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WIENER–HOPF FACTORIZATION FOR TIME-INHOMOGENEOUS MARKOV CHAINS AND ITS APPLICATION

BY

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Abstract. We derive the Wiener–Hopf factorization for a finite-state time-inhomogeneous Markov chain. Considered as the first step in the direction of the Wiener–Hopf factorization for time-inhomogeneous Markov chains, this work deals only with a special, but important class of time-inhomogeneous Markovian generators, namely piecewise constant generators, which allows us to use an appropriately tailored randomization technique.

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1. INTRODUCTION

The Wiener–Hopf factorization (WHf) for finite-state Markov chains was originally derived in [BRW80] for the time-homogeneous case; see also [LMRW82] and [Wil91]. For the WHf in the case of time-homogeneous Feller Markov processes we refer to [Wil08]. For some related applied work see [APU03], which deals with the ruin problem, and [Asm95], [Rog94], [RS94] that study fluid models. In addition, [KW90] investigates so called “noisy” Wiener–Hopf factorizations; for applications see [Asm95], [Rog94], [RS94], [JR06], [JP08], [MP11], [JP12], [Hie14], [HSZ18].

In this paper we derive the Wiener–Hopf factorization for a finite-state *time-inhomogeneous* Markov chain. Besides the mathematical importance of the Wiener–Hopf factorization methodology, there is also an important computational aspect: it allows for efficient computation of important functionals of both time-homogeneous and time-inhomogeneous Markov chains. In many practical situations, time-inhomogeneous Markov chains provide more appropriate models than time-homogeneous ones do.

We stress that even though the classical WHf of [BRW80] can be applied to the generator matrix, say G_t, of a time-inhomogeneous Markov chain X at every time t, these factorizations do not have any probabilistic meaning with regard to the process X. In particular, they are of no use for computing functionals such as (2.1)–(2.4) below. Thus, derivation of the relevant WHf for a time-inhomogeneous Markov chain requires a different approach than just directly applying the results of [BRW80] to each G_t, $t \geqslant 0$.

As far as we know, our study is the first attempt to investigate the Wiener–Hopf factorization for time-inhomogeneous Markov chains. The derivation of the WHf for time-inhomogeneous Markov chains is highly non-trivial, and in this work we initiate this study by providing derivation of the WHf in the piecewise constant generator case. This case is of practical importance, because time-inhomogeneous Markov chains with piecewise constant generators are convenient models for seasonal phenomena (see [YZ14]), Erlang loss systems with moving boundaries (see [N14]) or structural breaks in credit migrations (see [XC18]).

Given the piecewise constant generator of a time-inhomogeneous Markov chain, we apply a specially devised randomization technique to construct a time-homogeneous Markov chain with finite state space. This allows us to "project" the WHf results from the time-homogeneous chain to the original chain by using the inverse Laplace transform. From another perspective, the results presented here allow us to carry out the first test on how the WHf performs with regard to the computation of functionals of Markov chains vis-à-vis the performance of Monte-Carlo simulations. The numerical results, one of which is presented in the paper, speak in favor of the WHf method.

The paper is organized as follows. In Section 2 we provide the motivation and set-up of the problem. In Section 3 we introduce a randomization method and we give our main results. Section 4 provides a numerical algorithm for computing our version of the WHf and its application to compute a relevant functional in a time-inhomogeneous stochastic fluid-flow model. Finally, we give some supporting technical results in the Appendix.

2. MOTIVATION AND PROBLEM SET-UP

Let E be a finite set, $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, and $X := (X_t)_{t\geqslant 0}$ be a *time-inhomogeneous* Markov chain on $(\Omega, \mathscr{F}, \mathbb{P})$ with state space E and generator function $\mathsf{G} = \{\mathsf{G}_t, t \geqslant 0\}$. In particular, each G_t is an $|E| \times |E|$ matrix. We assume that $\mathbb{P}(X_0 = i) > 0$ for each $i \in E$ and we let $\mathbb{P}^i$ be the probability measure on $(\Omega, \mathscr{F})$ defined by

$$\mathbb{P}^i(A) := \mathbb{P}(A \,|\, X_0 = i), \qquad A \in \mathscr{F},$$

with $\mathbb{E}^i$ denoting the associated expectation.

In this paper we assume that the generator G is piecewise constant:

$$\mathsf{G}_t = \sum_{k=1}^{n} \mathsf{G}_k \mathbf{1}_{[s_{k-1},s_k)}(t) + \mathsf{G}_{n+1}\mathbf{1}_{[s_n,\infty)}(t)$$

for some $n \in \mathbb{N}$ and $0 = s_0 < s_1 < \cdots < s_n$. Without loss of generality we assume that $\mathsf{G}_1, \ldots, \mathsf{G}_{n+1}$ are not sub-Markovian, that is, the row sums of G_k are zero for any $k = 1, \ldots, n+1$. The results of this paper carry over to the sub-Markovian case by the standard augmentation of the state space.

Next, we consider a function $v : E \to \mathbb{R} \setminus \{0\}$ and we put

$$E^+ := \{i \in E \mid v(i) > 0\} \quad \text{and} \quad E^- := \{i \in E \mid v(i) < 0\}.$$

We also define an additive functional and the corresponding first passage times as

$$\varphi_t := \int_0^t v(X_u)\,\mathrm{d}u, \qquad \tau_t^{\pm} := \inf\{r \geqslant 0 \mid \pm\varphi_r > t\}, \qquad t \geqslant 0.$$

The main goal of this paper is to apply the Wiener–Hopf factorization technique, which we work out in Section 3, to compute the following expectations:

$$\Pi_c^+(i,j;s_1,\ldots,s_n) := \mathbb{E}(e^{-c\tau_0^+}\mathbf{1}_{\{X_{\tau_0^+}=j\}} \mid X_0 = i),\ i \in E^-,\ j \in E^+, \tag{2.1}$$

$$\Psi_c^+(\ell,i,j;s_1,\ldots,s_n) := \mathbb{E}(e^{-c\tau_\ell^+}\mathbf{1}_{\{X_{\tau_\ell^+}=j\}} \mid X_0 = i),\ i,j \in E^+,\ \ell > 0, \tag{2.2}$$

$$\Pi_c^-(i,j;s_1,\ldots,s_n) := \mathbb{E}(e^{-c\tau_0^-}\mathbf{1}_{\{X_{\tau_0^-}=j\}} \mid X_0 = i),\ i \in E^+,\ j \in E^-, \tag{2.3}$$

$$\Psi_c^-(\ell,i,j;s_1,\ldots,s_n) := \mathbb{E}(e^{-c\tau_\ell^-}\mathbf{1}_{\{X_{\tau_\ell^-}=j\}} \mid X_0 = i),\ i,j \in E^-,\ \ell > 0. \tag{2.4}$$

We focus on the computation of $\Pi_c^+(i,j;s_1,\ldots,s_n)$ and $\Psi_c^+(\ell,i,j;s_1,\ldots,s_n)$. By symmetry, analogous results can be obtained for $\Pi_c^-(i,j;s_1,\ldots,s_n)$ and for $\Psi_c^-(\ell,i,j;s_1,\ldots,s_n)$. To simplify notation, we frequently write $\Pi_c^+(i,j)$ and $\Psi_c^+(\ell,i,j)$ for $\Pi_c^+(i,j;s_1,\ldots,s_n)$ and $\Psi_c^+(\ell,i,j;s_1,\ldots,s_n)$, respectively.

3. A RANDOMIZATION METHOD AND WIENER–HOPF FACTORIZATION

In this section we construct a *time-homogeneous* Markov chain associated to X, by randomizing the discontinuity times $s_1, \ldots, s_n$ of the generator G. This key construction will allow us to compute the expectations (2.1) and (2.2) using analogous expectations corresponding to this time-homogeneous chain. The latter can be computed using Wiener–Hopf factorization theory of [BRW80].

Define $\mathbb{Z}_n := \{0, \ldots, n\}$, $\widetilde{E} := \mathbb{Z}_n \times E$, and let $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}})$ be a complete probability space. Next, we consider a *time-homogeneous Markov chain* $Z = (N, Y) :=$

$(N_t, Y_t)_{t\geqslant 0}$, defined on $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}})$, taking values in $\widetilde{E}$ and with generator matrix

$$\widetilde{\mathsf{G}} = \begin{array}{c} \scriptstyle \{0\}\times E \\ \scriptstyle \{1\}\times E \\ \vdots \\ \scriptstyle \{n-1\}\times E \\ \scriptstyle \{n\}\times E \end{array} \overset{\begin{array}{ccccc} \scriptstyle \{0\}\times E & \scriptstyle \{1\}\times E & \scriptstyle \cdots & \scriptstyle \{n-1\}\times E & \scriptstyle \{n\}\times E \end{array}}{\begin{bmatrix} \mathsf{G}_1 - q_1\mathsf{I} & q_1\mathsf{I} & \cdots & 0 & 0 \\ 0 & \mathsf{G}_2 - q_2\mathsf{I} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathsf{G}_n - q_n\mathsf{I} & q_n\mathsf{I} \\ 0 & 0 & \cdots & 0 & \mathsf{G}_{n+1} \end{bmatrix}}, \tag{3.1}$$

where $q_1, \ldots, q_n > 0$ are constants and I is the identity matrix. For each $i \in E$, we define a probability measure $\widetilde{\mathbb{P}}^i$ on $(\widetilde{\Omega}, \widetilde{\mathscr{F}})$ by $\widetilde{\mathbb{P}}^i(A) := \widetilde{\mathbb{P}}(A \,|\, Z_0 = (0, i))$, $A \in \widetilde{\mathscr{F}}$.

The next result regards the Markov property of the process N.

PROPOSITION 3.1. *For any $i \in E$, the process N is a time-homogeneous Markov chain under $\widetilde{\mathbb{P}}^i$, with generator matrix*

$$\widetilde{\mathsf{G}}_N = \begin{array}{c} \scriptstyle 0 \\ \scriptstyle 1 \\ \vdots \\ \scriptstyle n-1 \\ \scriptstyle n \end{array} \overset{\begin{array}{ccccc} \scriptstyle 0 & \scriptstyle 1 & \scriptstyle \cdots & \scriptstyle n-1 & \scriptstyle n \end{array}}{\begin{bmatrix} -q_1 & q_1 & \cdots & 0 & 0 \\ 0 & -q_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -q_n & q_n \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}}.$$

Proof. We will proceed in three steps.

Step 1. We start by showing that, for any $j_1 \in E$, $k \in \mathbb{N}$, and $n_1, n_2 \in \mathbb{Z}_n$,

$$\sum_{j_2 \in E} \widetilde{\mathsf{G}}^k((n_1, j_1), (n_2, j_2)) = \widetilde{\mathsf{G}}_N^k(n_1, n_2). \tag{3.2}$$

In particular, the left-hand side of (3.2) does not depend on j_1. We will prove (3.2) by induction in k. Clearly (3.2) holds true for $k = 1$. Next, assume that it holds for some $k = \ell \in \mathbb{N}$. Then

$$\begin{aligned} \sum_{j_2 \in E} & \widetilde{\mathsf{G}}^{\ell+1}((n_1, j_1), (n_2, j_2)) \\ &= \sum_{j_2 \in E} \sum_{m=0}^{n} \sum_{j \in E} \widetilde{\mathsf{G}}^{\ell}((n_1, j_1), (m, j)) \widetilde{\mathsf{G}}((m, j), (n_2, j_2)) \\ &= \sum_{m=0}^{n} \sum_{j \in E} \widetilde{\mathsf{G}}^{\ell}((n_1, j_1), (m, j)) \widetilde{\mathsf{G}}_N(m, n_2) \\ &= \sum_{m=0}^{n} \widetilde{\mathsf{G}}_N^{\ell}(n_1, m) \widetilde{\mathsf{G}}_N(m, n_2) = \widetilde{\mathsf{G}}_N^{\ell+1}(n_1, n_2), \end{aligned}$$

where we have used the inductive assumptions for $k = 1$ and $k = \ell$ in the second and the third equalities, respectively. Hence (3.2) is established.

Step 2. We will show that, for any $t, s \geqslant 0$, $j \in E$, and $0 \leqslant n_1 \leqslant n_2 \leqslant n$,

$$\widetilde{\mathbb{P}}^i(N_{t+s} = n_2 \,|\, N_t = n_1) = \widetilde{\mathbb{P}}^i(N_{t+s} = n_2 \,|\, N_t = n_1,\, Y_t = j) = e^{s\widetilde{\mathsf{G}}_N}(n_1, n_2). \tag{3.3}$$

In particular, the left-hand side of (3.3), and thus $\widetilde{\mathbb{P}}^i(N_{t+s} = n_2 \,|\, N_t = n_1)$, does not depend on t. We start by checking the second equality in (3.3):

$$\begin{aligned}\widetilde{\mathbb{P}}^i(N_{t+s} = n_2 \,|\, N_t = n_1,\, Y_t = j) &= \sum_{k\in E} \widetilde{\mathbb{P}}^i(N_{t+s} = n_2, Y_{t+s} = k \,|\, N_t = n_1,\, Y_t = j)\\ &= \sum_{k\in E} e^{s\widetilde{\mathsf{G}}}((n_1, j), (n_2, k)) = e^{s\widetilde{\mathsf{G}}_N}(n_1, n_2),\end{aligned}$$

where the last equality follows by expanding $e^{s\widetilde{\mathsf{G}}}$ in a power series and using (3.2). In particular, $\widetilde{\mathbb{P}}^i(N_{t+s} = n_2 \,|\, N_t = n_1,\, Y_t = j)$ does not depend on $j \in E$.

As for the first equality in (3.3), we have

$$\begin{aligned}&\widetilde{\mathbb{P}}^i(N_{t+s} = n_2 \,|\, N_t = n_1)\\ &\quad= \frac{\sum_{j\in E} \widetilde{\mathbb{P}}^i(N_{t+s} = n_2 \,|\, N_t = n_1,\, Y_t = j)\widetilde{\mathbb{P}}^i(N_t = n_1,\, Y_t = j)}{\sum_{j\in E} \widetilde{\mathbb{P}}^i(N_t = n_1,\, Y_t = j)}\\ &\quad= e^{s\widetilde{\mathsf{G}}_N}(n_1, n_2).\end{aligned}$$

Step 3. To complete the proof, we observe that, for any $m \in \mathbb{N}$, $0 = t_0 \leqslant t_1 < \cdots < t_m$, and $0 \leqslant n_1 \leqslant \cdots \leqslant n_m \leqslant n$,

$$\begin{aligned}&\widetilde{\mathbb{P}}^i(N_{t_m} = n_m \;\, | \, N_{t_{m-1}} = n_{m-1}, \ldots, N_{t_1} = n_1)\\ &= \frac{\sum_{j_1,\ldots,j_m\in E} \widetilde{\mathbb{P}}^i(N_{t_1} = n_1,\, Y_{t_1} = j_1; \ldots; N_{t_m} = n_m,\, Y_{t_m} = j_m)}{\sum_{j_1,\ldots,j_{m-1}\in E} \widetilde{\mathbb{P}}^i(N_{t_1} = n_1,\, Y_{t_1} = j_1; \ldots; N_{t_{m-1}} = n_{m-1},\, Y_{t_m} = j_{m-1})}\\ &= \frac{\sum_{j_1,\ldots,j_m\in E} \prod_{k=1}^{m} \widetilde{\mathbb{P}}^i(N_{t_k} = n_k,\, Y_{t_k} = j_k \,|\, N_{t_{k-1}} = n_{k-1},\, Y_{t_{k-1}} = j_{k-1})}{\sum_{j_1,\ldots,j_{m-1}\in E} \prod_{k=1}^{m-1} \widetilde{\mathbb{P}}^i(N_{t_k} = n_k,\, Y_{t_k} = j_k \,|\, N_{t_{k-1}} = n_{k-1},\, Y_{t_{k-1}} = j_{k-1})}\\ &= \widetilde{\mathbb{P}}^i(N_{t_m} = n_m \,|\, N_{t_{m-1}} = n_{m-1},\, Y_{t_{m-1}} = j_{m-1}) = e^{(t_m - t_{m-1})\widetilde{\mathsf{G}}_N}(n_{m-1}, n_m),\end{aligned}$$

where we have used the Markov property of $Z = (N, Y)$ under $\widetilde{\mathbb{P}}^i$ in the second equality, and (3.3) in the last equality. The proof is complete. ∎

Let $\widetilde{\mathbb{F}}^Y = (\widetilde{\mathscr{F}}_t^Y)_{t\geqslant 0}$ be the filtration generated by the process Y, and let $\widetilde{\mathscr{F}}_\infty^Y = \sigma(\bigcup_{t\geqslant 0} \widetilde{\mathscr{F}}_t^Y)$. For each $i \in E$, we will construct a probability measure $\overline{\mathbb{P}}^i$ on $(\widetilde{\Omega}, \widetilde{\mathscr{F}}_\infty^Y)$ such that the law of Y under $\overline{\mathbb{P}}^i$ is the same as the law of X under $\mathbb{P}^i$.

Moreover, we will establish a connection between $\overline{\mathbb{P}}^i$ and $\widetilde{\mathbb{P}}^i$. Note that $\widetilde{\mathbb{P}}^i$ is defined on $(\widetilde{\Omega}, \widetilde{\mathscr{F}})$, while $\overline{\mathbb{P}}^i$ will be defined on $(\widetilde{\Omega}, \widetilde{\mathscr{F}}^Y_\infty)$. We first let

$$S_0 := 0, \quad S_k := \inf\{t \geqslant 0 \,|\, N_t = k\}, \quad k \in \mathbb{N}_n := \{1, \dots, n\}.$$

We now derive the joint density of N and $(S_1, \dots, S_n)$ under $\widetilde{\mathbb{P}}^i$. For that, we set

$$T_k := S_k - S_{k-1}, \quad k \in \mathbb{N}_n. \tag{3.4}$$

It is shown in [Sys92, Section 1.1.4] that T_k's are independent and

$$\widetilde{\mathbb{P}}^i(T_1 > t_1, \dots, T_n > t_n) = \prod_{k=1}^n e^{-q_k t_k}, \quad t_1, \dots, t_n > 0,$$

which implies that the joint density of $(T_1, \dots, T_n)$ is given by

$$f_{T_1,\dots,T_n}(t_1, \dots, t_n) = \prod_{k=1}^n q_k e^{-q_k t_k}, \quad t_1, \dots, t_n > 0. \tag{3.5}$$

Combining (3.4) and (3.5), we deduce that

$$f_{S_1,\dots,S_n}(s_1, \dots, s_n) = \prod_{k=1}^n q_k e^{-q_k(s_k - s_{k-1})}, \quad (s_1, \dots, s_n) \in \Delta_n,$$

where $\Delta_n := \{(s_1, \dots, s_n) \in \mathbb{R}^n \mid 0 < s_1 < \dots < s_n\}$.

THEOREM 3.1. *For any $i \in E$, any $0 < s_1 < \dots < s_n$, and any cylinder set $A \in \widetilde{\mathscr{F}}^Y_\infty$ of the form*

$$A = \{(Y_{u_1}, \dots, Y_{u_m}) \in B\}, \quad 0 \leqslant u_1 < \dots < u_m,\ B \subseteq E^m,\ m \in \mathbb{N},$$

the limit

$$\overline{\mathbb{P}}^i(A; s_1, \dots, s_n) := \lim_{\Delta s_k \to 0,\, k \in \mathbb{N}_n} \frac{\widetilde{\mathbb{P}}^i(A,\ S_k \in (s_k, s_k + \Delta s_k],\ k \in \mathbb{N}_n)}{\widetilde{\mathbb{P}}^i(S_k \in (s_k, s_k + \Delta s_k],\ k \in \mathbb{N}_n)} \tag{3.6}$$

exists, and can be extended to a probability measure $\overline{\mathbb{P}}^i(\cdot; s_1, \dots, s_n)$ on $(\widetilde{\Omega}, \widetilde{\mathscr{F}}^Y_\infty)$. Moreover, for any $A \in \widetilde{\mathscr{F}}^Y_\infty$, the function $\overline{\mathbb{P}}^i(A; \dots)$ is Borel measurable on Δ_n, and

$$\widetilde{\mathbb{P}}^i(A) = \int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty \overline{\mathbb{P}}^i(A; s_1, \dots, s_n) \prod_{k=1}^n \left(q_k e^{-q_k(s_k - s_{k-1})}\right) \mathrm{d}s_n \cdots \mathrm{d}s_2 \, \mathrm{d}s_1. \tag{3.7}$$

In the proof of the theorem we will use the following lemma.

LEMMA 3.1. *Fix $i \in E$, $(s_1, \ldots, s_n) \in \Delta_n$, and let $0 = k_{(0)} < k_{(1)} < \cdots < k_{(n+1)}$ be positive integers. In addition, let $0 = u_0 < u_1 < \cdots < u_{k_{(1)}} \leqslant s_1 < u_{k_{(1)}+1} < \cdots < u_{k_{(2)}} \leqslant s_2 < \cdots \leqslant s_n < u_{k_{(n)}+1} < \cdots < u_{k_{(n+1)}}$, $i_0 = i$, and $i_1, \ldots, i_{k_{(n+1)}} \in E$. Then, for any cylinder set $A \in \widetilde{\mathscr{F}}^Y_\infty$ of the form*

$$A = \bigcap_{j=0}^{n} \{Y_{u_{k_{(j)}+1}} = i_{k_{(j)}+1}, \ldots, Y_{u_{k_{(j+1)}}} = i_{k_{(j+1)}}\} \tag{3.8}$$

we have

$$\begin{aligned}\lim_{\Delta s_\ell \to 0,\, \ell \in \mathbb{N}_n} & \frac{\widetilde{\mathbb{P}}^i(A, S_k \in (s_k, s_k + \Delta s_k],\, \ell \in \mathbb{N}_n)}{\widetilde{\mathbb{P}}^i(S_k \in (s_k, s_k + \Delta s_k],\, \ell \in \mathbb{N}_n)} \\ &= \prod_{\ell=0}^{n}\Big(\prod_{m=k_{(\ell)}+1}^{k_{(\ell+1)}} e^{(u_m - u_{m-1})\mathsf{G}_\ell}(i_{m-1}, i_m)\Big) \\ &\quad \cdot \sum_{j_1,\ldots,j_n \in E} \prod_{\ell=1}^{n} e^{(s_\ell - u_{k_{(\ell)}})\mathsf{G}_{\ell-1}}(i_{k_{(\ell)}}, j_\ell) e^{(u_{k_{(\ell)}+1} - s_\ell)\mathsf{G}_\ell}(j_\ell, i_{k_{(\ell)}+1}).\end{aligned} \tag{3.9}$$

In particular, for any $A \in \widetilde{\mathscr{F}}^Y_\infty$ of the form (3.8)*, the above limit is Borel measurable with respect to $(s_1, \ldots, s_n)$ in Δ_n.*

Proof. For $\ell \in \mathbb{N}_n$, choose $\Delta s_\ell > 0$ so that $s_\ell + \Delta s_\ell \leqslant u_{k_{(\ell)}+1}$. Then

$$\begin{aligned}&\widetilde{\mathbb{P}}^i(A,\, S_\ell \in (s_\ell, s_\ell + \Delta s_\ell],\, \ell \in \mathbb{N}_n) \\ &= \sum_{j_\ell, j'_\ell \in E,\, \ell \in \mathbb{N}_n} \widetilde{\mathbb{P}}^i\big(Z_{u_{k_{(\ell)}+1}} = (\ell, i_{k_{(\ell)}+1}),\, \ldots,\, Z_{u_{k_{(\ell+1)}}} = (\ell, i_{k_{(\ell+1)}}),\, \ell \in \mathbb{Z}_n; \\ &\qquad\qquad Z_{s_\ell} = (\ell - 1, j_\ell),\, Z_{s_\ell + \Delta s_\ell} = (\ell, j'_\ell),\, \ell \in \mathbb{N}_n\big) \\ &= \sum_{\substack{j_\ell, j'_\ell \in E \\ \ell \in \mathbb{N}_n}} \Big[\prod_{\ell=0}^{n}\Big(\prod_{m=k_{(\ell)}+1}^{k_{(\ell+1)}} e^{(u_m - u_{m-1})\widetilde{\mathsf{G}}}((\ell, i_{m-1}), (\ell, i_m))\Big)\Big] \\ &\qquad \cdot \Big(\prod_{\ell=1}^{n} e^{\Delta s_\ell \widetilde{\mathsf{G}}}((\ell - 1, j_\ell), (\ell, j'_\ell))\Big) \\ &\cdot \Big(\prod_{\ell=1}^{n} e^{(s_\ell - u_{k_{(\ell)}})\widetilde{\mathsf{G}}}((\ell - 1, i_{k_{(\ell)}}), (\ell - 1, j_\ell)) e^{(u_{k_{(\ell)}+1} - s_\ell - \Delta s_\ell)\widetilde{\mathsf{G}}}((\ell, j'_\ell), (\ell, i_{k_{(\ell)}+1}))\Big).\end{aligned}$$

In the above summation, the first product in brackets provides the transition probabilities of the evolutions of Z between times $u_{k_{(\ell)}}$ and $u_{k_{(\ell+1)}}$, for each $\ell \in \mathbb{Z}_n$, the second product gives the transition probabilities between times s_ℓ and $s_\ell + \Delta s_\ell$, for each $\ell \in \mathbb{N}_n$, and the third product denotes the transition probabilities between

$u_{k_{(\ell)}}$ and s_ℓ, and between $s_\ell + \Delta s_\ell$ and $u_{k_{(\ell)}+1}$, for each $\ell \in \mathbb{N}_n$. Hence, by (3.1),

$$
\begin{aligned}
\text{(3.10)} \quad \lim_{\Delta s_\ell \to 0,\, \ell \in \mathbb{N}_n} \frac{1}{\Delta s_1 \cdots \Delta s_n} \widetilde{\mathbb{P}}^i(A,\, S_\ell \in (s_\ell, s_\ell + \Delta s_\ell],\, \ell \in \mathbb{N}_n) \\
= \prod_{\ell=0}^{n} \Big(\prod_{m=k_{(\ell)}+1}^{k_{(\ell+1)}} e^{(u_m - u_{m-1})\widetilde{\mathsf{G}}}((\ell, i_{m-1}), (\ell, i_m)) \Big) \\
\cdot \sum_{j_1,\dots,j_n \in E} \prod_{\ell=1}^{n} \big(q_\ell e^{(s_\ell - u_{k_{(\ell)}})\widetilde{\mathsf{G}}}((\ell-1, i_{k_{(\ell)}}), (\ell-1, j_\ell)) \\
\cdot e^{(u_{k_{(\ell)}+1} - s_\ell)\widetilde{\mathsf{G}}}((\ell, j_\ell), (\ell, i_{k_{(\ell)}+1})) \big),
\end{aligned}
$$

and for $t \geqslant 0$,

$$
\begin{aligned}
& e^{t\widetilde{\mathsf{G}}}((\ell, j_1), (\ell, j_2)) = e^{t(\mathsf{G}_\ell - q_{\ell+1}\mathsf{I})}(j_1, j_2) = e^{-q_{\ell+1}t} e^{t\mathsf{G}_\ell}(j_1, j_2), \qquad \ell \in \mathbb{Z}_{n-1}, \\
& e^{t\widetilde{\mathsf{G}}}((n, j_1), (n, j_2)) = e^{t\mathsf{G}_n}(j_1, j_2).
\end{aligned}
$$

This, together with (3.10), implies that

$$
\begin{aligned}
\lim_{\Delta s_\ell \to 0,\, \ell = 1,\dots,n} \frac{1}{\Delta s_1 \cdots \Delta s_n} \widetilde{\mathbb{P}}^i(A,\, S_\ell \in (s_\ell, s_\ell + \Delta s_\ell],\, \ell \in \mathbb{N}_n) \\
= \Big(\prod_{\ell=1}^{n} q_\ell e^{-q_\ell (s_\ell - s_{\ell-1})} \Big) \cdot \Big[\prod_{\ell=0}^{n} \Big(\prod_{m=k_{(\ell)}+1}^{k_{(\ell+1)}} e^{(u_m - u_{m-1})\mathsf{G}_\ell}(i_{m-1}, i_m) \Big) \Big] \\
\cdot \sum_{j_1,\dots,j_n \in E} \prod_{\ell=1}^{n} \big(e^{(s_\ell - u_{k_{(\ell)}})\mathsf{G}_{\ell-1}}(i_{k_{(\ell)}}, j_\ell) e^{(u_{k_{(\ell)}+1} - s_\ell)\mathsf{G}_\ell}(j_\ell, i_{k_{(\ell)}+1}) \big).
\end{aligned}
$$

Finally, in view of the above and the fact that

$$
\text{(3.11)} \quad \lim_{\Delta s_\ell \to 0,\, \ell \in \mathbb{N}_n} \frac{\widetilde{\mathbb{P}}^i(S_\ell \in (s_\ell, s_\ell + \Delta s_\ell],\, \ell \in \mathbb{N}_n)}{\Delta s_1 \cdots \Delta s_n} = \prod_{\ell=1}^{n} q_\ell e^{-q_\ell (s_\ell - s_{\ell-1})},
$$

we obtain (3.9). The proof is complete. ∎

Proof of Theorem 3.1. Let $\mathcal{C}$ be the collection of all cylinder sets in $\widetilde{\mathscr{F}}_\infty^Y$ of the form

$$
C = \{(Y_{u_1}, \dots, Y_{u_m}) \in B\}, \quad 0 \leqslant u_1 < \cdots < u_m,\, B \subseteq E^m,\, m \in \mathbb{N}.
$$

Clearly, $\mathcal{C}$ is an algebra.

We first show that for any $C \in \mathcal{C}$ the limit in (3.6) exists and that an explicit formula for it can be derived. In fact, Lemma 3.1 shows that the limit in (3.6) exists, and belongs to $[0, 1]$, for all cylinder sets of the form (3.8). For $C \in \mathcal{C}$, an explicit formula for that limit can be obtained as follows. First, we refine the partition $0 \leqslant u_1 < \cdots < u_m$ so that each subinterval of the partition $0 < s_1 < \cdots < s_n$

contains at least one u_i. Clearly, since B_m is finite, A can be decomposed into a finite union of disjoint cylinder sets. Moreover, (3.9) provides an explicit formula for the limit in (3.6) for each of those cylinder sets. Finally, taking the finite sum over all those limits, we obtain the limit in (3.6) for C. In particular, for every cylinder set C, the limit in (3.6) is Borel measurable with respect to $(s_1, \ldots, s_n)$ in Δ_n.

In the second step we will demonstrate that the limit in (3.6) can be extended to a probability measure on $\sigma(\mathcal{C}) = \widetilde{\mathscr{F}}^Y_\infty$. We start by verifying the countable additivity of $\overline{\mathbb{P}}^i(\cdot; s_1, \ldots, s_n)$ on $\mathcal{C}$ for any fixed $(s_1, \ldots, s_n) \in \Delta_n$.

Since E is a finite set, if $(C_k)_{k\in\mathbb{N}}$ is a sequence of disjoint cylinder sets in $\mathcal{C}$ whose union also belongs to $\mathcal{C}$, then only finitely many of them are non-empty. Therefore, it suffices to verify the finite additivity of $\overline{\mathbb{P}}^i(\cdot; s_1, \ldots, s_n)$ on $\mathcal{C}$. Let $C_1, \ldots, C_k \in \mathcal{C}$ be disjoint cylinder sets. Then there exist $m \in \mathbb{N}$ and $0 \leqslant u_1 < \cdots < u_m$ such that $C_\ell = \{(Y_{u_1}, \ldots, Y_{u_m}) \in B_\ell\}$ for some $B_\ell \subseteq E^m$, $\ell = 1, \ldots, k$. Each $\overline{\mathbb{P}}^i(C_\ell; s_1, \ldots, s_n)$ can be represented as

$$\overline{\mathbb{P}}^i(C_\ell; s_1, \ldots, s_n) = \sum_{A_\ell \in \mathcal{C}_\ell} \overline{\mathbb{P}}^i(A_\ell; s_1, \ldots, s_n), \qquad \ell = 1, \ldots, k,$$

where $\mathcal{C}_1, \ldots, \mathcal{C}_k$ are disjoint classes of disjoint simple cylinder sets. Thus we have

$$\sum_{\ell=1}^k \overline{\mathbb{P}}^i(C_\ell; s_1, \ldots, s_n) = \sum_{A \in \bigcup_{\ell=1}^k \mathcal{C}_\ell} \overline{\mathbb{P}}^i(A; s_1, \ldots, s_n) = \overline{\mathbb{P}}^i\Big(\bigcup_{\ell=1}^k C_\ell; s_1, \ldots, s_n\Big).$$

Note that $\overline{\mathbb{P}}^i(C; s_1, \ldots, s_n) \leqslant 1$ for all $C \in \mathcal{C}$. By the Carathéodory extension theorem, for any $(s_1, \ldots, s_n) \in \Delta_n$, $\overline{\mathbb{P}}^i(\cdot; s_1, \ldots, s_n)$ can be uniquely extended to a probability measure on $(\widetilde{\Omega}, \widetilde{\mathscr{F}}^Y_\infty)$.

Let $\mathcal{D}_1 := \{A \in \widetilde{\mathscr{F}}^Y_\infty \mid \overline{\mathbb{P}}^i(A; \cdot, \cdots, \cdot)$ is Borel measurable on $\Delta_n\}$. We will show that $\mathcal{D}_1 = \widetilde{\mathscr{F}}^Y_\infty$. Towards this end, we first observe that (3.6) and (3.9) imply that, for any $A \in \mathcal{C}$, $\overline{\mathbb{P}}^i(A; \cdot, \cdots, \cdot)$ is Borel measurable with respect to $(s_1, \ldots, s_n)$ on Δ_n, and thus $\mathcal{D}_1 \supset \mathcal{C}$. Next, we show that $\mathcal{D}_1$ is a monotone class. For this, let $(A_k)_{k\in\mathbb{N}} \subset \mathcal{D}_1$ be an increasing sequence of events, so that, for any $(s_1, \ldots, s_n) \in \Delta_n$, we have

$$\overline{\mathbb{P}}^i\Big(\bigcup_{k=1}^\infty A_k; s_1, \ldots, s_n\Big) = \lim_{m\to\infty} \overline{\mathbb{P}}^i(A_m; s_1, \ldots, s_n).$$

Thus, $\overline{\mathbb{P}}^i(\bigcup_k A_k; \cdot, \cdots, \cdot)$, being the limit of a sequence of Borel measurable functions on Δ_n, is Borel measurable on Δ_n, and hence $\bigcup_k A_k \in \mathcal{D}_1$. Similarly, one can show that if $(A_k)_{k\in\mathbb{N}} \subset \mathcal{D}_1$ is a decreasing sequence of events, then $\bigcap_k A_k \in \mathcal{D}_1$. Therefore, $\mathcal{D}_1$ is a monotone class, and by the monotone class theorem, $\mathcal{D}_1 = \sigma(\mathcal{C}) = \widetilde{\mathscr{F}}^Y_\infty$.

It remains to show that (3.7) holds true for any $A \in \widetilde{\mathscr{F}}^Y_\infty$. In view of (3.6) and (3.11), for any $A \in \mathcal{C}$,

$$\int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty \overline{\mathbb{P}}^i(A; s_1, \ldots, s_n) \prod_{k=1}^n \left(q_k e^{-q_k(s_k - s_{k-1})}\right) \mathrm{d}s_1 \cdots \mathrm{d}s_n$$
$$= \int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty \frac{\partial^n}{\partial s_1 \cdots \partial s_n} \widetilde{\mathbb{P}}^i(A,\, S_k \leqslant s_k,\, k = 1, \ldots, n)\, \mathrm{d}s_1 \cdots \mathrm{d}s_n = \widetilde{\mathbb{P}}^i(A),$$

and thus $\mathcal{C} \subset \mathcal{D}_2$, where $\mathcal{D}_2 := \{A \in \widetilde{\mathscr{F}}^Y_\infty \,|\, \text{(3.7) holds for } A\}$. Next, for any increasing sequence of events $(A_k)_{k\in\mathbb{N}} \subset \mathcal{D}_2$, we have

$$\widetilde{\mathbb{P}}^i\Big(\bigcup_{k=1}^\infty A_k\Big) = \lim_{k\to\infty} \widetilde{\mathbb{P}}^i(A_k)$$
$$= \lim_{k\to\infty} \int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty \overline{\mathbb{P}}^i(A_k; s_1, \ldots, s_n) \prod_{\ell=1}^n \left(q_\ell e^{-q_\ell(s_\ell - s_{\ell-1})}\right) \mathrm{d}s_1 \cdots \mathrm{d}s_n$$
$$= \int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty \overline{\mathbb{P}}^i\Big(\bigcup_{k=1}^\infty A_k; s_1, \ldots, s_n\Big) \prod_{\ell=1}^n \left(q_\ell e^{-q_\ell(s_\ell - s_{\ell-1})}\right) \mathrm{d}s_1 \cdots \mathrm{d}s_n,$$

where the last equality follows from the dominated convergence theorem. Hence, $\bigcup_k A_k \in \mathcal{D}_2$. Similarly, one can show that if $(A_k)_{k\in\mathbb{N}} \subset \mathcal{D}_2$ is a decreasing sequence, then $\bigcap_k A_k \in \mathcal{D}_2$. Therefore, $\mathcal{D}_2$ is a monotone class, and by the monotone class theorem, $\mathcal{D}_2 = \sigma(\mathcal{C}) = \widetilde{\mathscr{F}}^Y_\infty$. This completes the proof. ■

Next, we will prove that the law of Y under $\overline{\mathbb{P}}^i$ is the same as that of X under $\mathbb{P}^i$. As usual, $\overline{\mathbb{E}}^i(\cdot; s_1, \ldots, s_n)$ will denote the expectation associated with $\overline{\mathbb{P}}^i(\cdot; s_1, \ldots, s_n)$ for $i \in E$ and $(s_1, \ldots, s_n) \in \Delta_n$. In what follows, if there is no ambiguity, we will omit the parameters $s_1, \ldots, s_n$ in $\overline{\mathbb{P}}^i$ and $\overline{\mathbb{E}}^i$.

THEOREM 3.2. *For any $i \in E$ and $(s_1, \ldots, s_n) \in \Delta_n$, under $\overline{\mathbb{P}}^i$, Y is a time-inhomogeneous Markov chain with generator $\mathsf{G} = \{\mathsf{G}_t, t \geqslant 0\}$. In particular, X and Y have the same law under respective probability measures $\mathbb{P}^i$ and $\overline{\mathbb{P}}^i$.*

Proof. Let $u_0, u_1, \ldots, u_m$ be such that

$$0 = u_0 \leqslant u_1 < \cdots < u_{k_1} \leqslant s_1 < u_{k_1+1} < \cdots < u_{k_2}$$
$$\leqslant s_2 < \cdots \leqslant s_n < u_{k_n+1} < \cdots < u_{k_{n+1}} = u_m.$$

Using (3.9) and a routine calculation we conclude that, for any $i_1, \ldots, i_m \in E$,

$$\overline{\mathbb{P}}^i(Y_{u_m} = i_m \,|\, Y_{u_{m-1}} = i_{m-1}, \ldots, Y_{u_1} = i_1) = e^{(u_m - u_{m-1})\mathsf{G}_n}(i_{m-1}, i_m),$$

and $\overline{\mathbb{P}}^i(Y_{u_m} = i_m \mid Y_{u_{m-1}} = i_{m-1}) = e^{(u_m - u_{m-1})\mathsf{G}_n}(i_{m-1}, i_m)$.

An analogous argument can be carried out for any $u_0 < u_1 < \cdots < u_m$, which completes the proof. ■

In analogy to φ_t and τ_t^+ we now define an additive functional ψ by $\psi_t := \int_0^t v(Y_u)\,\mathrm{d}u$, $t \geqslant 0$, and we consider the first passage time $\rho_t^+ := \inf\{r \geqslant 0 \mid \psi_r > t\}$, $t \geqslant 0$.

With this definition, we have the following corollary to Theorem 3.2.

COROLLARY 3.1. *For any* $(s_1, \ldots, s_n) \in \Delta_n$ *and* $c, t > 0$,

$$\Pi_c^+(i, j; s_1, \ldots, s_n) = \overline{\mathbb{E}}^i(e^{-c\rho_0^+}\mathbf{1}_{\{Y_{\rho_0^+}=j\}}; s_1, \ldots, s_n),\ i \in E^-,\ j \in E^+, \tag{3.12}$$

$$\Psi_c^+(t, i, j; s_1, \ldots, s_n) = \overline{\mathbb{E}}^i(e^{-c\rho_t^+}\mathbf{1}_{\{Y_{\rho_t^+}=j\}}; s_1, \ldots, s_n),\quad i \in E^+,\ j \in E^+. \tag{3.13}$$

In particular, $\Pi_c^+(i, j; \ldots)$ *and* $\Psi_c^+(t, i, j; \ldots)$ *are Borel measurable on* Δ_n.

3.1. Wiener–Hopf factorization for $Z = (N, Y)$. This subsection is devoted to computing the expectations on the right-hand side in (3.12) and (3.13). This will be done by computing the corresponding expectations relative to the *time-homogeneous* Markov chain $Z = (N, Y)$. The latter computation will be done using the classical results for finite-state time-homogeneous Markov chains, originally derived in [BRW80].

We begin with a restatement of the classical Wiener–Hopf factorization applied to Z. Towards this end, we let $\widetilde{E}^+ := \mathbb{Z}_n \times E^+$, $\widetilde{E}^- := \mathbb{Z}_n \times E^-$, and we take $\widetilde{v} : \widetilde{E} \to \mathbb{R} \setminus \{0\}$ defined by $\widetilde{v}(k, i) = v(i)$ for all $(k, i) \in \widetilde{E}$. We reorder the entries of $\widetilde{\mathsf{G}}$ defined in (3.1) so that the upper-left (respectively, lower-right) block of $\widetilde{\mathsf{G}}$ is on $\widetilde{E}^+$ (respectively, $\widetilde{E}^-$). Next, we define an additive functional $\widetilde{\varphi}$ and the corresponding first passage times by

$$\widetilde{\varphi}_t := \int_0^t \widetilde{v}(Z_u)\,\mathrm{d}u, \qquad \widetilde{\tau}_t^\pm := \inf\{r \geqslant 0 \mid \pm\widetilde{\varphi}_r > t\}, \qquad t \geqslant 0.$$

Let $\widetilde{V} := \operatorname{diag}\{\widetilde{v}(k, i) : (k, i) \in \widetilde{E}\}$ (a diagonal matrix). We denote by $\widetilde{\mathsf{I}}^\pm$ the identity matrix of dimension $|\widetilde{E}^\pm|$. Finally, $\mathcal{Q}(m)$ will stand for the set of $m \times m$ generator matrices (i.e., matrices with non-negative off-diagonal entries and non-positive row sums), and $\mathcal{P}(m, \ell)$ will be the set of $m \times \ell$ matrices whose rows are subprobability vectors.

THEOREM 3.3 ([BRW80, Theorems 1 & 2]). *Fix* $c > 0$. *Then*

(i) *there exists a unique quadruple of matrices* $(\widetilde{\Lambda}_c^+, \widetilde{\Lambda}_c^-, \widetilde{\mathsf{G}}_c^+, \widetilde{\mathsf{G}}_c^-)$, *where* $\widetilde{\Lambda}_c^+ \in \mathcal{P}(|\widetilde{E}^-|, |\widetilde{E}^+|)$, $\widetilde{\Lambda}_c^- \in \mathcal{P}(|\widetilde{E}^+|, |\widetilde{E}^-|)$, $\widetilde{\mathsf{G}}_c^+ \in \mathcal{Q}(|\widetilde{E}^+|)$, *and* $\widetilde{\mathsf{G}}_c^- \in \mathcal{Q}(|\widetilde{E}^-|)$, *such that*

$$\widetilde{V}^{-1}(\widetilde{\mathsf{G}} - c\,\widetilde{\mathsf{I}})\begin{pmatrix} \widetilde{\mathsf{I}}^+ & \widetilde{\Lambda}_c^- \\ \widetilde{\Lambda}_c^+ & \widetilde{\mathsf{I}}^- \end{pmatrix} = \begin{pmatrix} \widetilde{\mathsf{I}}^+ & \widetilde{\Lambda}_c^- \\ \widetilde{\Lambda}_c^+ & \widetilde{\mathsf{I}}^- \end{pmatrix}\begin{pmatrix} \widetilde{\mathsf{G}}_c^+ & 0 \\ 0 & -\widetilde{\mathsf{G}}_c^- \end{pmatrix}; \tag{3.14}$$

(ii) *the matrices* $\widetilde{\Lambda}_c^{\pm}$ *and* $\widetilde{\mathsf{G}}_c^{\pm}$ *admit the following probabilistic representations:*

$$\widetilde{\Lambda}_c^{+}((k,i),(\ell,j)) = \widetilde{\mathbb{E}}(e^{-c\widetilde{\tau}_0^+}\mathbf{1}_{\{Z_{\widetilde{\tau}_0^+}=(\ell,j)\}} \mid Z_0=(k,i)),\ (k,i)\in\widetilde{E}^-,\ (\ell,j)\in\widetilde{E}^+,$$

$$\widetilde{\Lambda}_c^{-}((k,i),(\ell,j)) = \widetilde{\mathbb{E}}(e^{-c\widetilde{\tau}_0^-}\mathbf{1}_{\{Z_{\widetilde{\tau}_0^-}=(\ell,j)\}} \mid Z_0=(k,i)),\ (k,i)\in\widetilde{E}^+,\ (\ell,j)\in\widetilde{E}^-,$$

$$e^{t\widetilde{\mathsf{G}}_c^{+}}((k,i),(\ell,j)) = \widetilde{\mathbb{E}}(e^{-c\widetilde{\tau}_t^+}\mathbf{1}_{\{Z_{\widetilde{\tau}_t^+}=(\ell,j)\}} \mid Z_0=(k,i)),\quad (k,i),(\ell,j)\in\widetilde{E}^+,$$

$$e^{t\widetilde{\mathsf{G}}_c^{-}}((k,i),(\ell,j)) = \widetilde{\mathbb{E}}(e^{-c\widetilde{\tau}_t^-}\mathbf{1}_{\{Z_{\widetilde{\tau}_t^-}=(\ell,j)\}} \mid Z_0=(k,i)),\quad (k,i),(\ell,j)\in\widetilde{E}^-,$$

for any $t\geqslant 0$.

In what follows we will use the "+" part of the above formulae and only for $k=0$. Accordingly, we define

$$\widetilde{\Pi}_c^{+}(i,j,\ell) := \widetilde{\Lambda}_c^{+}((0,i),(\ell,j)) = \widetilde{\mathbb{E}}^i(e^{-c\widetilde{\tau}_0^+}\mathbf{1}_{\{Z_{\widetilde{\tau}_0^+}=(\ell,j)\}}),\quad i\in E^-, j\in E^+, \tag{3.15}$$

$$\widetilde{\Psi}_c^{+}(t,i,j,\ell) := e^{t\widetilde{\mathsf{G}}_c^{+}}((0,i),(\ell,j)) = \widetilde{\mathbb{E}}^i(e^{-c\widetilde{\tau}_t^+}\mathbf{1}_{\{Z_{\widetilde{\tau}_t^+}=(\ell,j)\}}),\ i,j\in E^+, \tag{3.16}$$

for any $\ell\in\mathbb{N}$ and $t\geqslant 0$. Note that, for any $t\geqslant 0$, $\widetilde{v}(Z_t)=v(Y_t)$, which implies that $\widetilde{\varphi}_t=\psi_t$, and so $\rho_t^+=\widetilde{\tau}_t^+$, $\rho_t^-=\widetilde{\tau}_t^-$. Hence, by summing over all $\ell\in\mathbb{N}$ in (3.15) and (3.16), we obtain

$$\widetilde{\mathbb{E}}^i(e^{-c\rho_0^+}\mathbf{1}_{\{Y_{\rho_0^+}=j\}}) = \sum_{\ell=0}^{n}\widetilde{\Pi}_c^{+}(i,j,\ell),\qquad i\in E^-,\ j\in E^+, \tag{3.17}$$

$$\widetilde{\mathbb{E}}^i(e^{-c\rho_t^+}\mathbf{1}_{\{Y_{\rho_t^+}=j\}}) = \sum_{\ell=0}^{n}\widetilde{\Psi}_c^{+}(t,i,j,\ell),\quad i,j\in E^+,\ t\geqslant 0. \tag{3.18}$$

Observe that, in view of (3.7), if $U:\widetilde{\Omega}\to\mathbb{R}$ is an $\widetilde{\mathscr{F}}_{\infty}^{Y}$-measurable bounded random variable, then for any $i\in E$,

$$\widetilde{\mathbb{E}}^i(U) = \int_0^{\infty}\int_{s_1}^{\infty}\cdots\int_{s_{n-1}}^{\infty}\overline{\mathbb{E}}^i(U;s_1,\ldots,s_n)\prod_{k=1}^{n}\left(q_k e^{-q_k(s_k-s_{k-1})}\right)\mathrm{d}s_n\cdots\mathrm{d}s_2\,\mathrm{d}s_1.$$

Therefore, in light of Corollary 3.1, (3.17) and (3.18), we have

$$\begin{aligned}\widehat{\Pi}_c^{+}(i,j;q_1,\ldots,q_n) &:= \sum_{\ell=0}^{n}\widetilde{\Pi}_c^{+}(i,j,\ell)\\ &= \int_0^{\infty}\int_{s_1}^{\infty}\ldots\int_{s_{n-1}}^{\infty}\Pi_c^{+}(i,j;s_1,\ldots,s_n)\prod_{k=1}^{n}\left(q_k e^{-q_k(s_k-s_{k-1})}\right)\mathrm{d}s_n\cdots\mathrm{d}s_2\,\mathrm{d}s_1,\end{aligned}$$

$$\widehat{\Psi}_c^+(t,i,j;q_1,\ldots,q_n) := \sum_{\ell=0}^{n} \widetilde{\Psi}_c^+(t,i,j,\ell)$$
$$= \int_0^\infty \int_{s_1}^\infty \ldots \int_{s_{n-1}}^\infty \Psi_c^+(t,i,j;s_1,\ldots,s_n) \prod_{k=1}^n \big(q_k e^{-q_k(s_k-s_{k-1})}\big)\, \mathrm{d}s_n \cdots \mathrm{d}s_2\, \mathrm{d}s_1.$$

By a change of variables, we obtain

$$\widehat{\Pi}_c^+(i,j;q_1,\ldots,q_n) = \int_0^\infty \ldots \int_0^\infty \Pi_c^+\Big(i,j;t_1,\ldots,\sum_{k=1}^n t_k\Big) \prod_{k=1}^n (q_k e^{-q_k t_k})\, \mathrm{d}t_1 \cdots \mathrm{d}t_n,$$
$$\widehat{\Psi}_c^+(i,j;q_1,\ldots,q_n) = \int_0^\infty \ldots \int_0^\infty \Psi_c^+\Big(i,j;t_1,\ldots,\sum_{k=1}^n t_k\Big) \prod_{k=1}^n (q_k e^{-q_k t_k})\, \mathrm{d}t_1 \cdots \mathrm{d}t_n.$$

The above two equalities together with the argument in Section 5 imply that

$$q_1^{-1} \cdots q_n^{-1} \widehat{\Pi}_c^+(i,j;q_1,\ldots,q_n), \qquad q_1^{-1} \cdots q_n^{-1} \widehat{\Psi}_c^+(i,j;q_1,\ldots,q_n)$$

are well-defined for all $q_k \in \mathbb{C}^+ := \{z \in \mathbb{C} \mid \Re(z) > 0\}$ with $k = 1,\ldots,n$, as being the Laplace transforms of $\Pi_c^+(i,j;t_1,\ldots,t_1+\cdots+t_n)$ and $\Psi_c^+(i,j;t_1,\ldots,t_1+\cdots+t_n)$, respectively.

All the above leads to the following result, which is our main theorem, and where we make use of the inverse multivariate Laplace transform. We refer to the Appendix for the definition and the properties of the inverse multivariate Laplace transform relevant to our set-up.

THEOREM 3.4. *We have*

$$\Pi_c^+(i,j;s_1,\ldots,s_n) = \mathcal{L}^{-1}\bigg(\frac{\widehat{\Pi}_c^+(i,j;q_1,\ldots,q_n)}{\prod_{k=1}^n q_k}\bigg)(s_1, s_2-s_1,\ldots,s_n-s_{n-1})$$

for any $i \in E^-$, $j \in E^+$, and

$$\Psi_c^+(t,i,j;s_1,\ldots,s_n)$$
$$= \mathcal{L}^{-1}\bigg(\frac{\widehat{\Psi}_c^+(t,i,j;q_1,\ldots,q_n)}{\prod_{k=1}^n q_k}\bigg)(s_1, s_2-s_1,\ldots,s_n-s_{n-1})$$

for any $t > 0$, $i,j \in E^+$, where $\mathcal{L}^{-1}$ is the inverse multivariate Laplace transform.

REMARK 3.1. It has to be stressed that we can compute the values of $\widehat{\Pi}_c^+(i,j;q_1,\ldots,q_n)$ and $\widehat{\Psi}_c^+(t,i,j;q_1,\ldots,q_n)$ only for positive values of q_i's. Thus, Theorem 3.4 cannot be directly applied to compute $\Pi_c^+(i,j;s_1,\ldots,s_n)$ and $\Psi_c^+(t,i,j;s_1,\ldots,s_n)$. However, we can approximate these functions, as explained in Section 5.1, using values of $\widehat{\Pi}_c^+(i,j;q_1,\ldots,q_n)$ and $\widehat{\Psi}_c^+(t,i,j;q_1,\ldots,q_n)$ for positive of q_i's only.

4. NUMERICAL EXAMPLE

In this section we illustrate our theoretical results with a simple, but telling example. We first describe a numerical method to approximate Π_c^+ and Ψ_c^+, and then we proceed with its application to a concrete example.

4.1. Numerical procedure to approximate Π_c^+ and Ψ_c^+. We only consider Π_c^+. The procedure to approximate Ψ_c^+ is analogous.

According to Theorem 3.4 and Section 5.1, to approximate Π_c^+, we need to compute $\widehat{\Pi}_c^+(i,j;q_1,\ldots,q_n)$ for any $q_1,\ldots,q_n > 0$, and then to use the Gaver–Stehfest algorithm. Note that $\widehat{\Pi}_c^+(i,j;q_1,\ldots,q_n)$ can be computed by solving (3.14) directly using the diagonalization method of [RS94]. However, because of the special structure of $\widetilde{\mathsf{G}}$, we can simplify the calculation by working with matrices of smaller dimensions. Towards this end we observe that matrices in (3.14) can be written in block form as follows:

$$
\widetilde{\Lambda}_c^+ = \begin{array}{c} \\ (0,E^-) \\ (1,E^-) \\ \vdots \\ (n-1,E^-) \\ (n,E^-) \end{array}
\begin{array}{c}
\begin{array}{ccccc} (0,E^+) & (1,E^+) & \cdots & (n-1,E^+) & (n,E^+) \end{array} \\
\begin{bmatrix}
\widetilde{\Lambda}^+_{c,00} & \widetilde{\Lambda}^+_{c,01} & \cdots & \widetilde{\Lambda}^+_{c,0,n-1} & \widetilde{\Lambda}^+_{c,0n} \\
0 & \widetilde{\Lambda}^+_{c,11} & \cdots & \widetilde{\Lambda}^+_{c,1,n-1} & \widetilde{\Lambda}^+_{c,1n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \widetilde{\Lambda}^+_{c,n-1,n-1} & \widetilde{\Lambda}^+_{c,n-1,n} \\
0 & 0 & \cdots & 0 & \widetilde{\Lambda}^+_{c,nn}
\end{bmatrix}
\end{array}, \tag{4.1}
$$

$$
\widetilde{\Lambda}_c^- = \begin{array}{c} \\ (0,E^+) \\ (1,E^+) \\ \vdots \\ (n-1,E^+) \\ (n,E^+) \end{array}
\begin{array}{c}
\begin{array}{ccccc} (0,E^-) & (1,E^-) & \cdots & (n-1,E^-) & (n,E^-) \end{array} \\
\begin{bmatrix}
\widetilde{\Lambda}^-_{c,00} & \widetilde{\Lambda}^-_{c,01} & \cdots & \widetilde{\Lambda}^-_{c,0,n-1} & \widetilde{\Lambda}^-_{c,0n} \\
0 & \widetilde{\Lambda}^-_{c,11} & \cdots & \widetilde{\Lambda}^-_{c,1,n-1} & \widetilde{\Lambda}^-_{c,1n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \widetilde{\Lambda}^-_{c,n-1,n-1} & \widetilde{\Lambda}^-_{c,n-1,n} \\
0 & 0 & \cdots & 0 & \widetilde{\Lambda}^-_{c,nn}
\end{bmatrix}
\end{array}, \tag{4.2}
$$

$$
\widetilde{\mathsf{G}}_c^+ = \begin{array}{c} \\ (0,E^+) \\ (1,E^+) \\ \vdots \\ (n-1,E^+) \\ (n,E^+) \end{array}
\begin{array}{c}
\begin{array}{ccccc} (0,E^+) & (1,E^+) & \cdots & (n-1,E^+) & (n,E^+) \end{array} \\
\begin{bmatrix}
\widetilde{\mathsf{G}}^+_{c,00} & \widetilde{\mathsf{G}}^+_{c,01} & \cdots & \widetilde{\mathsf{G}}^+_{c,0,n-1} & \widetilde{\mathsf{G}}^+_{c,0n} \\
0 & \widetilde{\mathsf{G}}^+_{c,11} & \cdots & \widetilde{\mathsf{G}}^+_{c,1,n-1} & \widetilde{\mathsf{G}}^+_{c,1n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \widetilde{\mathsf{G}}^+_{c,n-1,n-1} & \widetilde{\mathsf{G}}^+_{c,n-1,n} \\
0 & 0 & \cdots & 0 & \widetilde{\mathsf{G}}^+_{c,nn}
\end{bmatrix}
\end{array}, \tag{4.3}
$$

$$\widetilde{\mathsf{G}}_c^- = \begin{array}{c} \\ (0,E^-) \\ (1,E^-) \\ \vdots \\ (n-1,E^-) \\ (n,E^-) \end{array} \begin{array}{c} \begin{array}{ccccc} (0,E^-) & (1,E^-) & \cdots & (n-1,E^-) & (n,E^-) \end{array} \\ \left[\begin{array}{ccccc} \widetilde{\mathsf{G}}^-_{c,00} & \widetilde{\mathsf{G}}^-_{c,01} & \cdots & \widetilde{\mathsf{G}}^-_{c,0,n-1} & \widetilde{\mathsf{G}}^-_{c,0n} \\ 0 & \widetilde{\mathsf{G}}^-_{c,11} & \cdots & \widetilde{\mathsf{G}}^-_{c,1,n-1} & \widetilde{\mathsf{G}}^-_{c,1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \widetilde{\mathsf{G}}^-_{c,n-1,n-1} & \widetilde{\mathsf{G}}^-_{c,n-1,n} \\ 0 & 0 & \cdots & 0 & \widetilde{\mathsf{G}}^-_{c,nn} \end{array}\right] \end{array}, \tag{4.4}$$

$$\widetilde{\mathsf{V}} = \begin{array}{c} \\ (0,E^+) \\ (1,E^+) \\ \vdots \\ (n,E^+) \\ (0,E^-) \\ (1,E^-) \\ \vdots \\ (n,E^-) \end{array} \begin{array}{c} \begin{array}{cccccccc} (0,E^+) & (1,E^+) & \cdots & (n,E^+) & (0,E^-) & (1,E^-) & \cdots & (n,E^-) \end{array} \\ \left[\begin{array}{cccccccc} \mathsf{V}^+ & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \mathsf{V}^+ & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathsf{V}^+ & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \mathsf{V}^- & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \mathsf{V}^- & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \mathsf{V}^- \end{array}\right] \end{array}, \tag{4.5}$$

and

$$\widetilde{\mathsf{G}} = \begin{array}{c} \\ (0,E^+) \\ (1,E^+) \\ \vdots \\ (n,E^+) \\ (0,E^-) \\ (1,E^-) \\ \vdots \\ (n,E^-) \end{array} \begin{array}{c} \begin{array}{cccccccc} (0,E^+) & (1,E^+) & \cdots & (n,E^+) & (0,E^-) & (1,E^-) & \cdots & (n,E^-) \end{array} \\ \left[\begin{array}{cccccccc} \mathsf{A}_1 & q_1\mathsf{I}^+ & \cdots & 0 & \mathsf{B}_1 & 0 & \cdots & 0 \\ 0 & \mathsf{A}_2 & \cdots & 0 & 0 & \mathsf{B}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathsf{A}_{n+1} & 0 & 0 & \cdots & \mathsf{B}_{n+1} \\ \mathsf{C}_1 & 0 & \cdots & 0 & \mathsf{D}_1 & q_1\mathsf{I}^- & \cdots & 0 \\ 0 & \mathsf{C}_2 & \cdots & 0 & 0 & \mathsf{D}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathsf{C}_{n+1} & 0 & 0 & \cdots & \mathsf{D}_{n+1} \end{array}\right] \end{array}$$

$$- \begin{array}{c} \\ (0,E^+) \\ (1,E^+) \\ \vdots \\ (n,E^+) \\ (0,E^-) \\ (1,E^-) \\ \vdots \\ (n,E^-) \end{array} \begin{array}{c} \begin{array}{cccccccc} (0,E^+) & (1,E^+) & \cdots & (n,E^+) & (0,E^-) & (1,E^-) & \cdots & (n,E^-) \end{array} \\ \left[\begin{array}{cccccccc} q_1\mathsf{I}^+ & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & q_2\mathsf{I}^+ & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{n+1}\mathsf{I}^+ & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & q_1\mathsf{I}^- & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & q_2\mathsf{I}^- & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & q_{n+1}\mathsf{I}^- \end{array}\right] \end{array},$$

where $q_{n+1} = 0$.

Plugging (4.5)–(4.4) into (3.14) and then comparing all the block entries on both sides gives an idea for a procedure to compute the factorization recursively.

In order to describe the procedure, in accordance to Theorem 3.3, for any generator matrix H and any constant $c > 0$, we denote by

$$(\Lambda_c^+(\mathsf{H}), \Lambda_c^-(\mathsf{H}), \mathsf{G}_c^+(\mathsf{H}), \mathsf{G}_c^-(\mathsf{H}))$$

the unique quadruple constituting the classical Wiener–Hopf factorization (cf. [BRW80]) corresponding to H with killing rate c. In addition, we let $c_k = q_k + c$, $k \in \mathbb{N}$.

1. **Compute the first diagonal:** for $k = 1, \ldots, n+1$, compute $\widetilde{\Lambda}^+_{c,k-1,k-1} = \Lambda^+_{c_k}(\mathsf{G}_k)$, using the diagonalization method in [RS94].
2. **Compute the second diagonal:** for $k = 1, \ldots, n$, solve the following linear system for $\widetilde{\Lambda}^+_{c,k-1,k}$ and $\widetilde{\mathsf{G}}^+_{c,k-1,k}$:

$$q_k \mathsf{I}^+ + \mathsf{B}_k \widetilde{\Lambda}^+_{c,k-1,k} = \mathsf{V}^+ \widetilde{\mathsf{G}}^+_{c,k-1,k},$$
$$[\mathsf{D}_k - c_k \mathsf{I}^-]\widetilde{\Lambda}^+_{c,k-1,k} + q_k \widetilde{\Lambda}^+_{c,k,k} = \mathsf{V}^- \widetilde{\Lambda}^+_{c,k-1,k-1} \widetilde{\mathsf{G}}^+_{c,k-1,k} + \mathsf{V}^- \widetilde{\Lambda}^+_{c,k-1,k} \widetilde{\mathsf{G}}^+_{c,k,k}.$$

3. **Compute the other diagonals:** for $r = 2, \ldots, n$ and $k = 0, \ldots, n-r$, solve the following linear system for $\widetilde{\Lambda}^+_{c,k,k+r}$ and $\widetilde{\mathsf{G}}^+_{c,k,k+r}$:

$$\mathsf{B}_{k+1} \widetilde{\Lambda}^+_{c,k,k+r} = \mathsf{V}^+ \widetilde{\mathsf{G}}^+_{c,k,k+r},$$
$$[\mathsf{D}_{k+1} - c_{k+1} \mathsf{I}^-]\widetilde{\Lambda}^+_{c,k,k+r} + q_{k+1} \widetilde{\Lambda}^+_{c,k+1,k+r} = \mathsf{V}^- \sum_{j=0}^{r} \widetilde{\Lambda}^+_{c,k,k+j} \widetilde{\mathsf{G}}^+_{c,k+j,k+r}.$$

4. **Compute:** for $q_1, \ldots, q_n > 0$,

$$P^+(q_1, \ldots, q_n) := q_1^{-1} \cdots q_n^{-1} \widehat{\Pi}_c^+(i, j; q_1, \ldots, q_n) = q_1^{-1} \cdots q_n^{-1} \sum_{\ell=0}^{n} \widetilde{\Lambda}_{c,0\ell}.$$

5. **Compute the approximate inverse Laplace transform of $P^+(q_1, \ldots, q_n)$ in order to approximate Π_c^+:** use the method discussed in Section 5.1.

REMARK 4.1. If $|E^+| = |E^-| = 1$, then the matrices in Steps 1–3 become numbers. Step 1 reduces to solving $n+1$ quadratic equations for a root in $[0, 1]$. In Steps 2 and 3, for each loop, the system reduces to a system of two linear equations in two unknowns. Moreover, in this case, P^+ has a closed-form representation for $q_1, \ldots, q_n > 0$, and hence for any $q_1, \ldots, q_n \in \mathbb{C}^+$, as mentioned in the previous section. This allows one to use general numerical inverse Laplace transform methods, and not necessarily the Gaver-Stehfest formula from Section 5.1. In particular, one can use the Talbot approximation formula (5.1) below, which is more efficient than the Gaver–Stehfest formula under fairly general assumptions (cf. [AW06]).

4.2. Application in fluid flow problems. The Wiener–Hopf factorization for a time-homogeneous finite Markov chain was applied in [Rog94] in the context of fluid models of queues. In this section, we will apply our results to a time-inhomogeneous Markov chain fluid flow problem.

First, we briefly review the classical fluid flow problem (see [Mit88] and [Rog94] for a detailed discussion). Suppose we have a large water tank with capacity $a \in (0, \infty]$. On the top of the tank, there are $I_t \in \mathcal{I}$ pipes open at time t, with each pipe pouring water into the tank at rate r^+. At the bottom of the tank, there are $O_t \in \mathcal{O}$ taps open at time t, with each tap allowing water to flow out at rate r^-. We assume that $\mathcal{I}$ and $\mathcal{O}$ are finite sets.

Then the volume ξ_t of water in the tank at time t satisfies

$$\frac{\mathrm{d}\xi_t}{\mathrm{d}t} = r^+ I_t - r^- O_t \quad \text{if } 0 < \xi_t < a. \tag{4.6}$$

Moreover, if $\xi_t = 0$, i.e. if the tank is empty, then the outflow ceases. If $\xi_t = a$, i.e. if the tank is full, then water flows over the top.

Let f be a real valued function on $\mathcal{I} \times \mathcal{O}$. We assume $X_t := f(I_t, O_t),\ t \geqslant 0$, is a (finite state) time-inhomogeneous Markov chain, and we denote by E the state space of X. Let $v(x) := V(r^+, r^-, x)$, $x \in E$, model the water outflow/inflow rate, in terms of the states of X, so that $v(X_t) = V(r^+, r^-, f(I_t, O_t))$, $t \geqslant 0$, represents the water outflow/inflow at time t. In the context of model (4.6) we would have $X_t = f(I_t, O_t) = (I_t, O_t)$ and $V(r^+, r^-, f(I, O)) = r^+ I - r^- O$.

It is well known that, typically, the dynamics of water volume in a retention tank is subject to seasonal and/or daily temporal influences. This is what calls for modeling the rate $v(X_t)$ in terms of a time-inhomogeneous Markov chain.

Let E^+ be the set of states of X such that the water tank has greater water inflow than outflow, and let E^- be the set of states of X with greater water outflow than inflow. The integral $\varphi_t = \int_0^t v(X_u)\,\mathrm{d}u$ is not exactly the water content at time t, because we should take into account those periods when the tank is full or empty. However, as noted in [Rog94], understanding φ_t, and the corresponding $\tau_t^{\pm}$ and $X_{\tau_t^{\pm}}$, allows us to easily express the quantities of interest for ξ_t in terms of Wiener–Hopf factorization, and to further compute these quantities once we compute the Wiener–Hopf factorization numerically.

We now assume that the tank has infinite capacity, $a = \infty$, and that it contains ℓ units of water at time $t = 0$. Thus, τ_ℓ^- represents the first time after $t = 0$ that the tank goes empty. We will compute the quantity

$$\Pi_c^-(i, j) = \mathbb{E}^i\big(e^{-c\tau_0^-}\mathbf{1}_{\{X_{\tau_0^-} = j\}}\big), \qquad i \in E^+,\ j \in E^-.$$

Towards this end, we further assume that the tank has either an aggregate water inflow at rate v^+ or an aggregate water outflow at rate v^-. In other words, $E^+ = \{e_+\}$, $E^- = \{e_-\}$, $v(e_+) = v^+$, and $v(e_-) = v^-$. Moreover, we assume that the time-inhomogeneous Markov chain X has the generator

$$\mathsf{G}_t = \mathsf{G}_1\mathbf{1}_{[s_0,s_1)}(t) + \mathsf{G}_2\mathbf{1}_{[s_1,s_2)}(t) + \mathsf{G}_3\mathbf{1}_{[s_2,\infty)}(t), \qquad 0 < s_1 < s_2.$$

We take the following inputs: $c = 0.5$, $v(e_+) = 2$, $v(e_-) = -3$, $s_1 = 2$, $s_2 = 8$,

$$\mathsf{G}_0 = \begin{matrix} & e_+ & e_- \\ e_+ & -2 & 2 \\ e_- & 1 & -1 \end{matrix}, \quad \mathsf{G}_1 = \begin{matrix} & e_+ & e_- \\ e_+ & -3 & 3 \\ e_- & 2 & -2 \end{matrix}, \quad \mathsf{G}_2 = \begin{matrix} & e_+ & e_- \\ e_+ & -5 & 5 \\ e_- & 3 & -3 \end{matrix}.$$

The following table compares our result and execution time with Monte-Carlo simulation (10000 paths).

Method	Wiener–Hopf	Monte-Carlo
$\Pi_c^-(e_+, e_-)$	0.6501	0.6462
Execution time	$0.15\,s$	$3.12\,s$

REMARK 4.2. One can also compute $\Pi_c^+(e_-, e_+)$, if it is the quantity of interest in the model. Note that if we change the labels of the states from $\{e_+, e_-\}$ to $\{e_-, e_+\}$ and modify the inputs accordingly, we can compute $\Pi_c^+(e_-, e_+)$ using the same algorithm that computes $\Pi_c^-(e_+, e_-)$.

5. APPENDIX: APPROXIMATION OF MULTIVARIATE INVERSE LAPLACE TRANSFORM

For the convenience of the reader, we briefly recall the basics of Laplace transform and its inverse. Then, we proceed with an important result regarding approximation of the multivariate inverse Laplace transform. The dimension n of the inverse Laplace transform is equal to the number of switches (changes) in the time-dependent generator function G_t.

Let $f : [0, \infty)^n \to [0, \infty)$ be a Borel measurable function such that

$$\int_0^\infty \cdots \int_0^\infty f(t_1, \ldots, t_n) \prod_{k=1}^n e^{-q_k t_k}\, \mathrm{d}t_1 \cdots \mathrm{d}t_n$$

exists for any $q_1, \ldots, q_n > 0$. Then the multivariate Laplace transform $\widehat{f}$ of f defined by

$$\widehat{f}(q_1, \ldots, q_n) = \mathcal{L}(f)(q_1, \ldots, q_n) := \int_0^\infty \cdots \int_0^\infty f(t_1, \ldots, t_n) \prod_{k=1}^n e^{-q_k t_k}\, \mathrm{d}t_1 \cdots \mathrm{d}t_n$$

is well defined for any $q_k \in \mathbb{C}^+$, $k = 1, \ldots, n$, where $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \Re(z) > 0\}$ with $\Re(z)$ denoting the real part of $z \in \mathbb{C}$. The inverse multivariate Laplace transform of $g : (\mathbb{C}^+)^n \to \mathbb{C}$ is the function $\breve{g}$ such that $\mathcal{L}(\breve{g}) = g$. We will also write $\breve{g} = \mathcal{L}^{-1}(g)$. The existence and uniqueness of the inverse Laplace transform is a well understood subject (see [Wid41]). Although there are explicit formulas for the inverse Laplace transform for many functions, in many practical situations the inverse Laplace transform of a function is computed by numerical approximation techniques. We refer the reader to [AW06], and the references therein, for a unified framework for numerically inverting the Laplace transform. For completeness, we here present one such method—the Talbot inversion formula—for one and two dimensions; the multidimensional case is done by analogy.

Assume that $\widehat{f}$ is the Laplace transform of a function $f : (0,\infty) \to \mathbb{C}$. The Talbot inversion formula to approximate f is given by

$$f_M^b(t) = \frac{2}{5t} \sum_{k=0}^{M-1} \Re\left(\gamma_k \widehat{f}\left(\frac{\delta_k}{t}\right)\right), \tag{5.1}$$

where (with $\mathrm{i} = \sqrt{-1}$)

$$\delta_0 = \frac{2M}{5}, \quad \delta_k = \frac{22k\pi}{5}\left(\cot\left(\frac{k\pi}{M}\right) + \mathrm{i}\right), \qquad 0 < k < M,$$

$$\gamma_0 = \frac{1}{2}e^{\delta_0}, \quad \gamma_k = \left(1 + \mathrm{i}\frac{k\pi}{M}\left(1 + \cot^2\left(\frac{k\pi}{M}\right)\right) - \mathrm{i}\cot\left(\frac{k\pi}{M}\right)\right)e^{\delta_k}, \quad 0 < k < M.$$

Analogously, given a Laplace transform $\widehat{g}$ of a complex-valued function g of two non-negative real variables, the Talbot inversion formula to compute $g(t_1, t_2)$ numerically is given by

$$g_M^b(t_1, t_2) = \frac{2}{25t_1t_2} \sum_{k_1=0}^{M-1} \Re\left\{\gamma_{k_1} \sum_{k_2=0}^{M-1}\left[\gamma_{k_2}\widehat{g}\left(\frac{\delta_{k_1}}{t_1}, \frac{\delta_{k_2}}{t_2}\right) + \bar{\gamma}_{k_2}\widehat{g}\left(\frac{\delta_{k_1}}{t_1}, \frac{\bar{\delta}_{k_2}}{t_2}\right)\right]\right\}.$$

5.1. A special case of numerical inverse Laplace transform. Let us consider a function $f : [0,\infty) \to [0,\infty)$ and its Laplace transform $\widehat{f}(q)$, for $q \in \mathbb{C}^+$. It turns out that the inverse Laplace transform of f can be approximated numerically by using only values of $\widehat{f}$ on the positive real line. One such approximation is the Gaver–Stehfest formula

$$f_n(t) = \frac{n \log 2}{t}\binom{2n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\widehat{f}\left(\frac{(n+k)\log 2}{t}\right). \tag{5.2}$$

For other methods and the comparison of their speeds of convergence we refer to [AW06]. Consecutive application of (5.2) leads to a multivariate Gaver–Stehfest formula.

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