{p}-1)^2, (\sqrt{p}+1)^2)}(x)dx.$$

By our assumption, (5.6) is also a probability measure. Equivalently,

$$1 = \frac{1}{p-1}(\alpha_1 m_0(\mathbf{fp}(p)) + \alpha_2 m_1(\mathbf{fp}(p))) = \frac{1}{p-1}(\alpha_1 + \alpha_2 p).$$

Therefore $\alpha_1 + \alpha_2 p = p - 1$. The converse is clear.

(2) Let $\mu_{p,\alpha_1,\alpha_2}$ be the measure (5.5) on the positive real line. It is a probability measure if and only if

$$1 = \int_{(\sqrt{p}-1)^2}^{(\sqrt{p}+1)^2} \mu_{p,\alpha_1,\alpha_2}(dx) = \alpha_1 m_0(\mathbf{fp}(p)) + \alpha_2 m_{-2}(\mathbf{fp}(p)).$$

The calculation (5.3) implies our conclusion. ∎

There is a criterion for free infinite divisibility of probability measures on $\mathbb{R}$ with compact support.

LEMMA 5.3 (see [14, Theorem 13.16]). *Let μ be a probability measure on $\mathbb{R}$ with compact support. The following conditions are equivalent:*

(1) *μ is FID.*

(2) *The sequence $\{\kappa_n(\mu)\}_{n\in\mathbb{N}}$ of free cumulants of μ is conditionally positive definite: for all $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ we have*

$$\sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \kappa_{i+j}(\mu) \geqslant 0.$$

In particular, if a compactly supported probability measure μ is FID, then the nth Hankel determinant $\det((\kappa_{i+j}(\mu))_{i,j=1}^n)$ of the sequence $\{\kappa_n(\mu)\}_{n\in\mathbb{N}}$ is nonnegative for all $n \in \mathbb{N}$.

First we give an example of the GPFP distribution which fails to satisfy the conclusion of Corollary 4.1.

EXAMPLE 5.1. Consider the GPFP distribution

$$\begin{aligned}\sigma_{\alpha_1,\alpha_2}(dx) &:= \mathrm{GPFP}\left(\frac{1}{(\sqrt{2}+1)^2}, \frac{1}{(\sqrt{2}-1)^2}, 2, \left(\frac{\alpha_1}{2\pi}, \frac{\alpha_2}{2\pi}\right), l = (1,2)\right) \\ &= \frac{\sqrt{\left(x - \frac{1}{(\sqrt{2}+1)^2}\right)\left(\frac{1}{(\sqrt{2}-1)^2} - x\right)}}{2\pi x} \left(\frac{\alpha_1}{x} + \frac{\alpha_2}{x^2}\right) 1_{\left(\frac{1}{(\sqrt{2}+1)^2}, \frac{1}{(\sqrt{2}-1)^2}\right)}(x)dx,\end{aligned}$$

where $\alpha_1, \alpha_2 > 0$ satisfy $\alpha_1 + 2\alpha_2 = 1$ $(0 < \alpha_2 < 1/2)$ (see Lemma 5.2(1)). The GPFP distribution $\sigma_{\alpha_1,\alpha_2}$ fails to satisfy the assumption of Corollary 4.1 since $l = (1,2) \notin [0,1]^2_<$.

If $X_{\alpha_1,\alpha_2} \sim \sigma_{\alpha_1,\alpha_2}$, then the distribution of $X_{\alpha_1,\alpha_2}^{-1}$ is not FID for all $\alpha_1, \alpha_2 > 0$ with $\alpha_1 + 2\alpha_2 = 1$. Therefore $\sigma_{\alpha_1,\alpha_2}$ does not satisfy the conclusion of Corollary 4.1.

Proof of Lemma 5.3. By Remark 4.1 (or (5.6)), we have

$$\begin{aligned}X_{\alpha_1,\alpha_2}^{-1} &\sim \mathrm{GPFP}\left((\sqrt{2}-1)^2, (\sqrt{2}+1)^2, 2, \left(\frac{\alpha_2}{2\pi}, \frac{\alpha_1}{2\pi}\right), (-1,0)\right) \\ &= \frac{\sqrt{(x-(\sqrt{2}-1)^2)((\sqrt{2}+1)^2 - x)}}{2\pi x} (\alpha_1 + \alpha_2 x) 1_{((\sqrt{2}-1)^2, (\sqrt{2}+1)^2)}(x)dx.\end{aligned}$$

From $\alpha_1 + 2\alpha_2 = 1$ and (5.2), we have

$$\begin{aligned}m_1(X_{\alpha_1,\alpha_2}^{-1}) &= (1-2\alpha_2)m_1(\mathbf{fp}(2)) + \alpha_2 m_2(\mathbf{fp}(2)) = 2(1+\alpha_2), \\ m_2(X_{\alpha_1,\alpha_2}^{-1}) &= (1-2\alpha_2)m_2(\mathbf{fp}(2)) + \alpha_2 m_3(\mathbf{fp}(2)) = 2(3+5\alpha_2), \\ m_3(X_{\alpha_1,\alpha_2}^{-1}) &= (1-2\alpha_2)m_3(\mathbf{fp}(2)) + \alpha_2 m_4(\mathbf{fp}(2)) = 2(11+23\alpha_2), \\ m_4(X_{\alpha_1,\alpha_2}^{-1}) &= (1-2\alpha_2)m_4(\mathbf{fp}(2)) + \alpha_2 m_5(\mathbf{fp}(2)) = 2(45+107\alpha_2).\end{aligned} \tag{5.7}$$

From (5.1) and (5.7), we have

$$\begin{aligned}
\kappa_2 &:= \kappa_2(X_{\alpha_1,\alpha_2}^{-1}) = -2(2\alpha_2^2 - \alpha_2 - 1),\\
\kappa_3 &:= \kappa_3(X_{\alpha_1,\alpha_2}^{-1}) = 2(8\alpha_2^3 - 6\alpha_2^2 - \alpha_2 + 1),\\
\kappa_4 &:= \kappa_4(X_{\alpha_1,\alpha_2}^{-1}) = -2(2\alpha_2 - 1)(20\alpha_2^3 - 10\alpha_2^2 - 3\alpha_2 + 1),
\end{aligned}$$

where κ_n is the nth free cumulant of $X_{\alpha_1,\alpha_2}^{-1}$. By an elementary calculation, the 2nd Hankel determinant $\kappa_2\kappa_4 - \kappa_3^2$ (a function of α_2) is negative for all $0 < \alpha_2 < 1/2$ with $\alpha_1 + 2\alpha_2 = 1$. By Lemma 5.3 for all $\alpha_1, \alpha_2 > 0$ with $\alpha_1 + 2\alpha_2 = 1$, the distribution of $X_{\alpha_1,\alpha_2}^{-1}$ is not FID. ∎

Next we give an example of the GPFP distribution which fails to satisfy the conclusion of Theorem 4.1.

EXAMPLE 5.2. Consider the GPFP distribution

$$\begin{aligned}
\eta_{\alpha_1,\alpha_2}(dx) &:= \mathrm{GPFP}\left((\sqrt{2}-1)^2,\ (\sqrt{2}+1)^2,\ 2,\ \left(\frac{\alpha_1}{2\pi}, \frac{\alpha_2}{2\pi}\right),\ l = (0,2)\right)\\
&= \frac{\sqrt{((\sqrt{2}+1)^2 - x)(x - (\sqrt{2}-1)^2)}}{2\pi x}\left(\alpha_1 + \frac{\alpha_2}{x^2}\right)1_{((\sqrt{2}-1)^2,(\sqrt{2}+1)^2)}(x)dx,
\end{aligned}$$

where $\alpha_1, \alpha_2 > 0$ satisfy $\alpha_1 + 2\alpha_2 = 1$ (see Lemma 5.2(2)). The GPFP distribution η_{α_1,α_2} fails to satisfy the assumption of Theorem 4.1 since $l = (0,2) \notin [t, t+1]^2_<$ for any $t \geqslant 0$.

There exist $\alpha_1, \alpha_2 > 0$ with $\alpha_1 + 2\alpha_2 = 1$ such that the GPFP distribution η_{α_1,α_2} is not FID. Therefore η_{α_1,α_2} does not satisfy the conclusion of Theorem 4.1.

Proof. From $\alpha_1 + 2\alpha_2 = 1$ and (5.2) and (5.3), we have

$$\begin{aligned}
m_1(\eta_{\alpha_1,\alpha_2}) &= (1-2\alpha_2)m_1(\mathbf{fp}(2)) + \alpha_2 m_{-1}(\mathbf{fp}(2)) = 2 - 3\alpha_2,\\
m_2(\eta_{\alpha_1,\alpha_2}) &= (1-2\alpha_2)m_2(\mathbf{fp}(2)) + \alpha_2 m_0(\mathbf{fp}(2)) = 6 - 11\alpha_2,\\
m_3(\eta_{\alpha_1,\alpha_2}) &= (1-2\alpha_2)m_3(\mathbf{fp}(2)) + \alpha_2 m_1(\mathbf{fp}(2)) = 2(11 - 21\alpha_2),\\
m_4(\eta_{\alpha_1,\alpha_2}) &= (1-2\alpha_2)m_4(\mathbf{fp}(2)) + \alpha_2 m_2(\mathbf{fp}(2)) = 6(15 - 29\alpha_2).
\end{aligned} \tag{5.8}$$

From (5.1) and (5.8), we have

$$\begin{aligned}
\kappa_2' &:= \kappa_2(\eta_{\alpha_1,\alpha_2}) = -9\alpha_2^2 + \alpha_2 + 2,\\
\kappa_3' &:= \kappa_3(\eta_{\alpha_1,\alpha_2}) = -54\alpha_2^3 + 9\alpha_2^2 + 6\alpha_2 + 2,\\
\kappa_4' &:= \kappa_4(\eta_{\alpha_1,\alpha_2}) = -405\alpha_2^4 + 90\alpha_2^3 + 34\alpha_2^2 + 10\alpha_2 + 2.
\end{aligned}$$

This implies that the 2nd Hankel determinant $\kappa_2'\kappa_4' - (\kappa_3')^2$ (a function of α_2) is negative for $0 < \alpha_2 < \tilde{\alpha}$, where $\tilde{\alpha} \in (0, 1/2)$ is the unique solution α_2 of the equation $\kappa_2'\kappa_4' - (\kappa_3')^2 = 0$. Thus η_{α_1,α_2} is not FID for $0 < \alpha_2 < \tilde{\alpha}$ with $\alpha_1 + 2\alpha_2 = 1$ by Lemma 5.3. ∎

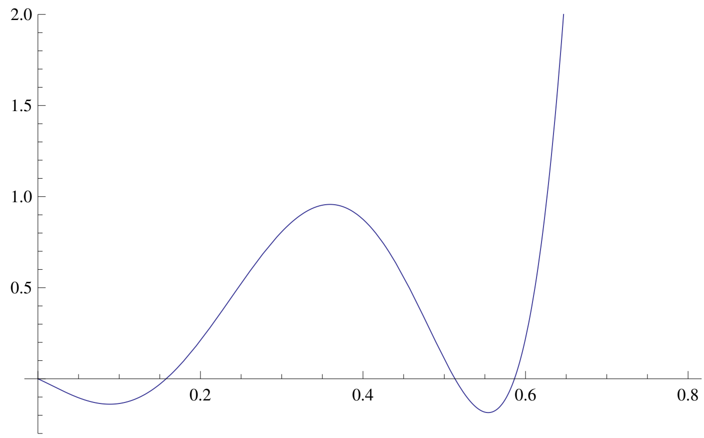

FIGURE 3. $\kappa_2'\kappa_4' - (\kappa_3')^2$ as a function of α_2. It is negative when $0 < \alpha_2 < \tilde{\alpha} \approx 0.157781$.

Using Mathematica, we get $\tilde{\alpha} \approx 0.157781$ (see Figure 3). For example, the GPFP distribution $\eta_{0.7,0.15}$ is not FID.

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