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ON A RELATION BETWEEN CLASSICAL AND FREE INFINITELY DIVISIBLE TRANSFORMS

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Abstract. We study two ways (two levels) of finding free-probability analogues of classical infinitely divisible measures. More precisely, we identify their Voiculescu transforms on the imaginary axis. For free-selfdecomposable measures we find a formula (a differential equation) for their background driving transforms. It is different from the one known for classical selfdecomposable measures. We illustrate our methods on hyperbolic characteristic functions. Our approach may produce new formulas for definite integrals of some special functions.

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1. INTRODUCTION

There are many notions of infinite divisibility that exhibit some similarities and also some differences. Here we study the classical infinite divisibility with respect to the convolution $*$ and the free-infinite divisibility for the box-plus $\boxplus$ operation (Theorem 2.1, Corollary 2.1). For free-selfdecomposable Voiculescu transforms we found an ordinary differential equation for their background driving transforms; (Theorem 3.1). It is different from the one we know for classical selfdecomposable measures. Finally, we illustrate Theorem 2.1 by introducing free-probability analogues of the Laplace (double exponential) and the hyperbolic distributions on the real line (see Sections 4–6; in Subsection 4.1 the necessary special functions and formulae are collected).

Hyperbolic distributions were studied from the infinite divisibility point of view by Pitman and Yor (2003). Here we utilize the fact that all of them are in a proper subclass of *selfdecomposable distributions* (also called *class L distributions*); see

Jurek (1996) and Jurek and Yor (2004).

The program of this study (that can be applied to other pairs of notions of infinite divisibility, e.g. Urbanik convolution, boolean convolution) may be viewed as a particular case of the following abstract set-up:

There are two abstract semigroups $(\mathcal{S}_1, \triangleleft)$ *and* $(\mathcal{S}_2, \diamond)$*, two* 1-1 *and* onto *operators* A *and* Z *acting on domains* $\mathcal{D}_1$ *and* $\mathcal{D}_2$*, respectively, and a* 1-1 *and* onto *mapping* j *between* $\mathcal{D}_1$ *and* $\mathcal{D}_2$*.*

That is, we have

$$j : \mathcal{D}_1 \to \mathcal{D}_2, \quad A : \mathcal{D}_1 \to \mathcal{S}_1 \quad \text{and} \quad Z : \mathcal{D}_2 \to \mathcal{S}_2.$$

Consequently, the diagram

$$\begin{array}{ccc} \mathcal{D}_1 & \xrightarrow{A} & (\mathcal{S}_1, \triangleleft) \\ {\scriptstyle j}\big\downarrow & & {\scriptstyle r}\big\downarrow \\ \mathcal{D}_2 & \xrightarrow{Z} & (\mathcal{S}_2, \diamond) \end{array}$$

allows us to define the identification r between $(\mathcal{S}_1, \triangleleft)$ and $(\mathcal{S}_2, \diamond)$.

We say that $\tilde{s} \in (\mathcal{S}_2, \diamond)$ is a $\diamond$-*analogue* or $\diamond$-*counterpart* of an $s \in (\mathcal{S}_1, \triangleleft)$ if there exists $x \in \mathcal{D}_1$ such that $A(x) = s$, $j(x) = y$ and $Z(y) = \tilde{s}$, that is, we have $r(A(x)) = r(s) = \tilde{s}$, or $Z(j(x)) = \tilde{s}$.

Similarly, $s \in (S_1, \triangleleft)$ is a $\triangleleft$-*analogue* of $\tilde{s} \in (\mathcal{S}_2, \diamond)$ if there exists $y \in \mathcal{D}_2$ such that $Z(y) = \tilde{s}$, $j^{-1}(y) = x$ and $A(x) = s$.

2. INFINITE DIVISIBILITY

2.1. In the setting of this paper, $(\mathcal{S}_1, \triangleleft) \equiv (\mathrm{ID}, *)$ is the (classical) convolution semigroup ID of all infinitely divisible probability measures μ on the real line with the convolution operation $*$. *Characteristic functions* (or Fourier transforms) are functions given as

$$\phi(t) := \int_{\mathbb{R}} e^{itx} \mu(dx), \quad t \in \mathbb{R}, \quad \text{for some probability measure } \mu.$$

Let

$$\mathcal{D}_1 := \{\phi \in \mathrm{ID} : t \mapsto \phi^{1/n}(t) \text{ is a characteristic function for } n = 2, 3, \dots\},$$

that is, $\mathcal{D}_1$ consists of all $*$-infinitely divisible characteristic functions.

Further, consider the family of pairs

$$\mathcal{D}_2 := \{[a, m] : a \in \mathbb{R} \text{ and } m \text{ is a finite Borel measure on } \mathbb{R}\}.$$

Because of the fundamental *Khinchin representation formula*, $\phi \in \mathcal{D}_1$ iff

$$(2.1) \quad \phi(t) = \exp\Big\{ita + \int_{\mathbb{R}} \Big(e^{itx} - 1 - \frac{itx}{1+x^2}\Big)\frac{1+x^2}{x^2}\, m(dx)\Big\}, \qquad t \in \mathbb{R},$$

for uniquely determined parameters a (a number) and m (a finite measure) (for instance, see Parthasarathy (1967) or Araujo and Giné (1980)), the mapping (isomorphism) $j : \mathcal{D}_1 \to \mathcal{D}_2$ given as

$$(2.2) \qquad j(\phi) := [a, m] \quad \text{iff} \quad \phi \text{ is of the form (2.1)}$$

is well defined.

REMARK 2.1. (i) In some situations and applications, instead of a finite measure m, in (2.1), one uses the σ-finite measure $M(dx) := \frac{1+x^2}{x^2} m(dx)$ on $\mathbb{R} \setminus \{0\}$ (equivalently, $m(dx) := \frac{x^2}{1+x^2} M(dx)$) and a slightly changed integrand as given below. Then the equality (2.1) can be rewritten as follows:

$$(2.1') \quad \phi(t) = \exp\Big\{itb - \frac{1}{2}t^2\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1 - itx\, 1_{\{|x|\leqslant 1\}}(x))\, M(dx)\Big\}$$

where $\sigma^2 := m(\{0\})$ and $b := a + \int_{\mathbb{R}} x[1_{\{|x|\leqslant 1\}}(x) - 1/(1+x^2)]\, M(dx)$.

(ii) The formula $(2.1')$ is called *the Lévy–Khinchin representation* of an infinitely divisible characteristic function (probability measure). The probability measure μ corresponding to $(2.1')$ is represented by the triple $\mu = [b, \sigma^2, M]$.

(iii) The measure M has the following stochastic meaning: $M(A)$ is the expected number of jumps that occur up to time 1 and are of sizes in the set A, of the corresponding Lévy process $(Y(t), t \geqslant 0)$, where ϕ is the characteristic function of the random variable $Y(1)$.

2.2. Further, $(\mathcal{S}_2, \diamond) \equiv (\mathrm{ID}, \boxplus)$ is the semigroup of all $\boxplus$ free-infinitely divisible probability measures. Namely, for a probability measure ν on $\mathbb{R}$, one introduces its Voiculescu transform V_ν (an analogue of a characteristic function ϕ) and an operation $\boxplus$ on measures in such a way that

$$V_{\mu\boxplus\nu}(z) = V_\mu(z) + V_\nu(z);$$

see Voiculescu (1999). This property allows us to introduce the notion of $\boxplus$-infinite divisibility and one has the following analogue of the Khinchin representation:

$$(2.3) \qquad \nu \in (\mathrm{ID}, \boxplus) \quad \text{iff} \quad V_\nu(z) = a + \int_{\mathbb{R}} \frac{1+zx}{z-x}\, m(dx), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

for a uniquely determined constant $a \in \mathbb{R}$ and finite (Borel) measure m; see Voiculescu (1999) or Bercovici and Voiculescu (1993) or Bercovici and Pata

(1999). However, for the uniqueness questions regarding the representation (2.3), it is enough to consider Voiculescu transforms only on the imaginary axis; see Jurek (2006, 2007) or Jankowski and Jurek (2012).

The formulas (2.1) and (2.2) suggest defining a mapping $A : \mathcal{D}_1 \to (\mathrm{ID}, *)$ by

$$A(\phi) := \mu \quad \text{iff} \quad \phi(t) = \int_{\mathbb{R}} e^{itx}\, \mu(dx), \quad t \in \mathbb{R}, \tag{2.4}$$

and $r : (\mathrm{ID}, *) \to (\mathrm{ID}, \boxplus)$ by

$$r(\mu) := \tilde{\mu} \quad \text{iff} \quad V_{\tilde{\mu}}(it) = it^2 \int_0^\infty \overline{\log \phi(s)}\, e^{-ts}\, ds, \quad t > 0. \tag{2.5}$$

Consequently, by (2.5), we get the composition $r \triangleleft A : \mathcal{D}_1 \to (\mathrm{ID}, \boxplus)$.

On the other hand, at the Z level in the diagram of Section 1, using the mapping j from (2.2) we define $Z : \mathcal{D}_2 \to (\mathrm{ID}, \boxplus)$ as

$$Z([a, m]) := \nu \quad \text{iff} \quad V_\nu(it) = a + \int_{\mathbb{R}} \frac{1 + itx}{it - x}\, m(dx). \tag{2.6}$$

So we have the following question:

(2.7) Does $j(\phi) = [a, m]$ imply that $r(A(\phi)) = Z(j(\phi))$, that is, $\tilde{\mu} = \nu$?

REMARK 2.2. The idea of inserting the same parameters a and m into two different integral kernels (2.1) and (2.3) is due to Bercovici and Pata (1999, Section 3). [In our notation, it is the mapping at the Z level.]

A different approach was proposed in Jurek (2006, 2007) and more recently in Jurek (2016). The key in those papers was the technique of random integral representation. Here, it is the mapping at the A level in the diagram.

2.3. Below we give straightforward connections between the formulas (2.1) and (2.2) and we prove the equality in (2.7). As in Jurek (2006), we consider the transforms V_ν only on the imaginary line.

THEOREM 2.1. *For each classical infinitely divisible $\mu \in (\mathrm{ID}, *)$ with characteristic function ϕ_μ there exists a unique free-infinitely divisible measure $\tilde{\mu} \in (\mathrm{ID}, \boxplus)$ whose Voiculescu transform $V_{\tilde{\mu}}$ is given as*

$$V_{\tilde{\mu}}(it) = it^2 \int_0^\infty \overline{\log \phi_\mu(s)}\, e^{-ts}\, ds, \quad t > 0. \tag{A}$$

Furthermore, if μ has a representation $\mu = [a, m]$ (in the Khinchin formula) then

$$V_{\tilde{\mu}}(it) = a + \int_{\mathbb{R}} \frac{1 + itx}{it - x}\, m(dx), \quad t > 0. \tag{Z}$$

For a symmetric $\mu = [0, m]$ (ϕ_μ *real*) *we have*

$$V_{\tilde{\mu}}(it) = it \int_0^\infty \int_{\mathbb{R}} (\cos(sx) - 1) \frac{1+x^2}{x^2} m(dx)\, te^{-st}\, ds = -it \int_{\mathbb{R}} \frac{1+x^2}{t^2+x^2} m(dx).$$

Proof. The fact that the function given in (A) indeed defines the Voiculescu transform of a free-infinitely divisible measure was already shown in Jurek (2007, Corollary 6).

On the other hand, formula (Z) is obviously the Voiculescu transform of a free-infinitely measure in view of the characterization (2.3) above.

In order to show that both give the same measure we will prove that

$$it^2 \int_0^\infty \overline{\log \phi_{[a,m]}(s)}\, e^{-ts}\, ds = a + \int_{\mathbb{R}} \frac{1+itx}{it-x} m(dx), \qquad t > 0. \tag{2.8}$$

Taking the Lévy exponent (the logarithm of the characteristic function) as given in the Khinchin formula (2.1), computing the Laplace transform of the shift part ita and then interchanging the order of integration we obtain

$$\begin{aligned}
\text{LHS} &= it^2\bigg(-\frac{ia}{t^2} + \int_{\mathbb{R}} \frac{1+x^2}{x^2}\bigg[\int_0^\infty \bigg(e^{-isx} - 1 + \frac{isx}{1+x^2}\bigg)e^{-st}\, ds\bigg]\, m(dx)\bigg) \\
&= it^2\bigg(-\frac{ia}{t^2} + \int_{\mathbb{R}} \frac{1+x^2}{x^2}\bigg[\frac{1}{ix+t} - \frac{1}{t} + \frac{ix}{1+x^2}\,\frac{1}{t^2}\bigg]\, m(dx)\bigg) \\
&= a + \int_{\mathbb{R}} \frac{1+x^2}{x^2} it^2 \bigg[\frac{-ix}{t(ix+t)} + \frac{ix}{(1+x^2)t^2}\bigg]\, m(dx) \\
&= a + \int_{\mathbb{R}} \frac{1+x^2}{x^2}\bigg[\frac{tx}{ix+t} - \frac{x}{1+x^2}\bigg]\, m(dx) \\
&= a + \int_{\mathbb{R}} \frac{tx-i}{ix+t}\, m(dx) = a + \int_{\mathbb{R}} \frac{1+itx}{it-x}\, m(dx) = \text{RHS},
\end{aligned}$$

which concludes the proof of (2.8) and of the first part of Theorem 2.1.

For the second part, we calculate as above, that is, we change the order of integration, utilize the fact that m is a symmetric measure and use the Laplace transform

$$\int_0^\infty \cos(as) e^{-st}\, ds = \frac{t}{t^2+a^2}.$$

This yields the second equality in Theorem 2.1. ∎

COROLLARY 2.1. *Let* $\mathcal{E}_t$, $t > 0$, *denote the exponential random variable with parameter* t *and probability density* $te^{-tx} 1_{(0,\infty)}(x)$. *Then*

$$\mathbb{E}[\log \phi_\mu(-\mathcal{E}_t)] = \int_0^\infty \overline{\log \phi_\mu(s)}\, (te^{-ts})\, ds = (it)^{-1} V_{\tilde{\mu}}(it) \qquad \text{for } t > 0.$$

Furthermore, if $\mu = [0, m]$ is symmetric (i.e., ϕ_μ is real) and m is the probability distribution of a random variable X that is stochastically independent of $\mathcal{E}_t$ then

$$(2.9)\qquad \mathbb{E}\left[(1-\cos(\mathcal{E}_t\, X))\frac{1+X^2}{X^2}\right] = \mathbb{E}\left[\frac{1+X^2}{t^2+X^2}\right], \quad t>0.$$

From the equality (A) in Theorem 2.1 we deduce

COROLLARY 2.2. *For $a \in \mathbb{R}$, $t > 0$ and $\mu, \nu \in (\mathrm{ID}, *)$ and their free analogues $\tilde{\mu}, \tilde{\nu} \in (\mathrm{ID}, \boxplus)$ we have*

(a) $V_{\widetilde{\mu*\nu}}(it) = V_{\tilde{\mu}}(it) + V_{\tilde{\nu}}(it) = V_{\tilde{\mu}\boxplus\tilde{\nu}}(it)$, (b) $V_{\widetilde{T_a\mu}}(it) = a\, V_{\tilde{\mu}}(it/a)$,

where $T_a\mu(\cdot) := \mu(a^{-1}(\cdot))$, and for the characteristic functions we have $\phi_{T_a\mu}(t) = \phi(at)$.

REMARK 2.3. The equality (A) in Theorem 2.1 can be potentially retrieved from Theorem 5.5 in Barndorff-Nielsen and Thorbjørnsen (2002). However, there one needs facts about Υ transforms (certain random integral mappings) and cumulants of $\boxplus$-infinitely divisible probability measures, while here we have straightforward, elementary calculations.

2.4. An application of Theorem 2.1. Let C stand for a *hyperbolic-cosh* variable or its probability distribution. It is $*$-infinitely divisible and its characteristic function is $\phi_C(t) = (\cosh t)^{-1}$. From Theorem 2.1, at the level of characteristic functions (the mapping A), the free-infinitely divisible analogue $\tilde{C}$ (of the hyperbolic-cosh) has the following Voiculescu transform:

$$(2.10)\qquad V_{\tilde{C}}(it) = -it^2 \int_0^\infty (\log\cosh(s))e^{-ts}\, ds = i[1 - t\beta(t/2)], \quad t>0,$$

where β is a special function defined in (4.1)(vii) below. The equality (2.10) is shown in Section 4.2.

On the other hand, at the level of the parameters $[a, m]$ (the mapping Z), the hyperbolic-cosh has $a = 0$ and $m(dx) = \frac{1}{2}\,\frac{|x|}{1+x^2}\,\frac{1}{\sinh(\pi|x|/2)}\, dx$. Consequently, by (2.8), its $\boxplus$-free infinitely divisible analogue $\tilde{C}$ has Voiculescu transform

$$(2.11)\qquad V_{\tilde{C}}(it) = -it \int_0^\infty \frac{|x|}{t^2+x^2}\,\frac{1}{\sinh(\pi|x|/2)}\, dx = i[t\beta(t/2+1) - 1];$$

for computational details see Section 4.2.

As a byproduct of (2.10) and (2.11) we get the following functional relation for the special function β:

$$(2.12)\qquad \beta(s) + \beta(s+1) = 1/s, \quad s>0,$$

However, the formula (2.12) can also be obtained from another known representation of the special function β: see (4.1)(ix).

REMARK 2.4. Our two ways (two levels, two mappings) of getting free-infinitely divisible analogues of classical infinitely divisible characteristic functions may produce new explicit relations between special functions.

3. SELFDECOMPOSABILITY

3.1. An important proper subclass of the class $(\mathrm{ID}, *)$ of all infinitely divisible measures is *the class L*, also known as the class of *selfdecomposable probability measures*; see Jurek and Vervaat (1983). A free analogue of a selfdecomposable distribution was given by Barndorff-Nielsen and Thorbjørnsen (2002).

Let us recall that the class L contains all stable probability measures, exponential distributions, t-Student distribution, chi-square, gamma, Laplace, hyperbolic-sine and hyperbolic-cosine measures (characteristic functions), etc.; see Jurek and Yor (2004).

For the purposes of this paper let us recall that for $\mu \in L$ (or equivalently for a characteristic function $\phi \in L$) there exists a unique $\nu \in \mathrm{ID}_{\log}$, i.e., infinitely divisible measure with finite logarithmic moment (or equivalently there exists a unique $\psi \in \mathrm{ID}_{\log}$) such that

$$(3.1) \qquad \log \psi(t) = t \frac{\phi'(t)}{\phi(t)}, \quad \text{equivalently,} \quad \log \phi(t) = \int_0^t \log \psi(s) \frac{ds}{s},$$

and hence

$$\log \psi(-t) = -t \frac{\phi'(-t)}{\phi(-t)} = t(\log \phi(-t))'.$$

The above relations follow from the *random integral representation of a selfdecomposable distribution*: for each $\mu \in L$ there exists a unique Lévy process $Y_\nu(s), s \geqslant 0$, such that

$$(3.2) \qquad \mu = \mathcal{L}\Big(\int_0^\infty e^{-s} dY_\nu(s)\Big), \quad \mathcal{L}(Y_\nu(1)) = \nu \in \mathrm{ID}_{\log};$$

see Jurek and Vervaat (1983).

The characteristic function ψ in (3.1) is referred to as the *background driving characteristic function* (BDCF) of $\phi \in L$, and Y_ν is the *background driving Lévy process* (BDLP) of μ.

REMARK 3.1. A very nice argument, based on the random integral representation (3.2), for the existence of densities for all real-valued selfdecomposable variables is due to Jacod (1985).

3.2. Here is a technical property of the Lévy exponent Φ of an infinitely divisible characteristic function with a real parameter a and a finite measure m, that is,

$$\Phi(t) := ita + \int_{\mathbb{R}} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) \frac{1+x^2}{x^2}\, m(dx) \tag{3.3}$$
$$= itb + \int_{\mathbb{R}} (e^{itx} - 1 - itx\, 1_{|x|\leqslant 1}(x)) \frac{1+x^2}{x^2}\, m(dx),$$

where $b := a + \int_{\mathbb{R}} x[1_{\{|x|\leqslant 1\}}(x) - 1/(1+x^2)] \frac{1+x^2}{x^2}\, m(dx)$ and the finiteness of the measure m guarantees the existence of the integral.

LEMMA 3.1. *For any constants $c_1, c_2 > 0$ and any Lévy exponent Φ we have*

$$\lim_{t\to\infty} t^{c_1} e^{-c_2 t}\, \Phi(t) = \lim_{t\to\infty} t^{c_1} e^{-c_2 t} \int_{\mathbb{R}} (e^{itx} - 1 - itx 1_{|x|\leqslant 1}(x)) \frac{1+x^2}{x^2}\, m(dx) = 0.$$

Proof. For a pure degenerate Φ, i.e., when $m = 0$ in (3.3), the assertion is obvious.

Assume that $b = 0$. Since

$$|e^{itx} - 1 - itx| \leqslant \min(|tx|^2/2, 2|tx|) \quad \text{and} \quad |e^{itx} - 1| \leqslant \min(|tx|, 2) \leqslant 2$$

(see, for instance, Billingsley (1986, pp. 352–353)), from (3.3) we get

$$t^{c_1} e^{-c_2 t}\, |\Phi(t)| \leqslant t^{c_1} e^{-c_2 t} \int_{|x|\leqslant 1} \frac{t^2}{2} x^2 \frac{1+x^2}{x^2}\, m(dx) + t^{c_1} e^{-c_2 t} \int_{|x|>1} 2 \frac{1+x^2}{x^2}\, m(dx)$$
$$\leqslant \frac{1}{2} t^{c_1+2} e^{-c_2 t} \int_{|x|\leqslant 1} (1+x^2)\, m(dx) + 2 t^{c_1} e^{-c_2 t} \int_{|x|>1} (1+x^{-2})\, m(dx) \to 0$$

as $t \to \infty$. ∎

Now we give a free-selfdecomposability analogue, for the background driving transforms, of the differential relations (3.1) known for classical selfdecomposability.

THEOREM 3.1. *Let $\tilde{\phi}$ and $\tilde{\psi}$ be free-probability analogues of a selfdecomposable characteristic function ϕ and its background driving characteristic function ψ, respectively. Then their Voiculescu transforms $V_{\tilde{\phi}}$ and $V_{\tilde{\psi}}$ satisfy the differential equation*

$$V_{\tilde{\psi}}(it) = V_{\tilde{\phi}}(it) - t \frac{d}{dt}[V_{\tilde{\phi}}(it)], \qquad t > 0. \tag{3.4}$$

Equivalently, in terms of $V_{\tilde{\psi}}$,

$$V_{\tilde{\phi}}(it) - t\, V_{\tilde{\phi}}(i) = -t \int_1^t s^{-2} V_{\tilde{\psi}}(is)\, ds = t \int_1^t V_{\tilde{\psi}}(is)\, d(s^{-1}), \qquad t > 0. \tag{3.5}$$

Proof. Note that using (A) from Theorem 2.1, the relation (3.1) for classical selfdecomposability and then Lemma 3.1 we have

$$
\begin{aligned}
V_{\tilde{\psi}}(it) &:= it^2 \int_0^\infty (\log \psi(-v)) e^{-tv} dv = it^2 \int_0^\infty (\log \phi(-v))' v e^{-tv} dv \\
&= it^2 \Big[(\log \phi(-v)) v e^{-tv} |_{v=0}^{v=\infty} - \int_0^\infty (\log \phi(-v))(1-tv) e^{-tv}\, dv \Big] \\
&= it^2 \Big[\int_0^\infty -(\log \phi(-v)) e^{-tv}\, dv + t \int_0^\infty (\log \phi(-v)) v e^{-tv}\, dv \Big] \\
&= -V_{\tilde{\phi}}(it) - it^3 \frac{d}{dt} \Big[\int_0^\infty \log \phi(-v) e^{-tv}\, dv \Big] = -V_{\tilde{\phi}}(it) - it^3 \frac{d}{dt} [(it^2)^{-1} V_{\tilde{\phi}}(it)] \\
&= -V_{\tilde{\phi}}(it) - t^3 \frac{d}{dt} [t^{-2} V_{\tilde{\phi}}(it)] = -V_{\tilde{\phi}}(it) - t^3 \Big[-2t^{-3} V_{\tilde{\phi}}(it) + t^{-2} \frac{d}{dt} V_{\tilde{\phi}}(it) \Big] \\
&= V_{\tilde{\phi}}(it) - t \frac{d}{dt} V_{\tilde{\phi}}(it),
\end{aligned}
$$

which proves (3.4).

For (3.5), note that (3.4) is a first-order linear differential equation that we can solve by the integrating factor method. More explicitly, (3.4) can be rewritten as

$$
t^{-1} V_{\tilde{\psi}}(it) = t^{-1} V_{\tilde{\phi}}(it) - \frac{d}{dt} [V_{\tilde{\phi}}(it)] = -t \frac{d}{dt} \Big[\frac{V_{\tilde{\phi}}(it)}{t} \Big].
$$

Hence, dividing by t and then integrating both sides over $[1, t]$ (or $[t, 1]$), we get

$$
\frac{V_{\tilde{\phi}}(it)}{t} - V_{\tilde{\phi}}(i) = - \int_1^t s^{-2} V_{\tilde{\psi}}(is)\, ds,
$$

which completes the proof of Theorem 3.1. ■

3.3. An application of Theorem 3.1. The hyperbolic-cosh function $\phi_C(t) = (\cosh t)^{-1}$ is selfdecomposable; see Jurek (1996). From (2.10) and (3.4) we obtain $V_{\tilde{\psi_C}}$, a free-probability analogue of the background driving characteristic function ψ_C, as

(3.6)
$$
V_{\tilde{\psi_C}}(it) = i \Big[1 + \frac{1}{2} t^2 \beta' \Big(\frac{1}{2} t \Big) \Big] = i \Big[1 + \frac{t^2}{2} \zeta \Big(2, \frac{t}{2} \Big) - \frac{t^2}{4} \zeta \Big(2, \frac{t}{4} \Big) \Big], \quad t > 0.
$$

For the first equality one puts (2.10) into (3.4) and then uses the formula that expresses β' in terms of the Riemann zeta function $\zeta(2, a)$; for details see Section 4.2. [Two more ways of getting the above formula are discussed in Section 5.]

4. FREE-PROBABILITY ANALOGUES OF HYPERBOLIC CHARACTERISTIC FUNCTIONS

4.1. For ease of reference we recall the definitions and basic properties of some special functions. All boldface numbers refer to Gradshteyn and Ryzhik (1994).

(4.1)

(i) (a) $\Gamma(z) := \int_0^\infty x^{z-1}e^{-x}dx, \quad \Re z > 0$ (Euler function);

(b) $\psi(z) := \frac{d}{dz}\log\Gamma(z), \quad \Re z > 0$ (digamma function);

(ii) $\psi_n(z) \equiv \psi^{(n)}(z) = (-1)^{n+1}n!\zeta(n+1,z)$ (**8.363**(8)) (nth derivative; also called polygamma function);

(iii) $\psi(2z) = \frac{1}{2}(\psi(z)+\psi(z+1/2)) + \log 2$ (**8.365**(6));

(iv) $\zeta(s,a) := \sum_{k=0}^\infty \frac{1}{(k+a)^s}, \quad \Re s > 1, \quad -a \notin \mathbb{N}$ (Riemann zeta function);

(v) $\zeta(s,a+1) = \zeta(s,a) - 1/a^s, \quad \zeta(s,a+1/2) = 2^s\zeta(s,2a) - \zeta(s,a)$;

(vi) $\zeta(2,t) - \frac{1}{4}\zeta\left(2,\frac{t}{2}\right) = \frac{1}{4}\zeta\left(2,\frac{t+1}{2}\right)$ (from (v));

(vii) $\beta(x) := \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right], \quad \beta(x) = \sum_{k=0}^\infty \frac{(-1)^k}{x+k}, \quad -x \notin \mathbb{N}$ (**8.732**(1));

(viii) $\beta'(x) = -\sum_{k=0}^\infty \frac{(-1)^k}{(x+k)^2} = \zeta(2,x) - \frac{1}{2}\zeta\left(2,\frac{x}{2}\right)$ (**8.374**);

(ix) $\beta(t) = \int_0^\infty \frac{1}{1+e^{-x}}e^{-tx}\,dx, \quad \Re t > 0$ (**8.371**(2));

(x) $\mathrm{ci}(x) \equiv \mathrm{Ci}(x) := -\int_x^\infty \frac{\cos u}{u}\,du$;

$\mathrm{si}(x) := -\int_x^\infty \frac{\sin u}{u}\,du = -\frac{\pi}{2} + \mathrm{Si}(x), \quad$ where $\quad \mathrm{Si}(x) := \int_0^x \frac{\sin u}{u}\,du.$

4.2. Hyperbolic-cosine random variable. Let C stand for the *standard hyperbolic cosine variable*, that is, the random variable with characteristic function

(4.2)

$$\begin{aligned}\phi_C(t) := \frac{1}{\cosh t} &= \exp\int_{\mathbb{R}}(\cos(tx)-1)\frac{1+x^2}{x^2}\left[\frac{1}{2}\,\frac{|x|}{1+x^2}\,\frac{1}{\sinh(\pi|x|/2)}\right](dx)\\ &= \exp\int_{\mathbb{R}}(\cos(tx)-1)\left[\frac{1}{2}\,\frac{1}{|x|\sinh(\pi|x|/2)}\right](dx),\end{aligned}$$

where the first bracket $[\ldots]$ is the density of the Khinchin finite measure m_C corresponding to ϕ_C in the representation (2.1), and the second one is the density of the σ-finite Lévy (spectral) measure M; see $(2.1')$ in Remark 2.1 and Jurek and Yor (2004).

COROLLARY 4.1. *The free-probability analogue of the hyperbolic-cosine characteristic function ϕ_C has the following Voiculescu transform:*

$$V_{\tilde{\phi_C}}(it) = i[1 - t\beta(t/2)], t > 0; \quad V_{\tilde{\phi_C}}(i) = -\,i(\pi/2 - 1) \approx -0.5707. \tag{4.3}$$

As a consequence,

$$\beta(s) + \beta(s+1) = 1/s, \quad s > 0.$$

Proof. First proof: We use the identity

$$\int_0^\infty e^{-\xi x} \log(\cosh x) dx = \frac{1}{\xi}[\beta(\xi/2) - 1/\xi], \quad \Re\xi > 0 \quad (\mathbf{4.342}(2)). \tag{4.4}$$

Hence and by (A) of Theorem 2.1,

$$\begin{aligned} V_{\tilde{\phi_C}}(it) &= it^2 \int_0^\infty (\log \phi_C(-v)) e^{-tv}\, dv = -it^2 \int_0^\infty (\log \cosh(v)) e^{-tv}\, dv \\ &= -it^2 \left(\frac{1}{t}(\beta(t/2) - 1/t) \right) = i[1 - t\beta(t/2)], \quad t > 0, \end{aligned}$$

proving (4.3).

Second proof: This time we use the integral identity

$$\int_0^\infty \frac{x\, dx}{(b^2 + x^2)\sinh(\pi x)} = \frac{1}{2b} - \beta(b+1), \quad b > 0 \quad (\mathbf{3.522}(2)). \tag{4.5}$$

From the first line in (4.2) we see that ϕ_C has finite Khinchin measure

$$m_C(dx) = \frac{1}{2} \frac{|x|}{1 + x^2} \frac{1}{\sinh(\pi|x|/2)}\, dx.$$

Consequently, from Theorem 2.1 we get

$$\begin{aligned} V_{\tilde{\phi_C}}(it) &= -it \int_{\mathbb{R}} \frac{1 + x^2}{t^2 + x^2} \frac{1}{2} \frac{|x|}{1 + x^2} \frac{1}{\sinh(\pi\, |x|/2)}\, dx \\ &= -it \int_0^\infty \frac{x}{t^2 + x^2} \frac{1}{\sinh(\pi x/2)}\, dx = -it \int_0^\infty \frac{y}{(t/2)^2 + y^2} \frac{1}{\sinh \pi y}\, dy \\ &= -it \left[\frac{1}{t} - \beta\left(\frac{t}{2} + 1 \right) \right] = i \left[t\beta\left(\frac{t}{2} + 1 \right) - 1 \right], \quad t > 0. \end{aligned}$$

From those two proofs we see that

$$(4.6)\qquad t\beta\left(\frac{t}{2}+1\right)-1=1-t\beta\left(\frac{t}{2}\right),\quad \text{or}\quad \beta(s)+\beta(s+1)=1/s;$$

and this completes the proof of Corollary 4.1. ∎

REMARK 4.1. (a) The identity (4.6) also follows from the fact that $\beta(t) = \int_0^\infty (1+e^{-x})^{-1}e^{-tx}\,dx$, $t>0$ (see (4.1)(ix)).

4.3. Hyperbolic-sine variable. Let S stand for the *standard hyperbolic-sine variable*, that is, the random variable with characteristic function

(4.7)

$$\phi_S(t) := \frac{t}{\sinh t} = \exp\int_{\mathbb{R}}(\cos(tx)-1)\frac{1+x^2}{x^2}\left[\frac{1}{2}\,\frac{|x|}{1+x^2}\,\frac{e^{-\pi|x|/2}}{\sinh(\pi|x|/2)}\right]dx$$
$$= \exp\int_{\mathbb{R}}(\cos(tx)-1)\left[\frac{e^{-\pi|x|/2}}{2|x|\sinh(\pi|x|/2)}\right]dx,$$

where the first bracket $[\ldots]$ is the density of the (Khinchin) finite measure m_S corresponding to ϕ_S in (2.1), and the second one is the density of the σ-finite Lévy (spectral) measure M_S in Remark 2.1, formula (2.1′).

COROLLARY 4.2. *The free-probability analogue $\tilde{\phi_S}$ of the hyperbolic-sine characteristic function ϕ_S has the following Voiculescu transform:*

$$(4.8)\qquad \begin{aligned} V_{\tilde{\phi_S}}(it) &= i[t\psi(t/2)-t\log(t/2)+1], \qquad t>0,\\ V_{\tilde{\phi_S}}(i) &= i(1-\gamma-\log 2)\approx -i\,0.2703 \end{aligned}$$

(γ *is the Euler–Mascheroni constant,* ≈ 0.577).

Proof. First proof: The key integral identity for (4.8) is

$$\int_0^\infty e^{-\xi x}(\log(\sinh x)-\log x)dx = \frac{1}{\xi}[\log(\xi/2)-1/\xi-\psi(\xi/2)],\qquad \Re\xi>0$$

(see **4.342**(3)). (Note the misprint there; comp. www.mathtable.com/errata/gr6_errata.pdf.)

Hence and from (A) of Theorem 2.1, we get

(4.9)

$$V_{\tilde{\phi_S}}(it) = it^2\int_0^\infty(\log\phi_S(-v))e^{-tv}\,dv = -it^2\int_0^\infty[\log(\sinh v)-\log v]e^{-tv}\,dv$$
$$= -it^2\frac{1}{t}[\log(t/2)-1/t-\psi(t/2)] = i[t\psi(t/2)-t\log(t/2)+1],$$

which proves (4.8).

Second proof: This time we need the formula

$$\int_0^\infty \frac{x\,dx}{(x^2+\beta^2)(e^{\mu x}-1)} = \frac{1}{2}\left[\log\left(\frac{\beta\mu}{2\pi}\right) - \frac{\pi}{\beta\mu} - \psi\left(\frac{\beta\mu}{2\pi}\right)\right], \qquad \Re\beta, \Re\mu > 0$$

(see **3.415**(1)). From the first line in (4.7) we find that the Khinchin (finite) measure m_S (for ϕ_S) is equal to

$$m_S(dx) = \frac{1}{2}\,\frac{|x|}{1+x^2}\,\frac{e^{-\pi|x|/2}}{\sinh(\pi|x|/2)}dx = \frac{|x|}{1+x^2}\,\frac{1}{e^{\pi|x|}-1}dx$$

(see also Jurek and Yor (2004)). Thus the above identity and (Z) of Theorem 2.1 give

$$\begin{aligned}(4.10)\quad V_{\tilde{\phi_S}}(it) &= -2it\int_0^\infty \frac{x}{t^2+x^2}\,\frac{1}{e^{\pi x}-1}\,dx = -it[\log(t/2) - 1/t - \psi(t/2)] \\ &= i[t\psi(t/2) - t\log(t/2) + 1], \qquad t > 0,\end{aligned}$$

which coincides with (4.8). This completes the proof of Corollary 4.2. ∎

4.4. The hyperbolic-tangent variable. Let T stand for the *standard hyperbolic-tangent variable*, that is, the random variable with characteristic function $\phi_T(t) = (\tanh t)/t$. Its Khinchin representation is

$$(4.11)\qquad \phi_T(t) = \frac{\tanh t}{t} = \exp\int_{-\infty}^{\infty}(\cos tx - 1)\left[\frac{1}{2}\,\frac{|x|}{1+x^2}\,\frac{e^{-\pi|x|/4}}{\cosh(\pi|x|/4)}\right]dx;$$

where [...] is the density of the finite Khinchin measure m_T from (2.1).

COROLLARY 4.3. *The free-probability analogue $\tilde{\phi_T}$ of the hyperbolic-tangent characteristic function has the following Voiculescu transform:*

(4.12)

$$V_{\tilde{\phi_T}}(it) = it[\log(t/2)-\beta(t/2)-\psi(t/2)] = it[\log(t/4)-\psi(t/4+1/2)], \quad t > 0,$$

$V_{\tilde{\phi_T}}(i) = \gamma-\pi/2+\log 2 \approx -i\,0.30,$ γ *is the Euler–Mascheroni constant.*

Consequently, we get an identity for Euler's digamma function:

$$(4.13)\qquad 2\psi(2s) - \psi(s) - \psi(s+1/2) = 2\log 2, \qquad s > 0.$$

Proof. First proof: From the equality $\phi_C(t) = \phi_S(t)\cdot\phi_T(t)$, (A) of Theorem 2.1, and Corollaries 4.1 and 4.2 we get

$$\begin{aligned}V_{\tilde{\phi_T}}(it) &= it^2\int_0^\infty[\overline{\log\phi_C(t)} - \overline{\log\phi_S(t)}]e^{-ts}\,ds = V_{\tilde{\phi_C}}(it) - V_{\tilde{\phi_S}}(it) \\ &= i[1-t\beta(t/2)] - i[t\psi(t/2) - t\log(t/2)+1] = it[\log(t/2)-\beta(t/2)-\psi(t/2)],\end{aligned}$$

which gives the first equality in (4.12).

Second proof: This time we need **3.415**(3), that is,

$$\int_0^\infty \frac{x}{(x^2+\beta^2)(e^{\mu x}+1)}\,dx = \frac{1}{2}\left[\psi\left(\frac{\beta\mu}{2\pi}+\frac{1}{2}\right) - \log\left(\frac{\beta\mu}{2\pi}\right)\right], \qquad \Re\beta, \Re\mu > 0.$$

Since $1-\tanh x = \frac{e^{-x}}{\cosh x} = \frac{2}{e^{2x}+1}$, using the above and (Z) of Theorem 2.1, we obtain

$$\text{(4.14)} \qquad V_{\tilde{\phi_T}}(it) = -it\int_{\mathbb{R}} \frac{1+x^2}{t^2+x^2}\,\frac{1}{2}\,\frac{|x|}{1+x^2}\,\frac{e^{-\pi|x|/4}}{\cosh(\pi|x|/4)}\,dx$$
$$= -2it\int_0^\infty \frac{x}{(t^2+x^2)(e^{\pi x/2}+1)}\,dx$$
$$= -2it\,\frac{1}{2}\left[\psi\left(\frac{t\pi/2}{2\pi}+\frac{1}{2}\right) - \log\left(\frac{t\pi/2}{2\pi}\right)\right] = it[\log(t/4)-\psi(t/4+1/2)],$$

which is the second equality in (4.12). Consequently, by Theorem 2.1,

$$\log(t/4)-\psi(t/4+1/2) = \log(t/2)-\beta(t/2)-\psi(t/2), \qquad t>0,$$

or equivalently

$$\text{(4.15)} \qquad \psi(2s) - \tfrac{1}{2}\psi(s) - \tfrac{1}{2}\psi(s+1/2) = \log 2, \qquad s>0,$$

which completes the proof. ■

REMARK 4.2. (a) Note that (4.15) coincides with **8.365**(6) for $n=2$ (see also (4.1)(iii)).

(b) By reasoning as in the second proofs of Corollaries 4.1 and 4.2, and using (4.14), we get, for $t>0$,

$$\int_0^\infty \frac{x}{t^2+x^2}(1-\tanh(\pi x))dx = \psi(2t)+\beta(2t)-\log(2t) = \psi(t+1/2)-\log t.$$

(c) In particular, $\int_0^\infty \frac{x}{1+x^2}(1-\tanh(\pi x))\,dx = \psi(3/2)$.

5. FREE-PROBABILITY ANALOGUES OF BACKGROUND DRIVING FUNCTIONALS OF HYPERBOLIC DISTRIBUTIONS

Since the three hyperbolic characteristic functions ϕ_C, ϕ_S and ϕ_T (of the random variables C, S and T) are selfdecomosable (in other words, in the Lévy class L), they admit the infinitely divisible background driving characteristic functions (for short, BDCF) ψ_C, ψ_S and ψ_T, respectively. Further, if N_C, N_S and N_T are their

Lévy spectral measures then, on $\mathbb{R}\setminus\{0\}$,

$$\psi_C(t) = \exp[-t\tanh t], \qquad N_C(dx) = \frac{\pi}{4}\,\frac{\cosh(\pi x/2)}{\sinh^2(\pi x/2)}dx, \tag{5.1}$$

$$\psi_S(t) = \exp[1-t\coth t], \qquad N_S(dx) = \frac{\pi}{4}\,\frac{1}{\sinh^2(\pi x/2)}dx, \tag{5.2}$$

$$\psi_T(t) = \exp\left[\frac{2t}{\sinh(2t)}-1\right], \quad N_T(dx) = \frac{\pi}{8}\,\frac{1}{\cosh^2(\pi x/4)}\,dx \tag{5.3}$$

(see Jurek and Yor (2004)).

REMARK 5.1. (a) Note that ψ_T is the characteristic function of a compound Poisson distribution.

(b) Elementary calculations give $\psi_C/\psi_S = \psi_T$.

Let $\tilde{\psi_C}, \tilde{\psi_S}$ and $\tilde{\psi_T}$ be the free-probability analogues of BDCF for ψ_C, ψ_S, ψ_T, respectively. We have three possible ways of finding them: two because of inequalities (A) and (Z) from Theorem 2.1 and, if possible, the third one by the differential equation (3.4).

COROLLARY 5.1. *Let $\tilde{\psi_C}, \tilde{\psi_S}$ and $\tilde{\psi_T}$ be the free-probability analogues of the corresponding BDCF. Then their Voiculescu transforms are as follows:*

$$\begin{aligned} V_{\tilde{\psi_C}}(it) &= i\left[\frac{t^2}{2}\zeta(2,t/2) - \frac{t^2}{4}\zeta(2,t/4)+1\right], \\ V_{\tilde{\psi_C}}(i) &= -i(2C-1) \approx -i\,0.83 \end{aligned} \tag{5.4}$$

(C denotes the Catalan constant $C := \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} \approx 0.9159$);

$$\begin{aligned} V_{\tilde{\psi_S}}(it) &= i[1+t-\tfrac{1}{2}t^2\zeta(2,t/2)], \\ V_{\tilde{\psi_S}}(i) &= -i(\pi^2/4-2) \approx -i\,0.4674; \end{aligned} \tag{5.5}$$

$$\begin{aligned} V_{\tilde{\psi_T}}(it) &= it[t\zeta(2,t/2) - (t/4)\zeta(2,t/4)-1] \\ &= it[(t/4)\zeta(2,(t+2)/4)-1], \\ V_{\tilde{\psi_T}}(i) &= -i(2C+1-\pi^2/4) \approx -i\,0.3645. \end{aligned} \tag{5.6}$$

Proof. From Theorem 2.1, using *Mathematica* and (4.1)(v), (vi) we get

$$\begin{aligned} V_{\tilde{\psi_C}}(it) &= -it^2\int_0^\infty (v\tanh v)e^{-tv}\,dv \\ &= -it^2\left[\tfrac{1}{8}\zeta(2,t/4+1) - \tfrac{1}{8}\zeta(2,t/4+1/2)+1/t^2\right] \\ &= -it^2\left[\tfrac{1}{4}\zeta(2,t/4) - \tfrac{1}{2}\zeta(2,t/2)-1/t^2\right] \\ &= i[(t^2/2)\zeta(2,t/2) - (t^2/4)\zeta(2,t/4)+1]. \end{aligned}$$

(Note that (5.4), in a different way, was already computed in (3.6).)

From **3.551**(3):

$$\int_0^\infty x^{\mu-1}e^{-\beta x}\coth x\,dx = \Gamma(\mu)[2^{1-\mu}\zeta(\mu,\beta/2)-\beta^{-\mu}], \qquad \Re\mu>1,\ \Re\beta>0,$$

putting $\mu=2$ and $\beta=t$ we get

$$V_{\tilde{\psi_S}}(it) = it^2\int_0^\infty(1-v\coth v)e^{-tv}dv = it^2\Big[t^{-1}-\int_0^\infty v\coth v\,e^{-vt}\,dv\Big]$$
$$= it^2\big[t^{-1}-\tfrac{1}{2}\zeta(2,t/2)+1/t^2\big] = i\big[t-\tfrac{1}{2}t^2\zeta(2,t/2)+1\big] = i\big[1+t-\tfrac{1}{2}t^2\zeta(2,t/2)\big],$$

which gives (5.5).

By Remark 5.1(b), $\log\psi_T = \log\psi_C - \log\psi_S$. Thus using (5.4) and (5.5) we get

$$\begin{aligned}V_{\tilde{\psi_T}}(it) &= -it^2\int_0^\infty[\log\psi_C(v)-\log\psi_S(v)]ve^{-tv}\,dv\\ &= \big[(t^2/2)\zeta(2,t/2)-(t^2/4)\zeta(2,t/4)+1\big]-i\big[1+t-\tfrac{1}{2}t^2\zeta(2,t/2)\big]\\ &= i[t^2\zeta(2,t/2)-(t^2/4)\zeta(2,t/4)-t] = it[t\zeta(2,t/2)-(t/4)\zeta(2,t/4)-1]\\ &= it\left[\frac{t}{4}\zeta\left(2,\frac{t+2}{4}\right)-1\right],\end{aligned}$$

which gives (5.6). (An alternative proof is by using the differential equation from Theorem 3.1 and the formulas in Corollary 4.3.) ■

Comparing (5.4), (5.5) and (5.6) yields

COROLLARY 5.2. *We have*

$$\tilde{\psi_C}(t) = \tilde{\psi_S}(t)\tilde{\psi_T}(t);\ \ V_{\tilde{\psi_C}}(it) = V_{\tilde{\psi_S}}(it)+V_{\tilde{\psi_T}}(it);\ \ V_{\tilde{\psi_C}}(i) = V_{\tilde{\psi_S}}(i)+V_{\tilde{\psi_T}}(i).$$

6. FREE LAPLACE (OR FREE DOUBLE EXPONENTIAL) MEASURE

All the hyperbolic characteristic functions ϕ_C,ϕ_S,ϕ_T are infinite products of the Laplace (also called double exponential) distributions; see Jurek (1996). Therefore we include the latter in this paper as well.

6.1. Recall that *the double exponential (2e)* (or *Laplace*) *distribution* has probability density $f(x):=2^{-1}e^{-|x|}$ ($x\in\mathbb{R}$) and characteristic function

$$(6.1)\qquad \phi_{2e}(t) = \frac{1}{1+t^2} = \exp\int_{\mathbb{R}}(\cos tx-1)\frac{1+x^2}{x^2}\left[\frac{e^{-|x|}|x|}{1+x^2}\right]dx$$
$$= \exp\int_{\mathbb{R}\setminus\{0\}}(\cos tx-1)\left[\frac{e^{-|x|}}{|x|}\right]dx,$$

where the first square bracket $[\ldots]$ is the finite Khinchin spectral measure m_{2e}, and the second one is the Lévy spectral measure M_{2e}.

COROLLARY 6.1. *The free-probability analogue $\tilde{\phi_{2e}}$ of the double exponential distribution has the following Voiculescu transform:*

$$V_{\tilde{\phi_{2e}}}(it) = 2it[\mathrm{ci}(t)\cos t + \mathrm{si}(t)\sin t] \quad \left(= -2it\int_0^\infty \frac{\cos w}{w+t}\,dw\right), \qquad t>0.$$

Proof. First proof (via (A) of Theorem 2.1): For $\Re\beta, \Re\xi > 0$ we have

$$\int_0^\infty e^{-\xi x}\log(\beta^2+x^2)\,dx = \frac{2}{\xi}[\log\beta - \mathrm{ci}(\beta\xi)\cos(\beta\xi) - \mathrm{si}(\beta\xi)\sin(\beta\xi)]$$

by **4.388**(1). Therefore

$$\begin{aligned}
(6.2)\qquad V_{\tilde{\phi_{2e}}}(it) &= it^2\int_0^\infty \overline{\log\phi_{2e}(v)}\,e^{-tv}\,dv = -it^2\int_0^\infty \log(1+v^2)e^{-tv}\,dv\\
&= -it^2[2t^{-1}(-\mathrm{ci}(t)\cos t - \mathrm{si}(t)\sin t)]\\
&= -2it\left(\cos t\int_t^\infty \frac{\cos x}{x}dx + \sin t\int_t^\infty \frac{\sin x}{x}\,dx\right)\\
&= -2it\int_t^\infty \frac{\cos t\cos x + \sin t\sin x}{x}\,dx = -2it\int_t^\infty \frac{\cos(x-t)}{x}\,dx \qquad (w := x-t)\\
&= -2it\int_0^\infty \frac{\cos w}{w+t}\,dw, \qquad t>0.
\end{aligned}$$

Second proof (via (Z) of Theorem 2.1): By the second part of Theorem 2.1 and **3.354**(2) we get

$$\begin{aligned}
(6.3)\qquad V_{\tilde{\phi_{2e}}}(it) &= -it\int_{\mathbb{R}} \frac{1+x^2}{t^2+x^2}\left[\frac{|x|}{1+x^2}e^{-|x|}\right]dx = -2it\int_0^\infty \frac{x}{t^2+x^2}e^{-x}\,dx\\
&= 2it[\mathrm{ci}(t)\cos t + \mathrm{si}(t)\sin t] = -2it\int_0^\infty \frac{\cos w}{w+t}\,dw.
\end{aligned}$$

6.2. The Laplace characteristic function $\phi_{2e} = (1+t^2)^{-1}$ is selfdecomposable. Therefore it has the background driving characteristic function ψ_{2e} related to ϕ_{2e} via (3.4). Hence

$$\begin{aligned}
(6.4)\qquad \psi_{2e}(t) &= \exp\left[t\frac{\phi'_{2e}(t)}{\phi_{2e}(t)}\right] = \exp\left(-\frac{2t^2}{1+t^2}\right)\\
&= \exp 2\left(\frac{1}{1+t^2} - 1\right) \qquad \text{(compound Poisson)}\\
&= \exp 2\int_{\mathbb{R}}(e^{itx}-1)\frac{1}{2}e^{-|x|}dx = \exp\int_{\mathbb{R}}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)e^{-|x|}\,dx,
\end{aligned}$$

from which we infer that $m_{\psi_{2e}}(dx) := \frac{x^2}{1+x^2}e^{-|x|}dx$ is its finite Khinchin measure in (2.1). Here is the Voiculescu transform of the free analogue of ψ_{2e}.

COROLLARY 6.2. *The free-probability analogue of the BDCF ψ_{2e} has Voiculescu transform*

$$V_{\tilde{\psi_{2e}}}(it) = 2it[t(\mathrm{ci}(t)\sin t - \mathrm{si}(t)\cos t) - 1]. \tag{6.5}$$

Proof. First proof: From Corollary 6.1, $V_{\tilde{\phi_{2e}}}(it) = 2it\alpha(t)$ where $\alpha(t) := \mathrm{ci}(t)\cos t + \mathrm{si}(t)\sin t$. Then Theorem 3.1 gives

$$\begin{aligned} V_{\tilde{\psi_{2e}}}(it) &= V_{\tilde{\phi_{2e}}}(it) - t\frac{d}{dt}[V_{\tilde{\phi_{2e}}}(it)] \\ &= 2it\alpha(t) - t(2i\alpha(t) + 2it\alpha'(t)) = -2it^2\alpha'(t) \\ &= -2it^2[t^{-1} - (\mathrm{ci}(t)\sin t - \mathrm{si}(t)\cos t)] \\ &= 2it[t(\mathrm{ci}(t)\sin t - \mathrm{si}(t)\cos t) - 1]. \end{aligned}$$

Second proof: Here we use the equality

$$\int_0^\infty \frac{e^{-\xi x}}{\beta^2 + x^2}\,dx = \frac{1}{\beta}[\mathrm{ci}(\xi\beta)\sin(\xi\beta) - \mathrm{si}(\xi\beta)\cos(\xi\beta)], \qquad \Re\xi, \Re\beta > 0$$

(see **3.354**(1)). Hence and from (A) of Theorem 2.1 and (6.4),

$$\begin{aligned} V_{\tilde{\psi_{2e}}}(it) &= it^2\int_0^\infty (\log\psi_{2e}(s))e^{-ts}\,ds = 2it^2\int_0^\infty \left[\frac{1}{1+s^2} - 1\right]e^{-st}\,ds \\ &= 2it^2\big([\mathrm{ci}(t)\sin t - \mathrm{si}(t)\cos t] - t^{-1}\big) \\ &= 2it[t(\mathrm{ci}(t)\sin t - \mathrm{si}(t)\cos t) - 1]. \end{aligned}$$

Third proof: Now we use (Z) of Theorem 2.1 and the finite Khinchin measures $m_\psi(dx) = \frac{x^2}{1+x^2}e^{-|x|}dx$ from (6.4) to get

$$\begin{aligned} V_{\tilde{\psi_{2e}}}(it) &= -it\int_{\mathbb{R}} \frac{x^2}{t^2+x^2}e^{-|x|}\,dx = -it\int_{\mathbb{R}}\left(1 - \frac{t^2}{t^2+x^2}\right)e^{-|x|}\,dx \\ &= 2it\left(t^2\int_0^\infty \frac{1}{t^2+x^2}e^{-x}\,dx - 1\right) = 2it\left(t^2\left[\frac{1}{t}(\mathrm{ci}(t)\sin t - \mathrm{si}(t)\cos t)\right] - 1\right) \\ &= 2it[t(\mathrm{ci}(t)\sin t - \mathrm{si}(t)\cos t) - 1]. \end{aligned}$$

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REFERENCES

A. Araujo and E. Giné (1980), *The Central Limit Theorem for Real and Banach Valued Random Variables*, Wiley, New York.

O. E. Barndorff-Nielsen and S. Thorbjørnsen (2002), *Self-decomposability and Lévy processes in free probability*, Bernoulli 8, 323–366.

H. Bercovici and V. Pata (1999), *Stable laws and domains of attraction in free probability theory*, Ann. of Math. 149, 1023–1060.

H. Bercovici and D. V. Voiculescu (1993), *Free convolution of measures with unbounded support*, Indiana Univ. Math. J. 42, 733–773.

P. Billingsley (1986), *Probability and Measure*, 2nd ed., Wiley, New York.

I. S. Gradshteyn and I. M. Ryzhik (1994), *Table of Integrals, Series and Products*, 5th ed., Academic Press, New York.

T. Hasebe, N. Sakuma and S. Thorbjørnsen (2019), *The normal distribution is freely self-decomposable*, Int. Math. Res. Notices, 1758–1787.

J. Jacod (1985), *Grossissements de filtration et processus d'Ornstein–Uhlenbeck generalisé*, in: Grossissements de filtrations: exemples et applications, T. Jeulin and M. Yor (eds.), Lecture Notes in Math. 1118, Springer, 37–44.

L. Jankowski and Z. J. Jurek (2012), *Remarks on restricted Nevanlinna transforms*, Demonstratio Math. 45, 297–307.

Z. J. Jurek (1996), *Series of independent exponential random variables*, in: Probability Theory and Mathematical Statistics (Tokyo, 1995), World Sci., 174–182.

Z. J. Jurek (2006), *Cauchy transforms of measures as some functionals of Fourier transforms*, Probab. Math. Statist. 26, 187–200.

Z. J. Jurek (2007), *Random integral representations for free-infinitely divisible and tempered stable distributions*, Statist. Probab. Lett. 77, 417–425.

Z. J. Jurek (2007), *On a method of introducing free-infinitely divisible probability measures*, Demonstratio Math. 49, 236–251.

Z. J. Jurek and W. Vervaat (1983), *An integral representation for self-decomposable Banach space valued random variables*, Z. Wahrsch. Verw. Gebiete 62, 247–262.

Z. J. Jurek and M. Yor (2004), *Selfdecomposable laws associated with hyperbolic functions*, Probab. Math. Statist. 24, 181–191.

K. R. Parthasarathy (1967), *Probability Measures on Metric Spaces*, Academic Press, New York.

J. Pitman and M. Yor (2003), *Infinitely divisible laws associated with hyperbolic functions*, Canad. J. Math. 55, 292–330.

D. Voiculescu (1999), *Lectures on free probability*, in: Lectures on Probability Theory and Statistics (Saint-Flour, 1998), Springer, 279–349.

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