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BORELL AND LANDAU–SHEPP INEQUALITIES FOR CAUCHY-TYPE MEASURES

BY

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Abstract. We investigate various inequalities for the one-dimensional Cauchy measure. We also consider analogous properties for one-dimensional sections of multidimensional isotropic Cauchy measures. The paper is a continuation of our previous investigations [2], where we found, among intervals with fixed measure, the ones with the extremal measure of the boundary. Here for the above mentioned measures we investigate inequalities that are analogous to those proved for Gaussian measures by Borell [1] and by Landau and Shepp [5]. We also consider a 1-symmetrization for Cauchy measures, analogous to the one defined for Gaussian measures by Ehrhard [3], and we prove the concavity of this operation.

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1. INTRODUCTION

Gaussian measures occupy a central place in various areas of mathematics. They satisfy important inequalities due to Prekopa–Leindler [8], Borell [1], Ehrhard [3] and Landau–Shepp [5].

The aim of this paper and the previous paper [2] is to find appropriate analogues of these inequalities for rotationally invariant, standard Cauchy measures. The first step is to examine the one-dimensional case. Even here the situation is different than in the Gaussian case, as half-lines are no longer minimal sets (in the sense of the measure of the boundary). It turns out that there are three types of minimal sets, depending on the measure (see [2]). Further, we consider one-dimensional sections of n-dimensional Cauchy measure (we call them "Cauchy-type measures") and

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carry out a suitable version of the Ehrhard symmetrization procedure (see [3]), which turns out to be a very effective tool for our goals. We prove that the 1-symmetrization is a concave operation on convex subsets of $\mathbb{R}^n$, which is the first step in the direction of n-dimensional setting.

The classical isoperimetric theorem on the plane states that among all Borel sets with fixed Lebesgue measure the disk has the smallest perimeter. The multidimensional version of the theorem states that in every finite dimension there exists a set with the smallest measure of the boundary and this minimum is attained for the ball. Here by "the measure of the boundary" we mean the following: if A is a Borel set and $B_h = \{x \in \mathbb{R}^n : \|x\| < h\}$ we put $A^h = A + B_h = \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) < h\}$. Then the measure of the boundary of A is $\limsup_{h \to 0^+} \frac{|A^h| - |A|}{h}$, where $|A|$ denotes the Lebesgue measure of A. For brevity we call this limit (whenever it exists, finite or not) the *perimeter* of the set A.

Let us start with the definition of the (additive) *perimeter* for probability measures. To avoid problems with the existence, we restrict our discussion to convex Borel sets. Let A be such a set. Put

$$\operatorname{per}(A) = \limsup_{h \to 0^+} \frac{\mu(A^h) - \mu(A)}{h},$$

whenever the limit is finite.

For a different (multiplicative) kind of measuring the boundary, and the corresponding isoperimetric problem, see the papers [4], [6], where the so-called S-hypothesis was solved.

Fifty years ago mathematicians generalized the isoperimetric theorem. Because the Gaussian distribution is one of the most important probability measures, this problem was first investigated for that measure. It turned out (see [9] and [1]) that among all convex Borel sets in $\mathbb{R}^n$ with the same fixed measure, the half-space, i.e. $\{x \in \mathbb{R}^n : x_n > a\}$, has the smallest Gaussian perimeter.

During investigation of these isoperimetric properties of Gaussian measures in $\mathbb{R}^n$ many interesting and useful inequalities were found. For instance C. Borell proved the following [1, Theorem 3.1]:

Let γ be the standard Gaussian measure in $\mathbb{R}^n$, let A be a Borel subset of $\mathbb{R}^n$ and let B be the unit ball. Let $\gamma(A) = \Phi(\alpha)$, where Φ is the distribution function of the standard one-dimensional Gaussian distribution $N(0, 1)$. Then for all $\varepsilon > 0$,

$$\gamma(A + \varepsilon B) \geqslant \Phi(\alpha + \epsilon).$$

Equivalently, using Φ^{-1}, the inverse function of Φ, one can formulate the above inequality as follows:

$$\Phi^{-1}(\gamma(A + \varepsilon B)) - \Phi^{-1}(\gamma(A)) \geqslant \epsilon.$$

H. J. Landau and L. A. Shepp proved the following [5, Theorem 4]:

Let γ be the Gaussian measure in $\mathbb{R}^n$, C a convex set, and s any number such that $\gamma(C) \geqslant \Phi(s)$. If $s > 0$ then for every $a > 1$,

$$\gamma(aC) \geqslant \Phi(as).$$

Equivalently, in terms of Φ^{-1}, the Landau–Shepp inequality can be formulated as follows:

$$\Phi^{-1}(\gamma(aC)) \geqslant a\Phi^{-1}(\gamma(C)).$$

The most complete approach to the Gaussian isoperimetric theory was presented in the paper of A. Ehrhard [3], who constructed a family of so-called k-symmetrizations, $1 \leqslant k \leqslant n$, in $\mathbb{R}^n$ equipped with the standard n-dimensional Gaussian distribution γ_n. Ehrhard established various basic properties of these symmetrizations, among other things, their convexity. He started with the Borell inequality for 1-dimensional Gaussian measure, applied this inequality to 1-symmetrization and then via a certain induction procedure with respect to k, he managed to transfer the inequality to k-symmetrizations for $k = 2, \dots, n$. This finally resulted in the Borell inequality for Gaussian measures in $\mathbb{R}^n$.

Ehrhard's symmetrization preserves the measure γ_n of a set and does not increase its perimeter. Using this symmetrization Ehrhard also proved the following deep result:

Let γ be the standard Gaussian measure in $\mathbb{R}^n$, A and B two Borel convex sets in $\mathbb{R}^n$ and let Φ^{-1} be the inverse of the distribution function of the standard one-dimensional Gaussian measure $N(0,1)$. Then for all $0 \leqslant \lambda \leqslant 1$,

$$\Phi^{-1}(\gamma(\lambda A + (1-\lambda)B)) \geqslant \lambda\Phi^{-1}(\gamma(A)) + (1-\lambda)\Phi^{-1}(\gamma(B)).$$

All the above-mentioned inequalities have interesting and deep consequences for Gaussian processes (see [1], [5], [3]). In this paper we examine analogous inequalities for the one-dimensional Cauchy and "Cauchy-type" measures. The latter arise as one-dimensional sections of the standard rotationally invariant multidimensional Cauchy distributions. Therefore, to generalize the one-dimensional case, we have to deal with such measures.

Our main results are given in Theorems 2.2 and 3.2 (analogues of the Borell inequality) and Theorems 2.3 and 3.3 (analogues of the Landau–Shepp inequality). In Theorems 3.4 and 3.5 we show that if we deal with the standard isotropic (rotation invariant) Cauchy measure μ_n in $\mathbb{R}^n$, then it is still possible to define an analogue of Ehrhard's 1-symmetrization for μ_n and that this symmetrization preserves convexity of sets.

For a different kind of measures and related interesting results, see [7].

2. CAUCHY MEASURES

The standard Cauchy distribution $\mu = \mu_1$ on the real line $\mathbb{R}^1$ has density function

$$f(x) = \frac{1}{\pi(1+x^2)}, \qquad x \in \mathbb{R},$$

and the rotationally invariant Cauchy distribution μ_n in $\mathbb{R}^n$ has the following density:

$$f_n(\mathbf{x}) = \frac{c_n}{(1+|\mathbf{x}|^2)^{(n+1)/2}}, \qquad \mathbf{x} \in \mathbb{R}^n, \qquad c_n = \frac{\pi^{n/2}}{\Gamma(n/2)}.$$

We will use the distribution function of the one-dimensional standard Cauchy measure

$$F(t) = \int_{-\infty}^{t} \frac{1}{\pi(1+x^2)}\,dx = \frac{1}{2} + \frac{1}{\pi}\arctan t \tag{2.1}$$

and its inverse, F^{-1}, given for $0 < t < 1$ by $F^{-1}(t) = -\cot(\pi t)$.

Let μ be the standard one-dimensional Cauchy measure. For $a < b$ we define $g := g(a,b)$ by the equality

$$\mu(-\infty, g) = \mu(a,b), \tag{2.2}$$

and $g^* := g^*(a,b)$ is defined by the identity

$$\mu(-g^*, g^*) = \mu(a,b). \tag{2.3}$$

We obtain

LEMMA 2.1. *We have*

$$g(a,b) = -\frac{1+ab}{b-a},$$

$$(g^*)^2(a,b) = \sqrt{1+g^2(a,b)} + g(a,b) = \frac{\sqrt{1+a^2}\sqrt{1+b^2} - 1 - ab}{b-a}.$$

Proof. By the definition of g we have $\mu(-\infty, g) = \mu(a,b)$, hence formula (2.1) implies $F(g) = F(b) - F(a)$. But $F(g) = \frac{1}{2} + \frac{1}{\pi}\arctan g$ and $F(b) - F(a) = \frac{1}{\pi}(\arctan b - \arctan a)$, hence

$$g = \tan\left(\arctan b - \arctan a - \frac{\pi}{2}\right) = -\cot(\arctan b - \arctan a) = -\frac{1+ab}{b-a}.$$

In order to prove the second formula, we observe that

$$2\arctan g^*(a,b) = \frac{\pi}{2} + \arctan g(a,b), \quad \text{hence} \quad \frac{2\,g^*(a,b)}{1-(g^*(a,b))^2} = -\frac{1}{g(a,b)}.$$

Solving for g^*, we get

$$(g^*)^2(a,b) = \sqrt{1+g^2(a,b)} + g(a,b) = \frac{\sqrt{1+a^2}\sqrt{1+b^2}-1-ab}{b-a}. \qquad \blacksquare$$

REMARK 2.1. Observe that $g(a,b) = F^{-1}(\mu(-\infty, g(a,b))) = F^{-1}(\mu(a,b))$, hence all facts concerning $g(a,b)$ could be formulated in terms of $F^{-1}(\mu(a,b))$.

For the standard Cauchy measure on $\mathbb{R}^1$ the extremality of intervals or half-lines was explained in [2, Theorem 2.1] as follows:

THEOREM 2.1 (Extremal intervals for Cauchy measure). • *If* $\mu(a,b) > 1/2$, *then*

$$\operatorname{per}(-g^*, g^*) < \operatorname{per}(a,b) < \operatorname{per}(-\infty, g).$$

• *If* $\mu(a,b) < 1/2$, *then*

$$\operatorname{per}(-\infty, g) < \operatorname{per}(a,b) < \operatorname{per}(-g^*, g^*).$$

• *If* $\mu(a,b) = 1/2$ *(and then* $-a = 1/b > 0$*), then*

$$\operatorname{per}(-\infty, 0) = \operatorname{per}(-1/b, b) = \operatorname{per}(-1,1) = 1/\pi.$$

2.1. Borell-type inequality. Now we prove an analogue of the Borell inequality for the standard Cauchy distribution in $\mathbb{R}$.

THEOREM 2.2. *For every* $a < b$ *and every* $r > 0$,

$$g(a-r, b+r) - g(a,b) \geqslant r/2. \tag{2.4}$$

If $\mu(a,b) < 1/2$, *then*

$$g(a-r, b+r) - g(a,b) \geqslant r \tag{2.5}$$

for all $r > 0$ *small enough. In fact, for* $r \leqslant 2/\sqrt{3}$, *the last inequality holds whenever* $\mu(a,b) < 1/3$.

An equivalent formulation of (2.4) *in terms of* F^{-1} *is*

$$F^{-1}(\mu(a-r, b+r)) - F^{-1}(\mu(a,b)) \geqslant r/2. \tag{2.6}$$

Proof. Taking into account the formula (2.2) we obtain

$$\begin{aligned} g(a-r, b+r) - g(a,b) &= -\frac{1+(a-r)(b+r)}{(b+r)-(a-r)} + \frac{1+ab}{b-a} \\ &= r\frac{(b+r)b + (a-r)a + 2}{((b+r)-(a-r))(b-a)}, \end{aligned}$$

hence it is enough to prove that

$$\frac{(b+r)b+(a-r)a+2}{((b+r)-(a-r))(b-a)} \geqslant \frac{(b+r)b+(a-r)a}{((b+r)-(a-r))(b-a)} \geqslant \frac{1}{2}.$$

The last inequality is true because

$$\begin{aligned} &2(b+r)b+2(a-r)a-((b+r)-(a-r))(b-a) \\ &\qquad = 2(b+r)b-(b+r)(b-a)+2(a-r)a+(a-r)(b-a) \\ &\qquad = (b+r)[2b-b+a]+(a-r)[2a+b-a] = (b+a)^2 \geqslant 0. \end{aligned}$$

To justify (2.5) we have to solve the inequality

$$\frac{(b+r)b+(a-r)a+2}{((b+r)-(a-r))(b-a)} \geqslant 1,$$

equivalent to

$$2\,\frac{1+ab}{b-a} \geqslant r.$$

This, by Lemma 2.1, is equivalent to

$$g(a,b) \leqslant -r/2,$$

which justifies (2.5), because $g(a,b) < 0$ for $\mu(a,b) < 1/2$.

Further if $r \leqslant 2/\sqrt{3}$, then $g(a,b) \leqslant -1/\sqrt{3}$ implies $g(a,b) \leqslant -r/2$, which yields (2.5). The inequality $g(a,b) \leqslant -1/\sqrt{3}$ is in turn equivalent to

$$\mu(a,b) \leqslant \int_{-\infty}^{-1/\sqrt{3}} \frac{dt}{\pi(1+t^2)} = \frac{1}{2} + \frac{1}{\pi}\left(\arctan\frac{-1}{\sqrt{3}}\right) = \frac{1}{3}. \qquad \blacksquare$$

Comparison between Gaussian and Cauchy cases. By the Borell inequality, for the standard n-dimensional Gaussian measure γ, every Borel set A and every $r \geqslant 0$ we have

$$\Phi^{-1}(\gamma(A+r)) - \Phi^{-1}(\gamma(A)) \geqslant r.$$

By contrast, for the one-dimensional Cauchy measure μ, every interval A and every $r \geqslant 0$,

$$F^{-1}(\mu(A+r)) - F^{-1}(\mu(A)) \geqslant r/2.$$

Comment. For Gaussian measures it is crucial that we have the same value r on both sides of the inequality for all Borel sets. Ehrhard [3] makes this property the cornerstone of his version of "Gaussian symmetrization". For the one-dimensional Cauchy measure and all intervals we only have "$r/2$" on the right.

2.2. Landau–Shepp-type inequality

THEOREM 2.3. *For every $a < b$ and every $r > 0$ the following holds:*

$$g(ra, rb) \geqslant rg(a, b) \quad \textit{if and only if} \quad r \geqslant 1.$$

Equivalently, for $r \geqslant 1$,

$$F^{-1}(\mu(ra, rb)) \geqslant rF^{-1}(\mu(a, b)).$$

Proof. By straightforward computation

$$g(ra, rb) - rg(a, b) = -\frac{1 + r^2 ab}{r(b - a)} + r\frac{1 + ab}{b - a} = \frac{r^2 - 1}{r(b - a)}. \qquad \blacksquare$$

2.3. Concavity of $g(a, b)$

THEOREM 2.4. *The function $g(a, b) = F^{-1}(\mu(a, b))$ is a concave function of (a, b) for $a < b$.*

Proof. Let us start with the explicit formulas for the second derivatives:

$$\frac{\partial^2 g}{\partial a^2} = -2\,\frac{1 + b^2}{(b - a)^3}, \quad \frac{\partial^2 g}{\partial b^2} = -2\,\frac{1 + a^2}{(b - a)^3}, \quad \frac{\partial^2 g}{\partial a \partial b} = 2\,\frac{1 + ab}{(b - a)^3}.$$

Computing the determinant of the Hessian matrix of $g(a, b)$, we obtain

$$\det \mathrm{Hess}(g)(a, b) = (b - a)^{-6}\,[(1 + a^2)(1 + b^2) - (1 + ab)^2] = (b - a)^{-4} \geqslant 0,$$

which, together with $\frac{\partial^2 g}{\partial a^2} < 0$, shows that the Hessian matrix is negative definite. $\blacksquare$

3. ONE-DIMENSIONAL SECTIONS OF MULTIDIMENSIONAL CAUCHY MEASURES

3.1. Concavity of $g(a, b)$. For a more general probability density function f we again use the earlier introduced (in a less general context) notation $g(a, b)$ for a function of intervals (a, b), $-\infty \leqslant a < b < \infty$, by

$$\int_a^b f(t)\,dt = \int_{-\infty}^{g(a,b)} f(t)\,dt. \tag{3.1}$$

Assume that f is differentiable and denote for simplicity

$$\chi(x) = (1/f(x))'. \tag{3.2}$$

In the lemma below we formulate a basic condition on the probability densities we investigate, which is equivalent to concavity of our function $g(a, b)$ with respect to variables (a, b).

LEMMA 3.1. *Assume that the probability density f is differentiable, strictly decreasing on $(0,\infty)$ and $f(-x)=f(x)$. Also assume that $1/f$ is strictly convex and denote by $\chi(x)=(1/f(x))'$ its derivative. Then the function $g(a,b)$ is strictly concave (as a function of a,b, for $a<b$) if and only if*

$$\frac{\chi(a)\,\chi(b)}{\chi(a)-\chi(b)} \geqslant \chi(g(a,b)). \tag{3.3}$$

Proof. Differentiating the defining equality (3.1), we obtain

$$\begin{aligned}
&\frac{\partial g}{\partial a}=\frac{-f(a)}{f(g(a,b))},\quad \frac{\partial g}{\partial b}=\frac{f(b)}{f(g(a,b))},\\
&f(g(a,b))\frac{\partial^2 g}{\partial a^2}=-f'(a)-f'(g(a,b))\,\frac{f^2(a)}{f^2(g(a,b))},\\
&f(g(a,b))\frac{\partial^2 g}{\partial b^2}=f'(b)-f'(g(a,b))\,\frac{f^2(b)}{f^2(g(a,b))},\\
&f(g(a,b))\frac{\partial^2 g}{\partial a\partial b}=f'(g(a,b))\,\frac{f(a)\,f(b)}{f^2(g(a,b))}.
\end{aligned}$$

We check that the Hessian matrix of the function g is negative definite.

For all $a<b$, by (3.1), we obtain $g(a,b)<b$. The strict convexity of $1/f$ implies that $-f'(x)/f^2(x)$ is strictly increasing, so that

$$-\frac{f'(b)}{f^2(b)}>-\frac{f'(g(a,b))}{f^2(g(a,b))},\quad \text{hence}\quad \frac{\partial^2 g}{\partial b^2}<0.$$

Moreover,

$$\begin{aligned}
&f^2(g(a,b))\det\mathrm{Hess}(g)(a,b)\\
&\qquad\qquad=\frac{f'(g(a,b))}{f^2(g(a,b))}[f'(a)\,f^2(b)-f'(b)\,f^2(a)]-f'(a)\,f'(b),
\end{aligned}$$

and the non-negativity of the above expression is equivalent to

$$\frac{f'(g(a,b))}{f^2(g(a,b))}\left[\frac{f'(a)}{f^2(a)}-\frac{f'(b)}{f^2(b)}\right]\geqslant \frac{f'(a)}{f^2(a)}\,\frac{f'(b)}{f^2(b)}.$$

Taking into account the definition of χ, we rewrite the above inequality as follows:

$$\chi(g(a,b))(\chi(a)-\chi(b))\geqslant\chi(a)\chi(b).$$

By the requirement that $1/f(x)$ is strictly convex, we see that $\chi(x)=-f'(x)/f^2(x)$ is strictly increasing, so the expression in brackets on the left-hand side above is negative. Dividing by this expression, we obtain (3.3), which concludes the proof. ∎

REMARK 3.1. Observe that if $a < 0 < b$ and $g(a,b) < 0$, then the left-hand side of (3.3) is positive, while the right-hand side is negative and the inequality holds automatically.

In all the remaining cases we have $\chi(g(a,b))\chi(a)\chi(b) \leqslant 0$ and

$$\frac{\chi(a) - \chi(b)}{\chi(g(a,b))\chi(a)\chi(b)} \geqslant 0.$$

Multiplying both sides of (3.3) by this expression, we obtain

$$\frac{1}{\chi(b)} - \frac{1}{\chi(a)} \leqslant \frac{1}{\chi(g(a,b))},$$

except when $a < 0 < b$ and simultaneously $g(a,b) < 0$.

The next theorem is crucial for all the presentation that follows. It consists of an indirect verification that the condition (3.3) holds for the one-dimensional sections of multidimensional isotropic Cauchy measures. The main difficulty to overcome is the lack of explicit formulas, which are available in the case of one-dimensional Cauchy measure.

We proceed in two steps. First, we reduce the problem, via an application of the Lagrange metod, to the case of symmetric intervals $(-p,p)$, $p \geqslant 0$. Next, we compare $g(-p,p)$ with an auxiliary function $x(p)$, introduced by the condition $\chi(x(p)) = \chi(p)/2$. Again, a direct comparison seems to be out of reach, and we apply compositions with a distribution function H.

THEOREM 3.1. *Suppose that $\nu_{\alpha,n}$, $\alpha \geqslant 0$, $n = 2, 3, \ldots$, is the probability measure with density*

$$\begin{aligned} f_{\alpha,n}(x) &= \frac{c_{\alpha,n}}{(1+\alpha^2+x^2)^{(n+1)/2}}, \\ c_{\alpha,n} &= (1+\alpha^2)^{n/2}\Gamma\left(\frac{n+1}{2}\right)\Big/\left(\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)\right). \end{aligned} \tag{3.4}$$

Then the function $g(a,b) := g_\alpha(a,b)$, defined by (3.1)*, is a concave function of two variables a, b for $a < b$.*

Proof. Denote $\chi(x) = \frac{d}{dx}\frac{1}{f_{\alpha,n}(x)}$. We will check that (3.3) holds. We rewrite it in the equivalent form

$$\left(\frac{1}{\chi(b)} - \frac{1}{\chi(a)}\right)^{-1} \geqslant \chi(g(a,b)). \tag{3.5}$$

The assumptions of the previous lemma are satisfied and

$$\chi(x) = \frac{1+n}{c_{\alpha,n}} x(1+\alpha^2+x^2)^{(n-1)/2}.$$

Observe that if we multiply both sides of (3.1) by a positive constant, then $g(a,b)$ remains unchanged. Moreover, the inequality (3.5) is homogeneous, that is, invariant under multiplication of χ by a positive constant. Therefore, we put $c_{\alpha,n} = 1+n$ in the remaining part of the proof. We note that $\lim_{x\to\infty} \chi(x) = \infty$.

First, let us observe that $\lim_{a\to-\infty} g(a,b) = g(-\infty,b) = b$ and $\lim_{a\to-\infty} \chi(a) = -\infty$, so we obtain equality in (3.5) for $a = -\infty$. Analogously, $\lim_{b\to\infty} g(a,b) = g(a,\infty) = -a$. Since $\chi(-a) = -\chi(a)$, we also get equality for $b = \infty$. For $a = b$ we have $g(a,a) = -\infty$, hence (3.5) obviously holds.

To prove (3.5) in full generality we use the Lagrange method to find extremal values of the function $F(a,b) = \frac{1}{\chi(b)} - \frac{1}{\chi(a)}$ under the condition $g(a,b) = t$, with $t \in (-\infty,\infty)$ fixed:

$$F(a,b) + \lambda g(a,b) = \frac{1}{\chi(b)} - \frac{1}{\chi(a)} + \lambda g(a,b).$$

We obtain

$$\frac{\partial F(a,b)}{\partial a} + \lambda \frac{\partial g(a,b)}{\partial a} = 0, \qquad \frac{\partial F(a,b)}{\partial b} + \lambda \frac{\partial g(a,b)}{\partial b} = 0.$$

By (3.1) we infer that

$$\frac{\partial g(a,b)}{\partial a} = -\frac{f_{\alpha,n}(a)}{f_{\alpha,n}(g(a,b))} \quad \text{and} \quad \frac{\partial g(a,b)}{\partial b} = \frac{f_{\alpha,n}(b)}{f_{\alpha,n}(g(a,b))},$$

hence

$$\frac{\left(\frac{1}{\chi(a)}\right)'}{f_{\alpha,n}(a)} = \frac{-\lambda}{f_{\alpha,n}(g(a,b))} \quad \text{and} \quad \frac{\left(\frac{1}{\chi(b)}\right)'}{f_{\alpha,n}(b)} = \frac{-\lambda}{f_{\alpha,n}(g(a,b))},$$

which implies

$$\frac{(1/\chi(a))'}{f_{\alpha,n}(a)} = \frac{(1/\chi(b))'}{f_{\alpha,n}(b)} = -\frac{\lambda}{f_{\alpha,n}(g(a,b))}.$$

Now we compute the explicit form of the function $\frac{(1/\chi(x))'}{f_{\alpha,n}(x)}$:

$$\begin{aligned}\frac{(1/\chi(x))'}{f_{\alpha,n}(x)} &= \left(\frac{1}{x(1+\alpha^2+x^2)^{(n-1)/2}}\right)' \cdot \frac{(1+\alpha^2+x^2)^{(n+1)/2}}{n+1} \\ &= -\frac{1+\alpha^2+nx^2}{(n+1)x^2}.\end{aligned}$$

Thus, this function is equal to $-n/(n+1) - (1+\alpha^2)/(x^2(n+1))$, and is obviously even and injective on $(0,\infty)$. Hence, the extremal values of F can only be attained at $a = \pm b$. Thus, it is sufficient to check the inequality for $a = -b$. By Remark 3.1 we may and will assume that $g(a,b) > 0$.

Denote $b = -a = p$ and $h(p) = g(-p, p)$. Since $\chi(-p) = -\chi(p)$, in order to prove (3.5) it is enough to show that for $p > 0$,

$$2\chi(h(p)) \leqslant \chi(p). \tag{3.6}$$

Observe that if $h(p) < 0$, then $\chi(h(p)) < 0$, while $\chi(p) \geqslant 0$, so that (3.6) is satisfied, hence we will assume further that $h(p) \geqslant 0$. We also note that for $n > 2$ no explicit formula for g (or h) is available and no direct verification of (3.6) seems to be possible. To overcome this difficulty, we introduce an auxiliary function $x(p)$ by the formula

$$\chi(x(p)) = \tfrac{1}{2}\chi(p). \tag{3.7}$$

Since χ is increasing on $(0, \infty)$ and $\chi(x) \geqslant 0$ for $x \geqslant 0$, we obtain

$$0 \leqslant x(p) \leqslant p. \tag{3.8}$$

Set $H(z) = \int_{-\infty}^{z} f_{\alpha,n}(t)\,dt$. By the definition of $h(p)$ we obtain

$$H(h(p)) = \int\limits_{-\infty}^{h(p)} f_{\alpha,n}(t)\,dt = \int\limits_{-p}^{p} f_{\alpha,n}(t)\,dt = 2\int\limits_{0}^{p} f_{\alpha,n}(t)\,dt. \tag{3.9}$$

In order to prove that $h(p) \leqslant x(p)$, it is enough to show that

$$H(h(p)) - H(x(p)) = \int\limits_{-\infty}^{h(p)} f_{\alpha,n}(t)\,dt - \int\limits_{-\infty}^{x(p)} f_{\alpha,n}(t)\,dt \leqslant 0.$$

If we prove that the derivative of the left-hand side is non-negative, then this will justify the above statement. Indeed, the value of the above difference at $p = 0$ is $(-1/2)$; at ∞ the value is 0, as both $h(p)$ and $x(p)$ tend to infinity $p \to \infty$.

From the definition of $\chi(p)$ and (3.9) we obtain

$$\frac{x(p)}{p} = \frac{1}{2}\left(\frac{1 + \alpha^2 + p^2}{1 + \alpha^2 + x^2(p)}\right)^{(n-1)/2}.$$

Let us rewrite this in the form

$$x(p)(1 + a^2 + x^2(p))^{(n-1)/2} = \tfrac{1}{2}(1 + a^2 + p^2))^{(n-1)/2}.$$

Differentiating the left-hand side, we obtain

$$\begin{aligned} x'(p)(1+a^2+x^2(p))^{(n-1)/2} + (n-1)x^2(p)x'(p)(1+a^2+x^2(p))^{(n-3)/2} \\ = x'(p)(1+a^2+x^2(p))^{(n-3)/2}(1+a^2+nx^2(p)), \end{aligned} \tag{3.10}$$

while the derivative of the right-hand side is

$$\tag{3.11} \begin{aligned} &\tfrac{1}{2}(1+a^2+x^2(p))^{(n-1)/2}+\tfrac{1}{2}(n-1)p^2(1+a^2+x^2(p))^{(n-3)/2}\\ &\qquad=\tfrac{1}{2}(1+a^2+p^2))^{(n-3)/2}(1+a^2+np^2). \end{aligned}$$

Comparing (3.10) and (3.11), we get

$$x'(p)=\frac{1}{2}\left(\frac{1+\alpha^2+p^2}{1+\alpha^2+x^2(p)}\right)^{(n-3)/2}\frac{1+\alpha^2+np^2}{1+\alpha^2+nx^2(p)}.$$

By the definition of $h(p)$,

$$\frac{d}{dp}H(h(p))=\frac{2}{(1+\alpha^2+p^2)^{(n+1)/2}}.$$

Taking into account the formulas for $x'(p)$ and for $x(p)$, we obtain

$$\begin{aligned} \frac{d}{dp}H(x(p))&=\frac{x'(p)}{(1+\alpha^2+x^2(p))^{(n+1)/2}}\\ &=\frac{1}{2}\,\frac{(1+\alpha^2+p^2)^{(n-3)/2}}{(1+\alpha^2+x^2(p))^{n-1}}\,\frac{1+\alpha^2+np^2}{1+\alpha^2+nx^2(p)}\\ &=2\frac{(1+\alpha^2+p^2)^{(n-3)/2}}{(1+\alpha^2+p^2)^{n-1}}\,\frac{x^2(p)}{p^2}\,\frac{1+\alpha^2+np^2}{1+\alpha^2+nx^2(p)}\\ &=2\frac{x^2(p)}{p^2(1+\alpha^2+p^2)^{(n+1)/2}}\,\frac{1+\alpha^2+np^2}{1+\alpha^2+nx^2(p)}. \end{aligned}$$

We thus obtain

$$\begin{aligned} \frac{d}{dp}[H(h(p))-H(x(p))]&=2\frac{p^2(1+\alpha^2+nx^2(p))-x^2(p)(1+\alpha^2+nx^2(p))}{p^2(1+\alpha^2+p^2)^{(n+1)/2}(1+\alpha^2+np^2)}\\ &=\frac{(1+\alpha^2)(p^2-x^2(p))}{p^2(1+\alpha^2+p^2)^{(n+1)/2}(1+\alpha^2+np^2)}\geqslant 0. \end{aligned}$$

In view of (3.8), that means $0\leqslant h(p)\leqslant x(p)\leqslant p$. The proof is now complete. ■

3.2. Borell-type inequality for measures with densities $f_{\alpha,n}$. In this section we prove the main result of the paper. It is the inequality (3.14) and the authors believe that it is the appropriate analogue of the Borell inequality for the measures $\nu_{\alpha,n}$. Observe that in the Gaussian case, on the right-hand side of the analogous inequality we have the value r, while in our case we only have $r/2^{1/n}$ and the inequality is strict.

THEOREM 3.2. *For $a < b$ and every $r > 0$,*

$$g_\alpha(a-r, b+r) - g_\alpha(a,b) \geqslant \frac{r}{2^{1/n}}, \tag{3.12}$$

where $g_\alpha := g_{\alpha,n}$ is defined by the density $f_{\alpha,n}$. If $\nu_\alpha(a,b) < 1/2$, then

$$g_\alpha(a-r, b+r) - g_\alpha(a,b) \geqslant r \tag{3.13}$$

for all $r > 0$ small enough.

In terms of F^{-1} we can write (3.12) as

$$F^{-1}(\nu_{\alpha,n}(a-r, b+r)) - F^{-1}(\nu_{\alpha,n}(a,b)) \geqslant \frac{r}{2^{1/n}}. \tag{3.14}$$

The proof of the theorem essentially goes along the lines indicated in the proof of Theorem 3.1. We first use Lagrange's method to reduce the problem to finding extremal values of the appropriate function. Next, we consider several cases; here again, we rely on the indirect method, introducing various auxiliary functions.

Proof of Theorem 3.2. We first prove the differential form of the inequalities:

$$-\frac{\partial g_\alpha}{\partial a} + \frac{\partial g_\alpha}{\partial b} \geqslant \frac{1}{2^{1/n}} \quad \text{or, if } \nu_\alpha(a,b) < 1/2, \text{ then} \quad -\frac{\partial g_\alpha}{\partial a} + \frac{\partial g_\alpha}{\partial b} \geqslant 1.$$

By the formulas for the partial derivatives of g_α we obtain the following form of these inequalities:

$$\begin{aligned} &f_{\alpha,n}(a) + f_{\alpha,n}(b) \geqslant \frac{1}{2^{1/n}} f_{\alpha,n}(g_\alpha(a,b)) \quad &&\text{for all } a < b, \\ &f_{\alpha,n}(a) + f_{\alpha,n}(b) \geqslant f_{\alpha,n}(g_\alpha(a,b)) &&\text{if } \nu_\alpha(a,b) < 1/2. \end{aligned} \tag{3.15}$$

Let $G(a,b) = f_{\alpha,n}(a) + f_{\alpha,n}(b)$. We seek extrema of G under the condition $g_\alpha(a,b) = t$, with $t \in (-\infty, \infty)$ fixed; in the second inequality we assume that $g_\alpha(a,b) = t < 0$. Using the Lagrange method, we obtain

$$\begin{gathered} \frac{\partial G}{\partial a} + \lambda \frac{\partial g_\alpha}{\partial a} = 0, \quad \frac{\partial G}{\partial b} + \lambda \frac{\partial g_\alpha}{\partial b} = 0 \quad \text{or, equivalently,} \\ f'(a) - \lambda \frac{f(a)}{f(g_\alpha(a,b))} = 0, \quad f'(b) + \lambda \frac{f(b)}{f(g_\alpha(a,b))} = 0. \end{gathered}$$

We thus obtain

$$\frac{f'(a)}{f(a)} = \frac{\lambda}{f(g_\alpha(a,b))} = -\frac{f'(b)}{f(b)}.$$

Since $\frac{f'(x)}{f(x)} = \frac{-(n+1)x}{1+\alpha^2+x^2}$, we get

$$\frac{a}{1+\alpha^2+a^2} = \frac{-b}{1+\alpha^2+b^2}, \quad \text{equivalently,} \quad (a+b)(1+\alpha^2+ab) = 0.$$

Now, we prove the first part of the theorem.

CASE 1: $-a = b = p > 0$, $h(p) = g_\alpha(-p, p)$. Our first inequality reduces to

$$h'(p) = 2\left(\frac{1+\alpha^2+h^2(p)}{1+\alpha^2+p^2}\right)^{(n+1)/2} \geqslant \frac{1}{2^{1/n}}, \tag{3.16}$$

or equivalently

$$\frac{1+\alpha^2+h^2(p)}{1+\alpha^2+p^2} \geqslant \frac{1}{2^{2/n}}.$$

Define $p_1 := p_1(\alpha)$ by

$$\frac{1+\alpha^2}{1+\alpha^2+p_1^2} = \frac{1}{2^{2/n}},$$

or more explicitly $p_1^2 = (1+\alpha^2)(2^{2/n}-1)$. Note that for $0 < p < p_1$ we obtain

$$2\left(\frac{1+\alpha^2+h^2(p)}{1+\alpha^2+p^2}\right)^{(n+1)/2} > 2\left(\frac{1+\alpha^2}{1+\alpha^2+p_1^2}\right)^{(n+1)/2} = \frac{1}{2^{1/n}}.$$

We thus assume that $p \geqslant p_1$. Define an auxiliary function $z := z(p) \geqslant 0$ such that

$$2\left(\frac{1+\alpha^2+z^2(p)}{1+\alpha^2+p^2}\right)^{(n+1)/2} = \frac{1}{2^{1/n}}.$$

It is enough to show that $z(p) \leqslant h(p)$. We obtain

$$h'(p) = 2\left(\frac{1+\alpha^2+h^2(p)}{1+\alpha^2+p^2}\right)^{(n+1)/2},$$

hence

$$\frac{h'(p)}{(1+\alpha^2+h^2(p))^{(n+1)/2}} = \frac{2}{(1+\alpha^2+p^2)^{(n+1)/2}}.$$

On the other hand, $z'(p) = p/(2^{2/n}z(p))$ and, by the definition of $z(p)$, we obtain

$$\frac{z'(p)}{(1+\alpha^2+z^2(p))^{(n+1)/2}} = \frac{p}{z(p)}\,\frac{2^{1-1/n}}{(1+\alpha^2+p^2)^{(n+1)/2}}.$$

Therefore,

$$\frac{d}{dp}\left[\int_{-\infty}^{h(p)} \frac{dt}{(1+\alpha^2+t^2)^{(n+1)/2}} - \int_{-\infty}^{z(p)} \frac{dt}{(1+\alpha^2+t^2)^{(n+1)/2}}\right]$$
$$= \frac{h'(p)}{(1+\alpha^2+h^2(p))^{(n+1)/2}} - \frac{z'(p)}{(1+\alpha^2+z^2(p))^{(n+1)/2}}$$
$$= \frac{2}{(1+\alpha^2+p^2)^{(n+1)/2}} - \frac{p}{z(p)}\,\frac{2^{1-1/n}}{(1+\alpha^2+p^2)^{(n+1)/2}}$$
$$= \frac{2}{(1+\alpha^2+p^2)^{(n+1)/2}}\left(1-\frac{1}{2^{1/n}}\,\frac{p}{z(p)}\right)$$
$$= \frac{2}{(1+\alpha^2+p^2)^{(n+1)/2}}\,\frac{2^{2/n}z^2(p)-p^2}{2^{1/n}z(p)(2^{1/n}z(p)+p)} < 0,$$

since $2^{2/n}z^2(p) - p^2 = -(1+\alpha^2)(2^{2/n}-1) < 0$. Taking into account that the value at ∞ of the function being differentiated is 0, that is,

$$\int_{-\infty}^{\infty} \frac{dt}{(1+\alpha^2+t^2)^{(n+1)/2}} - \int_{-\infty}^{\infty} \frac{dt}{(1+\alpha^2+t^2)^{(n+1)/2}} = 0,$$

we obtain $h(p) \geqslant z(p) \geqslant 0$ for $p \geqslant p_1$. This ends the proof of Case 1 and shows that

$$h'(p) \geqslant \frac{1}{2^{1/n}}. \tag{3.17}$$

We note that the above observation also yields

$$h(p_1) \geqslant 0, \quad \text{hence} \quad \nu_\alpha(-p_1, p_1) \geqslant 1/2.$$

CASE 2: $ab = -(1+\alpha^2)$. Put $a = -(1+\alpha^2)/b,\ b > 0$. Then the left-hand side of (3.15) takes the form

$$f_{\alpha,n}(-(1+\alpha^2)/b) + f_{\alpha,n}(b),$$

while the right-hand side is $f(g_{\alpha,n}((1+\alpha^2)/b, b))$. We multiply both sides of (3.15) by the constant $(1+\alpha^2)^{(n+1)/2}$ and put $p = b/\sqrt{1+\alpha^2}$, $h(p) = g_{\alpha,n}(-1/p, p)$. Taking into account the scaling property of G, we obtain the following form of the inequality:

$$(1+p^{n+1})\left(\frac{1+h^2(p)}{1+p^2}\right)^{(n+1)/2} \geqslant \frac{1}{2^{1/n}}. \tag{3.18}$$

Define

$$\phi(p) = \frac{1+p^{n+1}}{(1+p^2)^{(n+1)/2}}.$$

Then

$$\phi'(p) = \frac{p(n+1)}{(1+p^2)^{(n+3)/2}}(p^{n-1}-1).$$

Therefore, ϕ is decreasing on $(0,1)$, increasing on $(1,\infty)$, attains its minimum at $p=1$ and $\phi(1) = 2/2^{(n+1)/2} \leqslant 1/2^{1/n}$ for $n = 2,3,\ldots$. Observe that the left-hand side of (3.18) is invariant with respect to the mapping $p \mapsto 1/p$. Therefore, we consider only $p \geqslant 1$. For such p we define yet another auxiliary function $y(p) \geqslant 0$ by the identity

$$(1+p^{n+1})\left(\frac{1+y^2(p)}{1+p^2}\right)^{(n+1)/2} = \frac{1}{2^{1/n}}.$$

Differentiating, we obtain

$$y'(p) = \frac{p}{y(p)}\,\frac{1+y^2(p)}{1+p^2}(1-p^{n-1}),$$

hence $y(p)$ is decreasing on $(1,\infty)$, while $h(p)$ is increasing, since

$$\frac{h'(p)}{(1+h^2(p))^{n+1}2} = \frac{1+p^{n+1}}{(1+p^2)^{(n+1)/2}}.$$

Moreover, for $p=1$ we deduce from Case 1 that $y(1) = z(1)$, so from the monotonicity of $y(p)$ and $h(p)$ we obtain

$$y(p) \leqslant y(1) = z(1) \leqslant h(1) \leqslant h(p),$$

which implies that

$$(1+p^{n+1})\left(\frac{1+h^2(p)}{1+p^2}\right)^{(n+1)/2} \geqslant (1+p^{n+1})\left(\frac{1+y^2(p)}{1+p^2}\right)^{(n+1)/2} = \frac{1}{2^{1/n}}.$$

This ends the proof of Case 2 for the differential form of inequality (3.12).

In order to prove (3.12) we use the concavity of the function g. Denote

$$\psi_{a,b}(r) = g(a-r, b+r).$$

By Theorem 2.4, the function $\psi_{a,b}(r)$ is concave for $r>0$. The concavity implies

$$\frac{\psi_{a,b}(r)-\psi_{a,b}(0)}{r} \geqslant \psi'_{a,b}(r) = \psi'_{a-r,b+r}(0).$$

However, by the expressions for the derivatives of g and (3.15), we obtain

$$\psi'_{a-r,b+r}(0) = \frac{f_{\alpha,n}(a-r)}{g(a-r,b+r)} + \frac{f_{\alpha,n}(b+r)}{g(a-r,b+r)} \geqslant \frac{1}{2^{1/n}},$$

which finally gives (3.12) and ends the proof of the first part of the theorem.

In order to prove (3.13), observe that from [2, Lemma 5.1] we know that if $ab = -(1+\alpha^2)$, then $\nu_\alpha(a,b) > 1/2$, so $g_\alpha(a,b) = t > 0$ for such pairs (a,b), thus we exclude that case from our further considerations. What remains is the case $-a = b = p > 0$ and, as in Case 1, we put $h(p) = g_\alpha(-p,p)$. We note that our inequality reduces to

$$h'(p) = 2\left(\frac{1+\alpha^2+h^2(p)}{1+\alpha^2+p^2}\right)^{(n+1)/2} \geqslant 1,$$

or equivalently

$$\frac{1}{(1+\alpha^2+h(p)^2)^{(n+1)/2}} \leqslant \frac{2}{(1+\alpha^2+p^2)^{(n+1)/2}}.$$

However, this means that

$$\operatorname{per}(-p,p) \geqslant \operatorname{per}(-\infty, g_\alpha(-p,p))$$

and the fundamental Lemma 5.2 in [2] proves that the above inequality holds whenever $\nu_\alpha(-p,p) < 1/2$, ending the proof of the second part of the theorem in the differential form. The general version can again be obtained from the concavity of g. ■

3.3. Landau–Shepp-type inequality for measures with densities $f_{\alpha,n}$

THEOREM 3.3. *For every $a < b$ and every $\alpha \geqslant 0$,*

$$g_\alpha(ra,rb) \geqslant r g_\alpha(a,b) \quad \textit{if and only if} \quad r \geqslant 1. \tag{3.19}$$

Proof. We write the differential form of the above inequality. To do this, we rewrite (3.19) in the form

$$\frac{g_\alpha(ra,rb) - g_\alpha(a,b)}{r-1} \geqslant g_\alpha(a,b)$$

and, when $r \to 1$, we obtain

$$\left.\frac{dg_\alpha(ra,rb)}{dr}\right|_{r=1} \geqslant g_\alpha(a,b),$$

or equivalently

$$\frac{\partial g_\alpha(a,b)}{\partial a} a + \frac{\partial g_\alpha(a,b)}{\partial b} b \geqslant g_\alpha(a,b). \tag{3.20}$$

Taking into account the formulas for partial derivatives of g_α, we obtain

$$\frac{-f_\alpha(a)}{f_\alpha(g(a,b))} a + \frac{f_\alpha(b)}{f_\alpha(g(a,b))} b \geqslant g_\alpha(a,b),$$

or equivalently

$$-f_\alpha(a)a + f_\alpha(b)b \geqslant g_\alpha(a,b) f_\alpha(g(a,b)). \tag{3.21}$$

We will show that (3.21) holds, again using the Lagrange method. We put

$$F(a,b) = -f_\alpha(a)a + f_\alpha(b)b + \lambda g_\alpha(a,b)$$

and obtain

$$\frac{\partial F(a,b)}{\partial a} = -f'_\alpha(a)a - f_\alpha(a) + \lambda \frac{\partial g_\alpha(a,b)}{\partial a} = 0,$$
$$\frac{\partial F(a,b)}{\partial b} = f'_\alpha(b)b + f_\alpha(b) + \lambda \frac{\partial g_\alpha(a,b)}{\partial b} = 0.$$

Taking again into account the formulas for partial derivatives of g_α, we obtain

$$-f'_\alpha(a)a - f_\alpha(a) - \lambda \frac{f(a)}{f(g_\alpha(a,b))} = 0,$$
$$f'_\alpha(b)b + f_\alpha(b) + \lambda \frac{f(b)}{f(g_\alpha(a,b))} = 0,$$

which implies

$$\frac{f'_\alpha(a)}{f(a)} a = \frac{f'_\alpha(b)}{f(b)} b.$$

Thus, if

$$\frac{-(n+1)a^2}{1+\alpha^2+a^2} = \frac{-(n+1)b^2}{1+\alpha^2+b^2},$$

then $a = \pm b$. We now put $p = -a = b > 0$ and $h(p) = g_\alpha(-p,p)$, and consider (3.21) for these values of a and b:

$$2f_\alpha(p)p \geqslant f_\alpha(h(p))h(p).$$

Taking into account the formula (3.16) for $h'(p)$, we obtain an equivalent form of the desired inequality:

$$h'(p) \geqslant \frac{h(p)}{p}.$$

We will show that

$$\frac{h(p)}{p} \leqslant \frac{1}{2^{1/n}}. \tag{3.22}$$

In view of (3.17), this will end the proof of the theorem.

The scaling property implies that it is enough to prove (3.22) only for $f_{0,n}$. For this purpose, define

$$\Lambda(p) = 2\int_0^p \frac{dt}{(1+t^2)^{(n+1)/2}} - \int_{-\infty}^{p/2^{1/n}} \frac{dt}{(1+t^2)^{(n+1)/2}}.$$

We obtain

$$\Lambda(0) < 0, \quad \Lambda(\infty) = 2\int_0^\infty \frac{dt}{(1+t^2)^{(n+1)/2}} - \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{(n+1)/2}} = 0.$$

Moreover,

$$\begin{aligned}\Lambda'(p) &= \frac{2}{(1+p^2)^{(n+1)/2}} - \frac{1}{2^{1/n}}\,\frac{1}{(1+p^2/2^{2/n})^{(n+1)/2}} \\ &= \frac{2}{(1+p^2)^{(n+1)/2}} - \frac{2}{(2^{2/n}+p^2)^{(n+1)/2}} \geqslant 0,\end{aligned}$$

hence $\Lambda(p) \leqslant 0$, which means that

$$\int_{-\infty}^{h(p)} \frac{dt}{(1+t^2)^{(n+1)/2}} = 2\int_0^p \frac{dt}{(1+t^2)^{(n+1)/2}} \leqslant \int_{-\infty}^{p/2^{1/n}} \frac{dt}{(1+t^2)^{(n+1)/2}},$$

and this proves (3.20).

To finish the proof, observe that (3.20) holds for all a, b with $a < b$. We rewrite this by putting ra in place of a and rb in place of b to obtain

$$\frac{\partial g_\alpha(ra, rb)}{\partial (ra)}(ra) + \frac{\partial g_\alpha(ra, rb)}{\partial (rb)}(rb) \geqslant g_\alpha(ra, rb).$$

The above inequality, however, can in turn be written as

$$\frac{d}{dr}\left[\frac{g_\alpha(ra, rb)}{r}\right] \geqslant 0,$$

which means that $g_\alpha(ra, rb)/r$ is increasing as a function of r. The proof of the theorem is now complete. ■

3.4. Concavity of the function $g_{\mathbf{y}}(a, b)$. Let $\mathbf{y} \in \mathbb{R}^{n-1}$ and consider the density

$$f_{|\mathbf{y}|,n}(t) = \frac{c}{(1+|\mathbf{y}|^2+t^2)^{(n+1)/2}},$$

which is a one-dimensional section of the n-dimensional isotropic Cauchy distribution in the direction of $\mathbf{y}$. We denote this density as $f_{\alpha,n}(t)$ with $\alpha = |\mathbf{y}|$. As before, for $z_1 < z_2$, we define the function $g(z_1, z_2) := g_\alpha(z_1, z_2)$ by the identity

$$\int_{z_1}^{z_2} \frac{dt}{(1+\alpha^2+t^2)^{(n+1)/2}} = \int_{-\infty}^{g_\alpha(z_1,z_2)} \frac{dt}{(1+\alpha^2+t^2)^{(n+1)/2}}.$$

Introducing a new variable u by the formula $t = \sqrt{1+\alpha^2}\, u$, we obtain the following important scaling identity for the functions g_α:

$$g_\alpha(z_1, z_2) = \sqrt{1+\alpha^2}\, g_0\left(\frac{z_1}{\sqrt{1+\alpha^2}}, \frac{z_2}{\sqrt{1+\alpha^2}}\right). \tag{3.23}$$

We prove the following:

THEOREM 3.4. *The function*

$$\mathbb{R}^{n-1} \times \mathbb{R}^2 \ni (\mathbf{y}, a, b) \mapsto g_{|\mathbf{y}|}(a, b), \qquad a < b,$$

is concave, as a function of $n+1$ variables, for $a < b$.

Proof. We begin by computing the derivatives, using the identity (3.23):

$$\left.\frac{\partial g_\alpha}{\partial z_1}\right|_{(z_1,z_2)} = \left.\frac{\partial g_0}{\partial z_1}\right|_{\left(\frac{z_1}{\sqrt{1+\alpha^2}}, \frac{z_2}{\sqrt{1+\alpha^2}}\right)}, \qquad \left.\frac{\partial g_\alpha}{\partial z_2}\right|_{(z_1,z_2)} = \left.\frac{\partial g_0}{\partial z_2}\right|_{\left(\frac{z_1}{\sqrt{1+\alpha^2}}, \frac{z_2}{\sqrt{1+\alpha^2}}\right)},$$

$$\frac{\partial g_\alpha}{\partial \alpha} = \frac{\alpha}{1+\alpha^2}\left[g_\alpha - z_1\frac{\partial g_\alpha}{\partial z_1} - z_2\frac{\partial g_\alpha}{\partial z_2}\right].$$

Differentiating once again with respect to α, we obtain

$$\begin{aligned}\frac{\partial^2 g_\alpha}{\partial \alpha^2} &= \frac{1-\alpha^2}{(1+\alpha^2)^2}\left[g_\alpha - z_1\frac{\partial g_\alpha}{\partial z_1} - z_2\frac{\partial g_\alpha}{\partial z_2}\right] + \frac{\alpha}{1+\alpha^2}\left[\frac{\partial g_\alpha}{\partial \alpha} - z_1\frac{\partial^2 g_\alpha}{\partial z_1^2}\,\frac{(-z_1\alpha)}{(1+\alpha^2)^{3/2}}\right.\\ &\quad \left. - z_1\frac{\partial^2 g_\alpha}{\partial z_1 \partial z_2}\,\frac{-z_2\alpha}{(1+\alpha^2)^{3/2}} - z_2\frac{\partial^2 g_\alpha}{\partial z_2^2}\,\frac{-z_2\alpha}{(1+\alpha^2)^{3/2}} - z_2\frac{\partial^2 g_\alpha}{\partial z_1 \partial z_2}\,\frac{-z_1\alpha}{(1+\alpha^2)^{3/2}}\right].\end{aligned}$$

Taking into account the formula for $\frac{\partial g_\alpha}{\partial \alpha}$, we obtain

$$\frac{\partial^2 g_\alpha}{\partial \alpha^2} = \frac{1}{\alpha(1+\alpha^2)}\,\frac{\partial g_\alpha}{\partial \alpha} + \frac{\alpha^2}{(1+\alpha^2)^2}\left[z_1^2\frac{\partial^2 g_\alpha}{\partial z_1^2} + 2z_1z_2\frac{\partial^2 g_\alpha}{\partial z_1 \partial z_2} + z_2^2\frac{\partial^2 g_\alpha}{\partial z_2^2}\right].$$

The above calculations enable us to write down the Hessian matrix of $g_\alpha(z_1, z_2)$, as a function of three variables, in the following form:

$$\begin{bmatrix} \frac{\partial^2 g_\alpha}{\partial z_1^2} & \frac{\partial^2 g_\alpha}{\partial z_1 \partial z_2} & A \\ \frac{\partial^2 g_\alpha}{\partial z_1 \partial z_2} & \frac{\partial^2 g_\alpha}{\partial z_2^2} & B \\ A & B & \frac{\partial^2 g_\alpha}{\partial \alpha^2} \end{bmatrix},$$

where $A = \frac{-\alpha}{1+\alpha^2}\left(z_1\frac{\partial^2 g_\alpha}{\partial z_1^2} + z_2\frac{\partial^2 g_\alpha}{\partial z_1 \partial z_2}\right)$ and $B = \frac{-\alpha}{1+\alpha^2}\left(z_2\frac{\partial^2 g_\alpha}{\partial z_2^2} + z_1\frac{\partial^2 g_\alpha}{\partial z_1 \partial z_2}\right)$.

We compute the determinant of the above matrix by multiplying the first row by $\frac{\alpha}{1+\alpha^2} z_1$ and adding to the third row; analogously, we multiply the second row by $\frac{\alpha}{1+\alpha^2} z_2$ and add to the third one. We thus get the determinant of the matrix

$$\begin{bmatrix} \frac{\partial^2 g_\alpha}{\partial z_1^2} & \frac{\partial^2 g_\alpha}{\partial z_1 \partial z_2} & \frac{-\alpha}{1+\alpha^2}\left(z_1 \frac{\partial^2 g_\alpha}{\partial z_1^2} + z_2 \frac{\partial^2 g_\alpha}{\partial z_1 \partial z_2}\right) \\ \frac{\partial^2 g_\alpha}{\partial z_1 \partial z_2} & \frac{\partial^2 g_\alpha}{\partial z_2^2} & \frac{-\alpha}{1+\alpha^2}\left(z_2 \frac{\partial^2 g_\alpha}{\partial z_2^2} + z_1 \frac{\partial^2 g_\alpha}{\partial z_1 \partial z_2}\right) \\ 0 & 0 & \frac{1}{\alpha(1+\alpha^2)} \frac{\partial g_\alpha}{\partial \alpha} \end{bmatrix}.$$

The determinant is the product of the determinant of the upper left 2×2 corner and the term $\frac{1}{\alpha(1+\alpha^2)} \frac{\partial g_\alpha}{\partial \alpha}$. Since we already know that $g_\alpha(z_1, z_2)$ is concave as a function of z_1, z_2, everything reduces to the proof that $\frac{\partial g_\alpha}{\partial \alpha} < 0$, that is, $g_\alpha(z_1, z_2)$ is decreasing as a function of α. This, however, follows from the Landau–Shepp inequality (Theorem 3.3): for $r \geqslant 1$,

$$g_0(rz_1, rz_2) \geqslant r g_0(z_1, z_2).$$

Indeed, assume that $0 < \alpha_1 < \alpha_2$. From the above property and the scaling property (3.23) of g, we get

$$\begin{aligned} g_{\alpha_1}(z_1, z_2) &= \sqrt{1+\alpha_1^2}\, g_0\left(\frac{z_1}{\sqrt{1+\alpha_1^2}}, \frac{z_2}{\sqrt{1+\alpha_1^2}}\right) \\ &= \sqrt{1+\alpha_1^2}\, g_0\left(\frac{z_1}{\sqrt{1+\alpha_2^2}}\sqrt{\frac{1+\alpha_2^2}{1+\alpha_1^2}}, \frac{z_2}{\sqrt{1+\alpha_2^2}}\sqrt{\frac{1+\alpha_2^2}{1+\alpha_1^2}}\right) \\ &\geqslant \sqrt{1+\alpha_1^2}\sqrt{\frac{1+\alpha_2^2}{1+\alpha_1^2}} g_0\left(\frac{z_1}{\sqrt{1+\alpha_2^2}}, \frac{z_2}{\sqrt{1+\alpha_2^2}}\right) \\ &= \sqrt{1+\alpha_2^2}\, g_0\left(\frac{z_1}{\sqrt{1+\alpha_2^2}}, \frac{z_2}{\sqrt{1+\alpha_2^2}}\right) \\ &= g_{\alpha_2}(z_1, z_2). \end{aligned}$$

The above inequality shows that $g_\alpha(a, b)$ is concave as a function of (α, a, b) for $\alpha \geqslant 0$ and $a < b$. Since the norm $\mathbf{y} \mapsto |\mathbf{y}| = \alpha$ is a convex function and $g_\alpha(a, b)$ is decreasing as a function of α, the theorem follows. ∎

Now, let $\mathbf{x} = (y, \overline{\mathbf{x}}) \in \mathbb{R} \times \mathbb{R}^{n-1}$. For a Borel subset D of $\mathbb{R}^n$ denote

$$D^{(\overline{\mathbf{x}})} = \{y \in \mathbb{R} : (y, \overline{\mathbf{x}}) \in D\}$$

and define

$$S(D) = \{(z, \overline{\mathbf{x}}) \in \mathbb{R} \times \mathbb{R}^{n-1} : g_{|\overline{\mathbf{x}}|}(D^{(\overline{\mathbf{x}})}) > z\} := \bigcup_{\overline{\mathbf{x}}; z} \{(z, \overline{\mathbf{x}}) : g_{|\overline{\mathbf{x}}|}(D^{(\overline{\mathbf{x}})}) > z\}.$$

$S(D)$ is a version of Ehrhard's 1-symmetrization, adapted to the standard isotropic n-dimensional Cauchy measure. Observe that the operation S is monotonic, that is, if D and C are Borel subsets of $\mathbb{R}^n$ such that $C \subseteq D$ then also $S(C) \subseteq S(D)$.

THEOREM 3.5 (Concavity of Cauchy 1-symmetrization). *The* 1*-symmetrization S is a concave operation on convex subsets of $\mathbb{R}^n$; that is, for any convex Borel subsets D_1, D_2 of $\mathbb{R}^n$ and every* $0 < \lambda < 1$ *we have*

$$S(\lambda D_1 + (1-\lambda)D_2) \supseteq \lambda S(D_1) + (1-\lambda)S(D_2).$$

If $D_1 = D_2 = D$ is convex, then $S(D)$ is a convex subset of $\mathbb{R}^n$, hence the operation S carries convex sets into convex sets.

Proof. Let D_1, D_2 be convex Borel subsets of $\mathbb{R}^n$. Because $y_i \in D_i^{(\overline{\mathbf{x}}_i)}$ if and only if $(y_i, \overline{\mathbf{x}}_i) \in D_i$, for $i = 1, 2$, and

$$\begin{aligned}(\lambda y_1 + (1-\lambda)y_2, \lambda\overline{\mathbf{x}}_1 + (1-\lambda)\overline{\mathbf{x}}_2)& \\ = \lambda(y_1, \overline{\mathbf{x}}_1) + (1-\lambda)&(y_2, \overline{\mathbf{x}}_2) \in \lambda D_1 + (1-\lambda)D_2,\end{aligned}$$

we have

$$\lambda y_1 + (1-\lambda)y_2 \in (\lambda D_1 + (1-\lambda)D_2)^{(\lambda\overline{\mathbf{x}}_1 + (1-\lambda)\overline{\mathbf{x}}_2)}$$

and so

$$\lambda D_1^{(\overline{\mathbf{x}}_1)} + (1-\lambda)D_2^{(\overline{\mathbf{x}}_2)} \subseteq (\lambda D_1 + (1-\lambda)D_2)^{(\lambda\overline{\mathbf{x}}_1 + (1-\lambda)\overline{\mathbf{x}}_2)}.$$

Observe that the function $g_{|\overline{\mathbf{x}}|}(z_1, z_2)$, as a function of the interval (z_1, z_2), is increasing in the following sense: if the measure of (z_1, z_2) is greater than the measure of (z_3, z_4), then $g_{|\overline{\mathbf{x}}|}(z_1, z_2) > g_{|\overline{\mathbf{x}}|}(z_3, z_4)$.

From the above inclusion, together with the concavity of g, we obtain

$$\begin{aligned}(3.24)\quad g_{|\lambda\overline{\mathbf{x}}_1 + (1-\lambda)\overline{\mathbf{x}}_2|}&((\lambda D_1 + (1-\lambda)D_2)^{(\lambda\overline{\mathbf{x}}_1 + (1-\lambda)\overline{\mathbf{x}}_2)}) \\ &\geqslant g_{|\lambda\overline{\mathbf{x}}_1 + (1-\lambda)\overline{\mathbf{x}}_2|}(\lambda D_1^{(\overline{\mathbf{x}}_1)} + (1-\lambda)D_2^{(\overline{\mathbf{x}}_2)}) \\ &\geqslant \lambda g_{|\overline{\mathbf{x}}_1|}(D_1^{(\overline{\mathbf{x}}_1)}) + (1-\lambda)g_{|\overline{\mathbf{x}}_2|}(D_2^{(\overline{\mathbf{x}}_2)}).\end{aligned}$$

The definition of S implies

$$S(\lambda D_1 + (1-\lambda)D_2) = \bigcup_{\overline{\mathbf{x}};z}\{(z, \overline{\mathbf{x}}) : g_{|\overline{\mathbf{x}}|}((\lambda D_1 + (1-\lambda)D_2)^{(\overline{\mathbf{x}})}) > z\}$$

$$\begin{aligned}= \bigcup_{\overline{\mathbf{x}}_1, \overline{\mathbf{x}}_2; z_1, z_2} &\{\lambda(z_1, \overline{\mathbf{x}}_1) + (1-\lambda)(z_2, \overline{\mathbf{x}}_2) : \\ &g_{|\lambda\overline{\mathbf{x}}_1 + (1-\lambda)\overline{\mathbf{x}}_2|}((\lambda D_1 + (1-\lambda)D_2)^{(\lambda\overline{\mathbf{x}}_1 + (1-\lambda)\overline{\mathbf{x}}_2)}) > \lambda z_1 + (1-\lambda)z_2\}\end{aligned}$$

because obviously, if $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ and z_1, z_2 run over $\mathbb{R}^{n-1}$ and $\mathbb{R}$, respectively, so do $\overline{\mathbf{x}} = \lambda\overline{\mathbf{x}}_1 + (1-\lambda)\overline{\mathbf{x}}_2$ and $z = \lambda z_1 + (1-\lambda)z_2$. Using (3.24), we finally obtain

$$\begin{aligned}
&\bigcup_{\overline{\mathbf{x}}_1,\overline{\mathbf{x}}_2,z_1,z_2} \{\lambda(z_1,\overline{\mathbf{x}}_1) + (1-\lambda)(z_2,\overline{\mathbf{x}}_2) : \\
&\qquad\qquad g_{|\lambda\overline{\mathbf{x}}_1+(1-\lambda)\overline{\mathbf{x}}_2|}((\lambda D_1 + (1-\lambda)D_2)^{(\lambda\overline{\mathbf{x}}_1+(1-\lambda)\overline{\mathbf{x}}_2)}) > \lambda z_1 + (1-\lambda)z_2\} \\
&\supseteq \bigcup_{\overline{\mathbf{x}}_1,\overline{\mathbf{x}}_2,z_1,z_2} \{\lambda(z_1,\overline{\mathbf{x}}_1) + (1-\lambda)(z_2,\overline{\mathbf{x}}_2) : \\
&\qquad\qquad \lambda g_{|\overline{\mathbf{x}}_1|}(D_1^{(\overline{\mathbf{x}}_1)}) + (1-\lambda)g_{|\overline{\mathbf{x}}_2|}(D_2^{(\overline{\mathbf{x}}_2)}) > \lambda z_1 + (1-\lambda)z_2\} \\
&\supseteq \bigcup_{\overline{\mathbf{x}}_1,\overline{\mathbf{x}}_2,z_1,z_2} \{\lambda(z_1,\overline{\mathbf{x}}_1) + (1-\lambda)(z_2,\overline{\mathbf{x}}_2) : \\
&\qquad\qquad \lambda g_{|\overline{\mathbf{x}}_1|}(D_1^{(\overline{\mathbf{x}}_1)}) > \lambda z_1,\ (1-\lambda)g_{|\overline{\mathbf{x}}_2|}(D_2^{(\overline{\mathbf{x}}_2)}) > (1-\lambda)z_2\} \\
&\supseteq \lambda \bigcup_{\overline{\mathbf{x}}_1,z_1} \{(z_1,\overline{\mathbf{x}}_1) : g_{|\overline{\mathbf{x}}_1|}(D_1^{(\overline{\mathbf{x}}_1)}) > z_1\} \\
&\quad + (1-\lambda) \bigcup_{\overline{\mathbf{x}}_2,z_2} \{(z_2,\overline{\mathbf{x}}_2) : g_{|\overline{\mathbf{x}}_2|}(D_2^{(\overline{\mathbf{x}}_2)}) > z_2\} \\
&= \lambda S(D_1) + (1-\lambda)S(D_2).
\end{aligned}$$

Thus, S is a concave operation on convex subsets of $\mathbb{R}^n$. If $D_1 = D_2 = D$ is convex, then $\lambda D_1 + (1-\lambda)D_2 \subseteq D$ and $S(D) \supseteq S(\lambda D_1 + (1-\lambda)D_2) \supseteq \lambda S(D) + (1-\lambda)S(D)$, hence $S(D)$ is a convex subset of $\mathbb{R}^n$, and this completes the proof. ∎

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