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WEYL MULTIFRACTIONAL ORNSTEIN–UHLENBECK PROCESSES MIXED WITH A GAMMA DISTRIBUTION

BY

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Abstract. The aim of this paper is to study the asymptotic behavior of aggregated Weyl multifractional Ornstein–Uhlenbeck processes mixed with Gamma random variables. This allows us to introduce a new class of processes, Gamma-mixed Weyl multifractional Ornstein–Uhlenbeck processes (GWmOU), and study their elementary properties such as Hausdorff dimension, local self-similarity and short-range dependence. We also prove that these processes approach the multifractional Brownian motion.

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1. INTRODUCTION

Fractional Ornstein–Uhlenbeck (fOU) processes are one of the most well studied and widely applied classes of stochastic processes [8]. Recently, in [10], an interesting class of processes, of interest for various applications, has been introduced employing sequences of fOU processes with random coefficients.

Let us first present a brief summary of their construction. Let $B^H = \{B^H(t), t \in \mathbb{R}\}$ be a fractional Brownian motion (fBm) with Hurst index $H > 1/2$, defined on a probability space $(\Omega_{B^H}, \mathcal{F}_{B^H}, \mathcal{P}_{B^H})$. Consider a sequence of stationary fOU processes X_k, $k \geqslant 1$, with random coefficients defined by the stochastic integral

$$X_t^k = \int_{-\infty}^{t} e^{\gamma_k (t-s)}\, dB_s^H, \qquad t \in \mathbb{R}, \tag{1.1}$$

with initial condition $X_0^k = \int_{-\infty}^0 e^{\gamma_k (t-s)}\, dB_s^H$. The random coefficients

$\gamma_k,\ k \geqslant 1$, are independent random variables on a probability space $(\Omega_\gamma, \mathcal{F}_\gamma, \mathcal{P}_\gamma)$ and for any $k \geqslant 1$, $-\gamma_k \sim \Gamma(1-h, \lambda)$ with $0 < h < 1 - H$ and $\lambda > 0$.

Assume that the family $\{\gamma_k,\ k \geqslant 1\}$ is independent of B^H. The processes $X_k, k \geqslant 1$, defined above are P_γ-almost surely fOU processes (see [8]). Let

$$Y_n(t) = \frac{1}{n} \sum_{k=1}^{n} X_k(t), \qquad t \in \mathbb{R}, \tag{1.2}$$

denote the so-called aggregated process. It has been proven that as $n \to \infty$, $(Y_n)_{n \geqslant 1}$ converges weakly and in $L^2(\Omega_{B^H})$ for fixed time, P_γ-almost surely to a stochastic process denoted by $Y^\lambda := \{Y^\lambda(t),\ t \in \mathbb{R}\}$, given by the stochastic integral

$$Y^\lambda(t) = \int_{-\infty}^{t} \left(\frac{\lambda}{\lambda + t - s} \right)^{1-h} dB_s^H, \qquad t \in \mathbb{R}. \tag{1.3}$$

The limiting process Y^λ is stationary, almost self-similar and exhibits long-range dependence (see [13] or [10]). The asymptotic behavior of Y^λ with respect to λ has also been studied, as λ varies between ∞ and 0. The process Y^λ ranges from a fBm with index H to a fBm with index $h + H$.

When B^H is a standard Brownian motion (i.e. $H = 1/2$), Gamma-mixed Ornstein–Uhlenbeck processes have been studied in [13].

Our goal is to construct a new kind of processes, called Gamma-mixed Weyl multifractional Ornstein–Uhlenbeck (GWmOU) processes, in analogy to the limiting procedure that leads to the process defined in (1.3). In our construction we replace the processes X_k, $1 \leqslant k \leqslant n$, in the aggregated process (1.2) by Weyl multifractional Ornstein–Uhlenbeck (WmOU) processes mixed with Gamma random variables defined by the Wiener integral

$$X^k_{\alpha(t)}(t) = \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^{t} (t-s)^{\alpha(t)-1} e^{\gamma_k (t-s)} \, dB_s, \qquad t \in \mathbb{R},$$

where $B = \{B(s),\ s \in \mathbb{R}\}$ is a Brownian motion on $(\Omega_B, \mathcal{F}_B, P_B)$, and $\gamma_k, k \geqslant 1$, are independent random variables on $(\Omega_\gamma, \mathcal{F}_\gamma, P_\gamma)$, also independent of B, and for any $k \geqslant 1$, $-\gamma_k \sim \Gamma(1-h, \lambda)$ with $0 < h < 1$ and $\lambda > 0$. Moreover, α is a Hölder continuous function with exponent $0 < \beta \leqslant 1$. The processes $X_k, k \geqslant 1$, are P_γ-almost surely WmOU processes (see Section 2).

We define a GWmOU, denoted Y^λ_α, by

$$Y^\lambda_{\alpha(t)}(t) = \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^{t} \left(\frac{\lambda}{\lambda + t - s} \right)^{1-h} (t-s)^{\alpha(t)-1} \, dB_s, \qquad t \in \mathbb{R}.$$

It is non-stationary, locally asymptotically self-similar and exhibits short-range dependence. We will also study the Hölder exponent and the box and Hausdorff dimension of the process Y^λ_α. In addition, we will investigate the asymptotic behavior of Y^λ_α with respect to λ; we will prove that Y^λ_α approaches the multifractional

Brownian motion (see [17]) as $\lambda \to \infty$, while its integrated renormalized process

$$\hat{Y}_\alpha^\lambda(t) = \lambda^{h-1} \int_0^t Y_\alpha^\lambda(s)\, ds, \quad t \geqslant 0,$$

(here we suppose that the function α is constant) converges to a fractional Brownian motion modulo a constant as $\lambda \to 0$.

The motivation of this work comes from two facts. On the one hand, Gamma-mixed processes are good models for various applications; for example, the limiting process Y^λ defined by (1.3) is a successful model of heart rate variability and could also be a good model of a lot of Gaussian stationary data with long-range dependence (see [10], [13] for more details). Moreover, the so-called Gamma-mixed Poisson processes (also named Pólya processes) have many practical applications, one of them being the study of reliability of engineering systems [9]. On the other hand, multifractional Ornstein–Uhlenbeck processes are omnipresent in physics. For further details and references, we refer the reader to [15]. Also, for more details about the construction and study of several classes of multifractional processes, see e.g. [3], [2], [5], [4], [17], [19]. The above motivate mixing multifractional Ornstein–Uhlenbeck processes (Weyl version) with Gamma random variables, in order to introduce GWmOU processes, as a counterpart of the limiting process Y^λ, a new candidate to model several short range, variable fractal dimension and non-stationary physical phenomena.

The paper is structured as follows. Section 2 presents a short summary of results on WmOU processes. In Section 3 we introduce GWmOU processes as limits of aggregated Weyl multifractional Ornstein–Uhlenbeck processes mixed with Gamma-distributed random variables. Finally, Section 4 contains some interesting properties of GWmOU processes including their asymptotic behavior.

2. PRELIMINARIES

WmOU processses have been introduced as a multifractional generalization of Weyl fractional Ornstein–Uhlenbeck processes (WfOU).

Let us begin with a brief review of WfOU processes (see [14]). First, we recall some elementary definitions of fractional calculus (see [16], [18]). The Weyl fractional derivative of order $\alpha > 0$, denoted by ${}_aD_t^\alpha$, for $a = -\infty$, can be defined by its inverse using the Weyl fractional integral,

$${}_aD_t^{-\alpha} f(t) = {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \quad t \geqslant a.$$

For $n-1 \leqslant \alpha < n$, ${}_aD_t^\alpha$ is defined as the ordinary derivative of order n of the Weyl fractional integral of order $n-\alpha$,

$${}_aD_t^\alpha = (d/dt)^n\, {}_aD_t^{\alpha-n}.$$

WfOUs are stochastic processes obtained as solutions of the fractional Langevin equation

$$({}_aD_t + w)^\alpha X(t) = W(t), \qquad \alpha > 0,\, w > 0,$$

where $W(t)$ is a Gaussian white noise. They are defined explicitly by the stochastic integral

$$X_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} e^{-w(t-s)}\, dB_s, \qquad t \in \mathbb{R},$$

where $B = \{B(s), s \in \mathbb{R}\}$ is the standard Brownian motion and $\alpha > 1/2$ to ensure that $X_\alpha(t)$ has finite variance.

Similarly to the generalization of fractional Brownian motion to multifractional Brownian motion (see [17]), an extension of WfOU processes is obtained by replacing the parameter α by a Hölder continuous function with exponent $0 < \beta \leqslant 1$, i.e. there exists a constant K such that

$$|\alpha(t) - \alpha(s)| \leqslant K|t-s|^\beta \qquad \forall s, t,$$

and $\alpha(t) > 1/2$ for all t.

Let us recall WmOU processes and their properties needed in what follows. For more details we refer the reader to [15].

A WmOU process is a Gaussian process defined by the Wiener integral

$$X_{\alpha(t)}(t) = \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^{t} (t-s)^{\alpha(t)-1} e^{-w(t-s)}\, dB_s, \qquad t \in \mathbb{R}.$$

We have

$$\begin{aligned}(2.1)\qquad & E_B[(X_{\alpha(t)}(t+s) - X_{\alpha(t)}(t))^2] \\ &\qquad = \frac{-|s|^{2\alpha(t)-1}}{\Gamma(2\alpha(t))\cos(\pi\alpha(t))} - 2|s|^2 w^{3-2\alpha(t)} S_{\alpha(t)}(w|s|),\end{aligned}$$

where $S_\vartheta(x)$ is a continuous function given explicitly by

$$\begin{aligned} S_\vartheta(x) = -\frac{\sqrt{\pi}}{8\Gamma(\vartheta)\cos(\pi\vartheta)} \Bigg[& \sum_{m=0}^{\infty} \frac{x^{2m}}{2^{2m}(m+1)!\Gamma(m+5/2-\vartheta)} \\ & - \left(\frac{x}{2}\right)^{2\vartheta-1} \sum_{m=0}^{\infty} \frac{x^{2m}}{2^{2m}(m+1)!\Gamma(m+3/2+\vartheta)} \Bigg] \end{aligned}$$

for every $x > 0$ and $1/2 < \vartheta < 3/2$. The relevant variance is equal to

$$E[X_{\alpha(t)}(t)^2] = \frac{(2w)^{1-2\alpha(t)}\Gamma(2\alpha(t)-1)}{\Gamma(\alpha(t))^2}.$$

On the other hand, for $s < t$ the covariance of the WmOU is given by

$$E[X_{\alpha(t)}(t)\, X_{\alpha(s)}(s)] = \frac{e^{-w(t-s)}(t-s)^{\alpha(t)+\alpha(s)-1}}{\Gamma(\alpha(t))}\psi\big(\alpha(s), \alpha(s)+\alpha(t); 2w(t-s)\big),$$

where $\psi(\alpha, \gamma; z)$ is the confluent hypergeometric function. The variance and the covariance functions are divergent when $w \to 0$. However, if we set $Z_{\alpha(t)}(t) = X_{\alpha(t)}(t) - X_{\alpha(t)}(0)$, it has been proven in [15] that for $\alpha(t) \in (1/2, 3/2)$ and by identifying $\alpha(t)$ with $H(t)+1/2$, when $w \to 0$ the process $Z_{\alpha(t)}(t)$ approaches (in the sense of finite-dimensional distributions) $B_{H(t)}(t)$, the multifractional Brownian motion (with moving average definition) defined in [17] by

$$B_{H(t)}(t) = \frac{1}{\Gamma(H(t)+1/2)} \times \Big(\int_{-\infty}^{0} [(t-s)^{H(t)-1/2} - (-s)^{H(t)-1/2}]\, dB_s + \int_{0}^{t} (t-s)^{H(t)-1/2}\, dB_s \Big).$$

For the basic properties of WmOU processes such as short-range dependence, local self-similarity and Hausdorff dimension, we refer the reader to [15].

Let us now recall a sufficient criterion for weak convergence, which will be needed in what follows. By Prokhorov's theorem, the convergence of finite-dimensional distributions and tightness yield weak convergence. For processes X, X_n, $n \geqslant 1$, with paths in $C([a, b], \mathbb{R})$, one has the following sufficient criterion (Billingsley [6, Theorem 12.3], or [7]).

THEOREM 2.1. *Suppose that the finite-dimensional distributions of the family* $(X_n)_{n\geqslant 1}$ *converge to those of* X. *If, in addition, there exist constants* $\zeta > 0$, $\theta > 1$ *and* $c_{\zeta,\theta}$, *depending only on* ζ *and* θ, *such that for all* $s, t \in [a, b]$ *with* $a,b \in \mathbb{R}$, $a < b$,

$$E[|X_n(t) - X_n(s)|^{\zeta}] \leqslant c_{\zeta,\theta}|t-s|^{\theta}$$

for all $n \geqslant 1$, *then the family* $(X_n)_{n\geqslant 1}$ *is tight and consequently*

$$X_n \to X \quad \text{weakly in } C[a, b] \text{ as } n \to \infty.$$

3. AGGREGATED WEYL MULTIFRACTIONAL ORNSTEIN–UHLENBECK PROCESSES MIXED WITH GAMMA DISTRIBUTION

Let us now consider a sequence of WmOU processes mixed with Gamma distribution random variables $X^k_{\alpha} := \{X^k_{\alpha(t)}(t), t \in \mathbb{R}\}$ defined by the following Wiener integral:

$$X^k_{\alpha(t)}(t) = \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^{t} (t-s)^{\alpha(t)-1} e^{\gamma_k(t-s)}\, dB_s, \quad t \in \mathbb{R}, \tag{3.1}$$

where $B = \{B(s), s \in \mathbb{R}\}$ is a Brownian motion defined on a probability space $(\Omega_B, \mathcal{F}_B, P_B)$ and for any $k \geqslant 1$, $-\gamma_k \sim \Gamma(1-h, \lambda)$ with $0 < h < 1$ and $\lambda > 0$ are independent random variables, also independent of B, defined on a probability space $(\Omega_\gamma, \mathcal{F}_\gamma, P_\gamma)$.

The processes X^k_α, $k \geqslant 1$, are P_γ-almost surely WmOU processes defined on $(\Omega_B, \mathcal{F}_B, P_B)$. We define their empirical mean by

$$Y^n_{\alpha(t)}(t) = \frac{1}{n} \sum_{k=1}^{n} X^k_{\alpha(t)}(t)$$

for every $t \in \mathbb{R}$ and $n \geqslant 1$.

Throughout the paper we assume that

$$1/2 < \alpha_{\text{inf}} \leqslant \alpha_{\text{sup}} < 3/2, \tag{3.2}$$

where $\alpha_{\text{inf}} := \inf_{t \in \mathbb{R}} \alpha(t)$ and $\alpha_{\text{sup}} := \sup_{t \in \mathbb{R}} \alpha(t)$.

We will also need the following notations:

- $m_\alpha[a, b] = \min\{\alpha(t) : t \in [a, b]\}$ and $M_\alpha[a, b] = \max\{\alpha(t) : t \in [a, b]\}$ for all real $a < b$. E_B and E_γ denote the expectations with respect to P_B and P_γ respectively.
- C denotes a generic constant depending only on $[a, b]$, λ and h.
- $C^{x,y}$ denotes a generic constant depending on $[a, b]$, λ, h, x and y such that $0 < x < 2m_\alpha[a, b] - 1$ and $0 < y < 3/2 - h - M_\alpha[a, b]$.
- $C^{x,y}_\eta$ denotes a generic constant depending on $[a, b]$, λ, h, x and y such that $0 < x < 2m_\alpha[a, b] - 1$, $0 < y < 3/2 - h - M_\alpha[a, b]$ and $0 \leqslant \eta < m_\alpha[a, b] - 1/2$.

3.1. The limit of aggregated processes. If $0 < h < 3/2 - \alpha_{\text{sup}}$, we define a zero mean Gaussian process $Y^\lambda_\alpha := \{Y^\lambda_{\alpha(t)}(t), t \in \mathbb{R}\}$ by

$$Y^\lambda_{\alpha(t)}(t) = \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^{t} \left(\frac{\lambda}{\lambda + t - s} \right)^{1-h} (t-s)^{\alpha(t)-1}\, dB_s, \quad t \in \mathbb{R}. \tag{3.3}$$

It is easy to see that the Wiener integral in (3.3) is well-defined. The process Y^λ_α will be called a *Gamma-mixed Weyl multifractional Ornstein–Uhlenbeck process*, abbreviated as GWmOU.

Given a compact interval $[a, b] \subset \mathbb{R}$, the following result proves that P_γ-a.s., as $n \to \infty$, $Y^n_{\alpha(t)}(t)$ converges to $Y^\lambda_{\alpha(t)}(t)$ in $L^2(\Omega_B)$, uniformly in $t \in [a, b]$.

THEOREM 3.1. *Fix real numbers a, b such that $a < b$. If $0 < h < 3/2 - M_\alpha[a, b]$, then P_γ-a.s.,*

$$Y^n_{\alpha(t)}(t) \xrightarrow[n \to \infty]{} Y^\lambda_{\alpha(t)}(t) \quad \text{in } L^2(\Omega_B) \tag{3.4}$$

uniformly in $t \in [a,b]$*. In particular, if* $0 < h < 3/2 - \alpha_{\sup}$*, then* P_γ*-a.s., for every* $t \in \mathbb{R}$*,*

$$Y^n_{\alpha(t)}(t) \xrightarrow[n\to\infty]{} Y^\lambda_{\alpha(t)}(t) \quad \textit{in } L^2(\Omega_B). \tag{3.5}$$

Proof. We prove (3.4). For every $x > 0$, $n \geqslant 1$, set

$$f_n(x) := \frac{1}{n}\sum_{k=1}^{n} e^{\gamma_k x}, \qquad c(x) := E_\gamma[e^{\gamma_1 x}] = \left(\frac{\lambda}{\lambda + x}\right)^{1-h}.$$

By the law of large numbers, we have P_γ-a.s., for every $x > 0$,

$$f_n(x) = \frac{1}{n}\sum_{k=1}^{n} e^{\gamma_k x} \xrightarrow[n\to\infty]{} c(x), \tag{3.6}$$

and for every $c > 0$ and $d < 3/2 - h$,

$$\frac{1}{n}\sum_{k=1}^{n} \frac{e^{\gamma_k c}}{(-\gamma_k)^{d-1/2}} \xrightarrow[n\to\infty]{} E_\gamma\left[\frac{e^{\gamma_1 c}}{(-\gamma_1)^{d-1/2}}\right] = \frac{\lambda^{1-h}\Gamma(3/2-d-h)}{\Gamma(1-h)(\lambda+c)^{3/2-d-h}}. \tag{3.7}$$

Using the change of variable $u = t - s$, we can write

$$\begin{aligned}
&E_B[(Y^n_{\alpha(t)}(t) - Y^\lambda_{\alpha(t)}(t))^2] \\
&\quad = \frac{1}{\Gamma(\alpha(t))^2} E_B\Big[\Big(\int_{-\infty}^{t} (t-s)^{\alpha(t)-1}\big(f_n(t-s) - c(t-s)\big)\, dB_s\Big)^2\Big] \\
&\quad = \frac{1}{\Gamma(\alpha(t))^2} \int_{-\infty}^{t} (t-s)^{2\alpha(t)-2}\big(f_n(t-s) - c(t-s)\big)^2\, ds \\
&\quad = \frac{1}{\Gamma(\alpha(t))^2} \int_{0}^{\infty} u^{2\alpha(t)-2}(f_n(u) - c(u))^2\, du.
\end{aligned}$$

Hence, for every $m \geqslant 2$ and $t \in [a,b]$,

$$\begin{aligned}
&E_B[(Y^n_{\alpha(t)}(t) - Y^\lambda_{\alpha(t)}(t))^2] \\
&= \frac{1}{\Gamma(\alpha(t))^2}\Big[\int_0^1 u^{2\alpha(t)-2}(f_n(u)-c(u))^2 du + \int_1^m u^{2\alpha(t)-2}(f_n(u)-c(u))^2\, du \\
&\qquad + \int_m^\infty u^{2\alpha(t)-2}(f_n(u)-c(u))^2\, du\Big] \\
&\leqslant K\Big[\int_0^1 u^{2m_\alpha[a,b]-2}(f_n(u)-c(u))^2 du + \int_1^m u^{2M_\alpha[a,b]-2}(f_n(u)-c(u))^2\, du \\
&\qquad + \int_m^\infty u^{2M_\alpha[a,b]-2}(f_n(u)-c(u))^2\, du\Big] \\
&:= K[A(n,m) + B(n,m) + C(n,m)],
\end{aligned} \tag{3.8}$$

where K is the maximum of the continuous function $z \mapsto 1/\Gamma(z)$ on the interval $[m_\alpha[a,b], M_\alpha[a,b]]$.

Combining (3.6), $f_n(u) \leqslant 1, c(u) \leqslant 1$ and (3.2) with Lebesgue's dominated convergence theorem, we conclude that P_γ-a.s, for every $m \geqslant 2$,

$$A(n,m) \xrightarrow[n\to\infty]{} 0, \quad B(n,m) \xrightarrow[n\to\infty]{} 0. \tag{3.9}$$

Now we will estimate $C(n,m)$ for all $m \geqslant 2$. We have

$$\begin{aligned} C(n,m) &= \int_m^\infty (f_n(u) - c(u))^2 u^{2M_\alpha[a,b]-2}\, du \\ &\leqslant 2\int_m^\infty f_n(u)^2 u^{2M_\alpha[a,b]-2}\, du + 2\int_m^\infty c(u)^2 u^{2M_\alpha[a,b]-2}\, du. \end{aligned}$$

Moreover, by the change of variable $v = (-\gamma_j - \gamma_k)u$ and $2\sqrt{(-\gamma_j)(-\gamma_k)} \leqslant -\gamma_j - \gamma_k$,

$$\begin{aligned} &\int_m^\infty f_n(u)^2 u^{2M_\alpha[a,b]-2}\, du = \frac{1}{n^2}\sum_{k,j=1}^n \int_m^\infty e^{\gamma_j u} e^{\gamma_k u} u^{2M_\alpha[a,b]-2}\, du \\ &\quad = \frac{1}{n^2}\sum_{k,j=1}^n \frac{1}{(-\gamma_j-\gamma_k)^{2M_\alpha[a,b]-1}} \int_{m(-\gamma_j-\gamma_k)}^\infty v^{2M_\alpha[a,b]-2} e^{-v}\, dv \\ &\quad \leqslant \frac{2^{1-2M_\alpha[a,b]}}{n^2}\sum_{k,j=1}^n \frac{e^{-\frac{m}{2}(-\gamma_j-\gamma_k)}}{[(-\gamma_j)(-\gamma_k)]^{M_\alpha[a,b]-1/2}} \int_{m(-\gamma_j-\gamma_k)}^\infty v^{2M_\alpha[a,b]-2} e^{-v/2}\, dv \\ &\quad \leqslant \Gamma(2M_\alpha[a,b]-1)\left(\frac{1}{n}\sum_{j=1}^n \frac{e^{-\frac{m}{2}(-\gamma_j)}}{(-\gamma_j)^{M_\alpha[a,b]-1/2}}\right)^2. \end{aligned}$$

Combining this with (3.7) we get, P_γ-a.s.,

$$\begin{aligned} &\limsup_{n\to\infty} \int_m^\infty f_n(u)^2 u^{2M_\alpha[a,b]-2}\, du \\ &\qquad \leqslant \Gamma(2M_\alpha[a,b]-1)\left(\frac{\lambda^{1-h}\Gamma(3/2 - M_\alpha[a,b] - h)}{\Gamma(1-h)(\lambda + m/2)^{3/2 - M_\alpha[a,b]-h}}\right)^2 \xrightarrow[m\to\infty]{} 0. \end{aligned}$$

On the other hand, since

$$\begin{aligned} &\int_0^\infty \left(\frac{\lambda}{\lambda+u}\right)^{2-2h} u^{2M_\alpha[a,b]-2}\, du \\ &\qquad = \lambda^{2M_\alpha[a,b]-1}\beta(3 - 2M_\alpha[a,b] - 2h, 2M_\alpha[a,b]-1) < \infty, \end{aligned}$$

we have

$$\int_m^\infty c(u)^2 u^{2M_\alpha[a,b]-2}\,du = \int_m^\infty \left(\frac{\lambda}{\lambda+u}\right)^{2-2h} u^{2M_\alpha[a,b]-2}\,du \xrightarrow[m\to\infty]{} 0.$$

which implies that P_γ-a.s.,

$$\limsup_{n\to\infty} C(n,m) \xrightarrow[m\to\infty]{} 0. \tag{3.10}$$

Therefore, by applying the convergences (3.9) and (3.10) in (3.8) we deduce that P_γ-a.s.,

$$\limsup_{n\to\infty} \sup_{t\in[a,b]} E_B[(Y^n_{\alpha(t)}(t) - Y^\lambda_{\alpha(t)}(t))^2] = 0,$$

which finishes the proof of (3.4).

Finally, the convergence (3.5) is a direct consequence of (3.4), (3.2) and $0 < h < 3/2 - \alpha_{\sup}$. ■

The weak convergence of the sequence $(Y^n_\alpha)_{n\geqslant 1}$ is established in our next theorem.

THEOREM 3.2. *Fix real $a < b$. Suppose that $0 < h < 3/2 - M_\alpha[a,b]$ and $\min\{2m_\alpha[a,b]-1, 2\beta\} < 1$. Then P_γ-a.s.,*

$$Y^n_\alpha \xrightarrow[n\to\infty]{} Y^\lambda_\alpha \quad \textit{in } C[a,b], \tag{3.11}$$

where $C[a,b]$ is the space of continuous functions on $[a,b]$.

Proof. First, since P_γ-almost surely, Y^n_α and Y^λ_α are zero mean Gaussian processes whose finite-dimensional distributions are determined by their covariances, (3.4) implies the convergence P_γ-almost surely of the finite-dimensional distributions of $(Y^n_\alpha)_{n\geqslant 1}$ to those of Y^λ_α. Thus, in order to prove (3.11) it remains to prove the P_γ-a.s. tightness of $(Y^n_\alpha)_{n\geqslant 1}$ by using Theorem 2.1.

Throughout the proof all the results are given P_γ-almost surely.

Let $t,\ t+\tau \in [a,b]$ be such that $|\tau| < \min(\lambda/2, 1)$. Then

$$\begin{aligned} &E_B[(Y^n_{\alpha(t+\tau)}(t+\tau) - Y^n_{\alpha(t)}(t))^2] \\ &\quad = E_B\left[\left(\frac{1}{n}\sum_{k=1}^n (X^k_{\alpha(t+\tau)}(t+\tau) - X^k_{\alpha(t)}(t))\right)^2\right] \\ &\quad \leqslant 2E_B\left[\left(\frac{1}{n}\sum_{k=1}^n U^k_t(\tau)\right)^2\right] + 2E_B\left[\left(\frac{1}{n}\sum_{k=1}^n V^k_t(\tau)\right)^2\right], \end{aligned} \tag{3.12}$$

where

$$U^k_t(\tau) := X^k_{\alpha(t)}(t+\tau) - X^k_{\alpha(t)}(t), \qquad V^k_t(\tau) := X^k_{\alpha(t+\tau)}(t+\tau) - X^k_{\alpha(t)}(t+\tau).$$

We will first prove that for every $n \geqslant 1$,

$$E_B\left[\left(\frac{1}{n}\sum_{k=1}^{n} U_t^k(\tau)\right)^2\right] \leqslant C|\tau|^{2m_\alpha[a,b]-1}. \tag{3.13}$$

To this end, by using Hölder's inequality and (2.1), we can write

$$\begin{aligned}E_B\left[\left(\frac{1}{n}\sum_{k=1}^{n} U_t^k(\tau)\right)^2\right] &\leqslant \frac{1}{n}\sum_{k=1}^{n} E_B[U_t^k(\tau)^2]\\ &= \frac{-|\tau|^{2\alpha(t)-1}}{\Gamma(2\alpha(t))\cos(\pi\alpha(t))} - 2|\tau|^2\frac{1}{n}\sum_{k=1}^{n}(-\gamma_k)^{3-2\alpha(t)}S_{\alpha(t)}(-\gamma_k|\tau|).\end{aligned}$$

Since $1/2 < \alpha(t) < 3/2$ and $\cos(\pi\alpha(t)) < 0$, we get

$$\begin{aligned}&-S_{\alpha(t)}(-\gamma_k|\tau|)\\ &= \frac{\sqrt{\pi}}{8\Gamma(\alpha(t))\cos(\pi\alpha(t))}\sum_{m=0}^{\infty}\frac{(-\gamma_k|\tau|)^{2m}}{2^{2m}(m+1)!\Gamma(m+5/2-\alpha(t))}\\ &\quad - \frac{\sqrt{\pi}}{8\Gamma(\alpha(t))\cos(\pi\alpha(t))}\left(\frac{-\gamma_k|\tau|}{2}\right)^{2\alpha(t)-1}\sum_{m=0}^{\infty}\frac{(-\gamma_k|\tau|)^{2m}}{2^{2m}(m+1)!\Gamma(m+3/2+\alpha(t))}\\ &\leqslant -\frac{\sqrt{\pi}}{8\Gamma(\alpha(t))\cos(\pi\alpha(t))}\left(\frac{-\gamma_k|\tau|}{2}\right)^{2\alpha(t)-1}\sum_{m=0}^{\infty}\frac{(-\gamma_k|\tau|)^{2m}}{2^{2m}(m+1)!\Gamma(m+3/2+\alpha(t))}\\ &\leqslant C(-\gamma_k|\tau|)^{2\alpha(t)-1}\sum_{m=0}^{\infty}\frac{(-\gamma_k|\tau|)^{2m}}{2^{2m}((m+1)!)^2},\end{aligned}$$

where the last inequality comes from $\Gamma(m+3/2+\alpha(t)) \geqslant (m+1)!$ and the fact that the functions $\Gamma(x)$ and $\cos(\pi x)$ are continuous at every x with $1/2 < m_\alpha[a,b] \leqslant x \leqslant M_\alpha[a,b] < 3/2$.

As a consequence,

$$\begin{aligned}E_B&\left[\left(\frac{1}{n}\sum_{k=1}^{n} U_t^k(\tau)\right)^2\right]\\ &\leqslant C|\tau|^{2\alpha(t)-1}\left(1+\frac{1}{n}\sum_{k=1}^{n}(-\gamma_k)^2\sum_{m=0}^{\infty}\frac{(-\gamma_k|\tau|)^{2m}}{2^{2m}((m+1)!)^2}\right).\end{aligned} \tag{3.14}$$

Moreover, by the law of large numbers, we obtain

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(-\gamma_k)^2\sum_{m=0}^{\infty}\frac{(-\gamma_k|\tau|)^{2m}}{2^{2m}((m+1)!)^2}\\ &= E_\gamma\left[(-\gamma_1)^2\sum_{m=0}^{\infty}\frac{(-\gamma_1|\tau|)^{2m}}{2^{2m}((m+1)!)^2}\right] = \frac{1}{\Gamma(1-h)\lambda^2}\sum_{m=0}^{\infty}\frac{\Gamma(2m+3-h)}{2^{2m}((m+1)!)^2}\left(\frac{|\tau|}{\lambda}\right)^{2m}\\ &\leqslant C\sum_{m=0}^{\infty}\frac{\Gamma(2m+3-h)}{2^{2m}((m+1)!)^2}\left(\frac{1}{2}\right)^{2m} < \infty,\end{aligned} \tag{3.15}$$

where we have used the fact that the radius of convergence of the power series $\sum_{m=0}^{\infty} \frac{\Gamma(2m+3-h)}{2^{2m}((m+1)!)^2} x^m$ is 1. By combining (3.14) and (3.15), we obtain (3.13).

Let us now turn to the second term in (3.12). It remains to prove that for every $n \geqslant 1$,

$$E_B\left[\left(\frac{1}{n}\sum_{k=1}^{n} V_t^k(\tau)\right)^2\right] \leqslant C^{\delta,\rho}|\tau|^{2\beta}. \tag{3.16}$$

To this end, from (3.1) we can write

$$V_t^k(\tau) = V_{t,1}^k(\tau) + V_{t,2}^k(\tau), \tag{3.17}$$

where

$$V_{t,1}^k(\tau) = \left(\frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))}\right)\int_{-\infty}^{t+\tau} (t+\tau-u)^{\alpha(t+\tau)-1} e^{\gamma_k(t+\tau-u)}\, dB_u,$$

$$V_{t,2}^k(\tau) = \frac{1}{\Gamma(\alpha(t))}\int_{-\infty}^{t+\tau} \big((t+\tau-u)^{\alpha(t+\tau)-1} - (t+\tau-u)^{\alpha(t)-1}\big) e^{\gamma_k(t+\tau-u)}\, dB_u.$$

Then

$$E_B\left[\left(\frac{1}{n}\sum_{k=1}^{n} V_t^k(\tau)\right)^2\right] \leqslant 2E_B\left[\left(\frac{1}{n}\sum_{k=1}^{n} V_{t,1}^k(\tau)\right)^2\right] + 2E_B\left[\left(\frac{1}{n}\sum_{k=1}^{n} V_{t,2}^k(\tau)\right)^2\right].$$

Combining the mean value theorem and the fact that any continuous function has a maximum on any compact interval, we get

$$\left|\frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))}\right|^2 \Gamma(2\alpha(t+\tau)-1) \leqslant C|\alpha(t+\tau)-\alpha(t)|^2.$$

Moreover, since α is β-Hölder continuous, and since $2\sqrt{(-\gamma_j)(-\gamma_k)} \leqslant -\gamma_j - \gamma_k$ and $1-2\alpha(t+\tau) < 0$, we have

$$\begin{aligned}
&E_B\left[\left(\frac{1}{n}\sum_{k=1}^{n} V_{t,1}^k(\tau)\right)^2\right] \\
&= \left(\frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))}\right)^2 \frac{1}{n^2}\sum_{j,k=1}^{n}\int_{-\infty}^{t+\tau} (t+\tau-u)^{2\alpha(t+\tau)-2} e^{(\gamma_j+\gamma_k)(t+\tau-u)}\, du \\
&= \left(\frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))}\right)^2 \Gamma(2\alpha(t+\tau)-1)\frac{1}{n^2}\sum_{j,k=1}^{n} (-\gamma_j-\gamma_k)^{1-2\alpha(t+\tau)} \\
&\leqslant C|\tau|^{2\beta}\left[\frac{1}{n}\sum_{k=1}^{n} (-\gamma_k)^{1/2-\alpha(t+\tau)}\right]^2.
\end{aligned}$$

Moreover,

$$\frac{1}{n}\sum_{k=1}^{n}(-\gamma_k)^{1/2-\alpha(t+\tau)} \xrightarrow[n\to\infty]{} \frac{\lambda^{\alpha(t+\tau)-1/2}}{\Gamma(1-h)}\Gamma(3/2-\alpha(t+\tau)-h)<\infty.$$

Thus, we conclude that for every $n \geqslant 1$,

$$E_B\left[\left(\frac{1}{n}\sum_{k=1}^{n} V_{t,1}^k(\tau)\right)^2\right] \leqslant C|\tau|^{2\beta}. \tag{3.18}$$

On the other hand, by the change of variable $t+\tau-u=x$, we have

$$\begin{aligned}
&E_B\left[\left(\frac{1}{n}\sum_{k=1}^{n} V_{t,2}^k(\tau)\right)^2\right]\\
&\quad= \frac{1}{\Gamma(\alpha(t))^2}\frac{1}{n^2}\\
&\quad\quad\times\sum_{j,k=1}^{n}\int_{-\infty}^{t+\tau}[(t+\tau-u)^{\alpha(t+\tau)-1}-(t+\tau-u)^{\alpha(t)-1}]^2 e^{(\gamma_j+\gamma_k)(t+\tau-u)}\,du\\
&\quad= \frac{1}{\Gamma(\alpha(t))^2}\frac{1}{n^2}\sum_{j,k=1}^{n}\int_0^{\infty}[x^{\alpha(t+\tau)-1}-x^{\alpha(t)-1}]^2 e^{(\gamma_j+\gamma_k)x}\,dx\\
&\quad= \frac{[\alpha(t+\tau)-\alpha(t)]^2}{\Gamma(\alpha(t))^2}\frac{1}{n^2}\sum_{j,k=1}^{n}\int_0^{\infty}[\log(x)x^{c_{t,\tau}^x-1}]^2 e^{(\gamma_j+\gamma_k)x}\,dx
\end{aligned}$$

for some $c_{t,\tau}^x \in (m_\alpha[a,b], M_\alpha[a,b])$, where the last equality comes from the mean value theorem.

Let $0<\delta<2m_\alpha[a,b]-1$, $0<\rho<3/2-M_\alpha[a,b]-h$ and define $\mu = 1/(2m_\alpha[a,b]-1-\delta)$. Since α is β-Hölder continuous, we can write

$$\begin{aligned}
&E_B\left[\left(\frac{1}{n}\sum_{k=1}^{n} V_{t,2}^k(\tau)\right)^2\right]\\
&= \frac{[\alpha(t+\tau)-\alpha(t)]^2}{\Gamma(\alpha(t))^2}\frac{1}{n^2}\sum_{j,k=1}^{n}\int_0^{\infty}\log(x)^2 x^{2c_{t,\tau}^x-2}e^{(\gamma_j+\gamma_k)x}\,dx\\
&= \frac{[\alpha(t+\tau)-\alpha(t)]^2}{\Gamma(\alpha(t))^2}\frac{1}{n^2}\sum_{j,k=1}^{n}\left(\int_0^{1}\log(x)^2 x^{2c_{t,\tau}^x-2}e^{(\gamma_j+\gamma_k)x}\,dx\right.\\
&\qquad\qquad\left.+\int_1^{\infty}\log(x)^2 x^{2c_{t,\tau}^x-2}e^{(\gamma_j+\gamma_k)x}\,dx\right)\\
&\leqslant C|\tau|^{2\beta}\left(\int_0^1 x^{2m_\alpha[a,b]-2-\delta}\,dx+\frac{1}{n^2}\sum_{j,k=1}^{n}\int_1^{\infty}x^{2M_\alpha[a,b]-2+2\rho}e^{(\gamma_j+\gamma_k)x}\,dx\right)\\
&\leqslant C|\tau|^{2\beta}\left(\mu+\frac{1}{n^2}\sum_{j,k=1}^{n}\int_0^{\infty}x^{2M_\alpha[a,b]-2+2\rho}e^{(\gamma_j+\gamma_k)x}\,dx\right)
\end{aligned}$$

$$\leqslant C|\tau|^{2\beta}\bigg(\mu+\frac{1}{n^2}\sum_{j,k=1}^{n}\frac{\Gamma(2M_\alpha[a,b]-1+2\rho)}{(-\gamma_j-\gamma_k)^{2M_\alpha[a,b]-1+2\rho}}\bigg)$$
$$\leqslant C|\tau|^{2\beta}\bigg(\mu+\frac{2^{1-2\rho-2M_\alpha[a,b]}}{n^2}\sum_{j,k=1}^{n}\frac{\Gamma(2M_\alpha[a,b]-1+2\rho)}{(\sqrt{(-\gamma_j)(-\gamma_k)})^{2M_\alpha[a,b]-1+2\rho}}\bigg)$$
$$= C|\tau|^{2\beta}$$
$$\times\bigg(\mu+2^{1-2\rho-2M_\alpha[a,b]}\Gamma(2M_\alpha[a,b]-1+2\rho)\bigg(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{(-\gamma_k)^{M_\alpha[a,b]-1/2+\rho}}\bigg)^2\bigg).$$

Combining this with

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{(-\gamma_k)^{M_\alpha[a,b]-1/2+\rho}}\xrightarrow[n\to\infty]{}\frac{\lambda^{M_\alpha[a,b]-1/2+\rho}}{\Gamma(1-h)}\Gamma(3/2-M_\alpha[a,b]-h-\rho)<\infty,$$

we deduce that for every $n \geqslant 1$,

$$E_B\bigg[\bigg(\frac{1}{n}\sum_{k=1}^{n}V_{t,2}^{k}(\tau)\bigg)^2\bigg]\leqslant C^{\delta,\rho}|\tau|^{2\beta}. \tag{3.19}$$

Thus, combining (3.18) and (3.19), we get (3.16).

Therefore, from (3.12), (3.13) and (3.16) we obtain, for every $n \geqslant 1$,

$$E_B[(Y^n_{\alpha(t+\tau)}(t+\tau)-Y^n_{\alpha(t)}(t))^2]\leqslant C^{\delta,\rho}|\tau|^{\min\{2m_\alpha[a,b]-1,2\beta\}}. \tag{3.20}$$

Let $a < b$. For $s < t \in [a,b]$, we can find $2k+2$ points $u_1,\dots,u_{2k+2}\in[s,t]$ with $b-a = k\min\{\lambda/2,1\}+c$, $0\leqslant c<\min\{\lambda/2,1\}$ and $0 < u_{i+1}-u_i < \min\{\lambda/2,1\}$ such that $[t,s]=\bigcup_{i=1}^{2k+2}[u_i,u_{i+1}]$.

Using Minkowski's inequality, (3.20) and Proposition 4.1 (because $0 < \min\{2m_\alpha[a,b]-1,2\beta\} < 1$) we conclude that for every $n \geqslant 1$ and $s,t\in[a,b]$,

$$E_B[(Y^n_{\alpha(t)}(t)-Y^n_{\alpha(s)}(s))^2]\leqslant C^{\delta,\rho}|t-s|^{\min\{2m_\alpha[a,b]-1,2\beta\}}.$$

Consequently, given $r > 0$ and using again the fact that Y^n_α is P_γ-almost surely Gaussian, there exists a constant C_r depending only on r such that

$$\begin{aligned}E_B[|Y^n_{\alpha(t)}(t)-Y^n_{\alpha(s)}(s)|^r] &= C_r\big(E_B[(Y^n_{\alpha(t)}(t)-Y^n_{\alpha(s)}(s))^2]\big)^{r/2}\\ &\leqslant C_r(C^{\delta,\rho})^{r/2}|t-s|^{r\min\{m_\alpha[a,b]-1/2,\beta\}}\end{aligned}$$

for all $n \geqslant 1$ and $s,t\in[a,b]$. If we choose so that $r\min\{m_\alpha[a,b]-1/2,\beta\}>1$, Theorem 2.1, implies that the family $(Y^n_\alpha)_{n\geqslant 1}$ is tight, as desired. ■

3.2. Properties of GWmOU processes and asymptotic behavior with respect to λ. In this section we study several interesting properties of the GWmOU process Y_α^λ, such as the Hölder exponent and short-range dependence. In addition, we investigate the asymptotic behavior of Y_α^λ when $\lambda \to \infty$ and when $\lambda \to 0$.

Let us first compute the variance and the covariance of Y_α^λ. An easy computation shows that for all $t \in \mathbb{R}$ the variance is given by

$$E_B[Y_{\alpha(t)}^\lambda(t)^2] = \frac{\lambda^{2-2h}}{\Gamma(\alpha(t))^2} \int_{-\infty}^{t} (\lambda + t - s)^{2h-2}(t-s)^{2\alpha(t)-2}\,ds = \frac{\lambda^{2\alpha(t)-1}}{\Gamma(\alpha(t))^2}\,\beta(3-2h-2\alpha(t), 2\alpha(t)-1), \tag{3.21}$$

where β is the beta function defined by $\beta(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}\,du$ for $x, y > 0$. Hence a GWmOU process is in general not stationary.

In addition, for $s < t$, using the change of variable $z = \lambda/(\lambda + s - u)$, the covariance of Y_α^λ is given by

$$\begin{aligned} &E_B[Y_{\alpha(t)}^\lambda(t) Y_{\alpha(s)}^\lambda(s)] \\ &= \frac{1}{\Gamma(\alpha(t))\Gamma(\alpha(s))} \\ &\quad\times \int_{-\infty}^{s} \left(\frac{\lambda}{\lambda + t - u}\right)^{1-h} \left(\frac{\lambda}{\lambda + s - u}\right)^{1-h} (t-u)^{\alpha(t)-1}(s-u)^{\alpha(s)-1}\,du \\ &= \frac{\lambda^{\alpha(t)+\alpha(s)-1}}{\Gamma(\alpha(t))\Gamma(\alpha(s))} G(\alpha(t), \alpha(s), h, t-s/\lambda), \end{aligned} \tag{3.22}$$

with

$$G(a,b,c,d) = \int_0^1 \frac{(1+dz)^{c-1}}{(1+[d-1]z)^{1-a}}(1-z)^{b-1} z^{2-[a+b]-2c}\,dz.$$

In order to study the local properties of GWmOU processes we will need the following result.

PROPOSITION 3.1. *Fix a compact interval $[a,b] \subset \mathbb{R}$.*

(1) *If $0 < h < 3/2 - M_\alpha[a,b]$, then there exists a constant $C^{\delta,\rho}$ such that*

$$E_B[(Y_{\alpha(t+\tau)}^\lambda(t+\tau) - Y_{\alpha(t)}^\lambda(t))^2] \leqslant C^{\delta,\rho}|\tau|^{\min\{2m_\alpha[a,b]-1, 2\beta\}} \tag{3.23}$$

for all $t, t+\tau \in [a,b]$ with $|\tau| < \min\{\lambda/2, 1\}$.

(2) *If $0 < h < 3/2 - M_\alpha[a,b]$, $M_\alpha[a,b] < 1$ and $\alpha(t) - 1/2 < \beta$ for all t, then*

(a) *there exist constants C_2 and $\epsilon < 1$ such that*

$$E_B[(Y_{\alpha(t+\tau)}^\lambda(t+\tau) - Y_{\alpha(t)}^\lambda(t))^2] \geqslant (C_2/2)|\tau|^{2M_\alpha[a,b]-1} \tag{3.24}$$

for all $t, t+\tau \in [a,b]$ with $|\tau| < \epsilon$,

(b) *as* $\tau \to 0$,

$$E_B[(Y^\lambda_{\alpha(t+\tau)}(t+\tau) - Y^\lambda_{\alpha(t)}(t))^2] = C_2|\tau|^{\alpha(t)-1/2} + O(|\tau|^{2\alpha(t)-1}). \tag{3.25}$$

Proof. The inequality (3.23) is a direct consequence of (3.20) and (3.4).

Let us now prove (3.24). For convenience, for all $t, t+\tau \in [a,b]$ with $|\tau| < 1$, we set

$$U^\lambda_t(\tau) = Y^\lambda_{\alpha(t)}(t+\tau) - Y^\lambda_{\alpha(t)}(t), \quad V^\lambda_t(\tau) = Y^\lambda_{\alpha(t+\tau)}(t+\tau) - Y^\lambda_{\alpha(t)}(t+\tau) = V^\lambda_{t,1}(\tau) + V^\lambda_{t,2}(\tau),$$

where

$$V^\lambda_{t,1}(\tau) = \left(\frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))}\right) \int_{-\infty}^{t+\tau} (t+\tau-u)^{\alpha(t+\tau)-1} \frac{\lambda^{1-h}}{(\lambda+t+\tau-u)^{1-h}}\, dB_u,$$

$$V^\lambda_{t,2}(\tau) = \frac{1}{\Gamma(\alpha(t))} \times \int_{-\infty}^{t+\tau} [(t+\tau-u)^{\alpha(t+\tau)-1} - (t+\tau-u)^{\alpha(t)-1}] \frac{\lambda^{1-h}}{(\lambda+t+\tau-u)^{1-h}}\, dB_u.$$

Hence

$$\begin{aligned} E_B[(Y^\lambda_{\alpha(t+\tau)}(t+\tau) - Y^\lambda_{\alpha(t)}(t))^2] &\geqslant E_B[U^\lambda_t(\tau)^2] + 2E_B[U^\lambda_t(\tau)V^\lambda_t(\tau)] \\ &\geqslant E_B[U^\lambda_t(\tau)^2] - 2E[U^\lambda_t(\tau)^2]^{1/2} E[V^\lambda_t(\tau)^2]^{1/2}. \end{aligned} \tag{3.26}$$

The last inequality follows from the Cauchy–Schwarz inequality. By Lemma 4.1 below and the inequality (4.10), there exist constants C_1 and C_2 depending only on $[a,b]$, λ and h such that

$$C_2|\tau|^{2\alpha(t)-1} \leqslant E[(U^\lambda_t)^2] \leqslant C_1|\tau|^{2\alpha(t)-1}. \tag{3.27}$$

On the other hand,

$$E_B[V^\lambda_t(\tau)^2] \leqslant 2\big(E_B[V^\lambda_{t,1}(\tau)^2] + E_B[V^\lambda_{t,2}(\tau)^2]\big).$$

A standard computation combined with the mean value theorem and the fact that any continuous function has a maximum on any compact interval, we obtain

$$\begin{aligned} &E_B[V^\lambda_{t,1}(\tau)^2] \\ &= \left(\frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))}\right)^2 \lambda^{2\alpha(t+\tau)-1} \beta\big(3 - 2\alpha(t+\tau) - 2h, 2\alpha(t+\tau) - 1\big) \\ &\leqslant C|\alpha(t+\tau) - \alpha(t)|^2 \leqslant C|\tau|^{2\beta}. \end{aligned}$$

Moreover, by the change of variable $x = t+\tau-u$, we have

$$\begin{aligned}
&E_B[V_{t,2}^{\lambda}(\tau)^2]\\
&\quad=\frac{\lambda^{2-2h}}{\Gamma(\alpha(t))^2}\int_{-\infty}^{t+\tau}[(t+\tau-u)^{\alpha(t+\tau)-1}-(t+\tau-u)^{\alpha(t)-1}]^2(\lambda+t+\tau-u)^{2h-2}\,du\\
&\quad=\frac{\lambda^{2-2h}}{\Gamma(\alpha(t))^2}\int_{0}^{\infty}[x^{\alpha(t+\tau)-1}-x^{\alpha(t)-1}]^2(\lambda+x)^{2h-2}\,dx\\
&\quad=\frac{\lambda^{2-2h}[\alpha(t+\tau)-\alpha(t)]^2}{\Gamma(\alpha(t))^2}\int_{0}^{\infty}\log(x)^2x^{2a_{t,\tau}^x-2}\,(\lambda+x)^{2h-2}\,du
\end{aligned}$$

for some $a_{t,\tau}^x \in (m_\alpha[a,b], M_\alpha[a,b])$, the last equality following from the mean value theorem. Let $0<\sigma<2m_\alpha[a,b]-1$ and $0<\varsigma<3/2-M_\alpha[a,b]-h$. Since α is β-Hölder continuous, for $c=3-2h-2M_\alpha[a,b]-2\varsigma$ and $d=2M_\alpha[a,b]-1+2\varsigma$ we have

$$\begin{aligned}
E_B[V_{t,2}^{\lambda}(\tau)^2]&=\frac{\lambda^{2-2h}[\alpha(t+\tau)-\alpha(t)]^2}{\Gamma(\alpha(t))^2}\Big(\int_0^1\log(x)^2x^{2c_{t,\tau}^x-2}(\lambda+x)^{2h-2}\,dx\\
&\qquad+\int_1^\infty\log(x)^2x^{2a_{t,\tau}^x-2}(\lambda+x)^{2h-2}\,dx\Big)\\
&\leqslant C|\tau|^{2\beta}\Big(\int_0^1 x^{2m_\alpha[a,b]-2-\sigma}\,dx+\int_1^\infty x^{2M_\alpha[a,b]-2+2\varsigma}(\lambda+x)^{2h-2}\,dx\Big)\\
&\leqslant C|\tau|^{2\beta}(1/(2m_\alpha[a,b]-1-\sigma)+\beta(c,d))\leqslant C^{\sigma,\varsigma}|\tau|^{2\beta}.
\end{aligned}$$

We then deduce that

$$E_B[V_t^{\lambda}(\tau)^2]\leqslant C^{\sigma,\varsigma}|\tau|^{2\beta}. \tag{3.28}$$

Combining (3.27), (3.28) and the Cauchy–Schwarz inequality yields

$$|E[U_t^{\lambda}(\tau)V_t^{\lambda}(\tau)]|\leqslant E[U_t^{\lambda}(\tau)^2]^{1/2}E[V_t^{\lambda}(\tau)^2]^{1/2}\leqslant C^{\sigma,\varsigma}|\tau|^{\beta+\alpha(t)-1/2}. \tag{3.29}$$

Thus, by plugging (3.27) and (3.29) in (3.26), we get

$$\begin{aligned}
E_B[(Y_{\alpha(t+\tau)}^{\lambda}(t+\tau)-Y_{\alpha(t)}^{\lambda}(t))^2]&\geqslant C_2|\tau|^{2\alpha(t)-1}-C^{\sigma,\varsigma}|\tau|^{\alpha(t)-1/2+\beta}\\
&\geqslant|\tau|^{2M_\alpha[a,b]-1}(C_2-C^{\sigma,\varsigma}|\tau|^{\beta-M_\alpha[a,b]+1/2}).
\end{aligned}$$

By assuming that $\alpha(t)-1/2\leqslant M_\alpha[a,b]-1/2<\beta$, the function

$$g:\tau\mapsto C_2-C^{\sigma,\varsigma}|\tau|^{\beta-M_\alpha[a,b]+1/2}$$

is continuous in τ and converges to C_2 when $\tau\to 0$. So there exists $\epsilon>0$ such that $g(\tau)>C=C_2$ for $|\tau|<\epsilon$, which gives the inequality (3.24).

On the other hand, by the assumption $\alpha(t) - 1/2 \leqslant M_\alpha[a,b] - 1/2 < \beta$ and using the equivalence (4.11), (3.28) and (3.29), we immediately obtain (3.25). ∎

In the following, we state interesting properties of GWmOU processes such as continuity, Hölder exponent at t, Hausdorff dimension and local asymptotic self-similarity. The same properties hold for WmOU processes, the proofs of which are based on [15, Lemma 3.1], of which Proposition 3.1 is the counterpart for GWmOU processes. Having Proposition 3.1 at hand, the proofs for GWmOU processes proceed analogously to those in [15]. Therefore, we omit them.

3.2.1. *Continuity*

PROPOSITION 3.2. *The process* $\{Y^\lambda_{\alpha(t)}(t), t \in \mathbb{R}\}$ *admits a continuous modification.*

In the following properties: Hölder exponent, Hausdorff dimension and local asymptotic self-similarity, we make the additional assumptions that $\alpha(t)-1/2 < \beta$ for all t in the domain of α and $M_\alpha[a,b] < 1$.

3.2.2. *Hölder exponent*

PROPOSITION 3.3. *Let* $[a,b] \subset \mathbb{R}$ *be an interval. For any* $0 \leqslant \eta < m_\alpha[a,b] - 1/2$*, with probability* 1*, there exists a constant* $C^{\delta,\rho}_\eta$ *such that*

$$|Y^\lambda_{\alpha(t)}(t) - Y^\lambda_{\alpha(s)}(s)| \leqslant C^{\delta,\rho}_\eta |t-s|^\eta \quad \forall t, s \in [a,b].$$

We now turn to the Hölder continuity of GWmOU processes. Let us first recall the following definition.

DEFINITION 3.1. A real-valued function is said to have *Hölder exponent* β at a point t_0 if

$$\begin{aligned} \lim_{h\to 0} \frac{|f(t_0+h)-f(t_0)|}{|h|^\gamma} &= 0 \qquad \text{for any } \gamma < \beta, \\ \limsup_{h\to 0} \frac{|f(t_0+h)-f(t_0)|}{|h|^\gamma} &= \infty \qquad \text{for any } \gamma > \beta. \end{aligned}$$

PROPOSITION 3.4. *With probability* 1*, the Hölder exponent of* $Y^\lambda_{\alpha(t)}(t)$ *at a point* t_0 *in the domain is* $\alpha(t_0) - 1/2$.

3.2.3. *Hausdorff dimension.* Let $\dim_H A$, $\underline{\dim}_B A$, and $\overline{\dim}_B A$ denote the Hausdorff dimension, the lower box dimension, and the upper box dimension of a set A in $\mathbb{R}^n$, respectively. Given a compact interval $[a,b] \subset \mathbb{R}$, $\mathbf{G}_\alpha[a,b] = \{(t, Y^\lambda_{\alpha(t)}(t)) : t \in [a,b]\}$ stands for the graph of the process $Y^\lambda_{\alpha(t)}(t)$ restricted to $[a,b]$. For more information on these notions see [11]. We now formulate our result.

PROPOSITION 3.5. *Let $[a,b]$ be an interval in the domain of definition of α. With probability* 1, $\dim_H \mathbf{G}_\alpha[a,b] = \underline{\dim}_B \mathbf{G}_\alpha[a,b] = \overline{\dim}_B \mathbf{G}_\alpha[a,b] = 5/2 - m_\alpha[a,b]$.

3.2.4. *Local asymptotic self-similarity.* WmOU processes are locally asymptotically self-similar, in the following sense defined in [4].

DEFINITION 3.2. Let $X(t)$ be a Gaussian process. We say that $X(t)$ is *locally asymptotically self-similar* with parameter H at a point t_0 if the limit process

$$\left\{\lim_{h\to 0^+} \frac{X(t_0+hu)-X(t_0)}{h^H},\, u\in\mathbb{R}\right\}$$

exists and is nontrivial for every t_0.

This property holds true for GWmOU processes. Before stating this result, let us first recall that a *fractional Brownian motion* with Hurst index H is a centered Gaussian process with covariance

$$E[B^H(t)B^H(s)] = \tfrac{1}{2}[|t|^{2H}+|s|^{2H}-|t-s|^{2H}].$$

PROPOSITION 3.6. *For any t_0 the stochastic process*

$$\left\{\lim_{h\to 0^+} \frac{Y^\lambda_{\alpha(t_0+hu)}(t_0+hu)-Y^\lambda_{\alpha(t_0)}(t_0)}{h^{\alpha(t_0)/2-1/4}},\, u\in\mathbb{R}\right\}$$

is, modulo a constant, a fractional Brownian motion with Hurst index $\alpha(t_0)/2 - 1/4$.

3.2.5. *Short-range dependence.* We are now interested in the strength of the dependence of GWmOU processes.

DEFINITION 3.3 ([12]). Let $X(t)$ be a Gaussian process with covariance denoted by $c(s,t)=\mathrm{cov}(X(s),X(t))$ and correlation $\rho(s,t)$ defined by

$$\rho(s,t)=\frac{c(s,t)}{\sqrt{c(t,t)c(s,s)}}.$$

We say that $X(t)$ is *long-range dependent* if

$$\int_0^\infty |\rho(t,t+\tau)|\,d\tau=\infty,$$

and it is *short-range dependent* if the integral is finite.

The following lemma provides an upper bound for the inverse of the variance of the process Y_α^λ with $0 < h < 3/2 - \alpha_{\sup}$ and $1/2 < \alpha(t)$ for all t.

LEMMA 3.1. *For all t the function* $t \mapsto 1/E_B[Y_{\alpha(t)}^\lambda(t)^2]$ *is upper bounded.*

Proof. From (3.21), we find that

$$\frac{1}{E_B[Y_{\alpha(t)}^\lambda(t)^2]} = \frac{\lambda^{1-2\alpha(t)}[2\alpha(t)-1]\Gamma(\alpha(t))^2\Gamma(2-2h)}{\Gamma(2\alpha(t))\Gamma(3-2h-2\alpha(t))}.$$

The functions $z \mapsto \lambda^{1-2z}$, $z \mapsto 2z-1$, $z \mapsto \Gamma(z)^2$, $z \mapsto \Gamma(2z)$ and $z \mapsto \Gamma(3-2h-2z)$ are continuous for $z \in [1/2, \alpha_{\sup}]$. As a consequence,

$$\frac{1}{E_B[Y_{\alpha(t)}^\lambda(t)^2]} \leqslant C. \quad \blacksquare \tag{3.30}$$

We are thus led to the following short-range dependence property of GWmOU processes.

PROPOSITION 3.7. *For* $0 < h < 1-\alpha_{\sup}$, *the GWmOU process is short-range dependent.*

Proof. Set $y = \tau/\lambda$. Using (3.22) and (3.30), we have

$$0 \leqslant \rho_\alpha(t, t+\tau) \leqslant CG(\alpha(t+\tau), \alpha(t), h, y).$$

Since $0 \leqslant u \leqslant 1$ and $1/2 < \alpha(t) < 1$ for all t, we obtain

$$\begin{aligned} &G(\alpha(t+\tau), \alpha(t), h, y) \\ &\quad = \int_0^1 u^{2-[\alpha(t)+\alpha(t+\tau)]-2h}(1-u)^{\alpha(t)-1}(1+yu)^{h-1}(yu+1-u)^{\alpha(t+\tau)-1}\,du \\ &\quad \leqslant y^{\alpha(t+\tau)-1}\,(y+1)^{h-1}\int_0^1 u^{-\alpha(t)-h}(1-u)^{\alpha(t)-1}\,du. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^\infty |\rho_\alpha(t,t+\tau)|d\tau &\leqslant C\int_0^\infty y^{\alpha(t+\lambda y)-1}(y+1)^{h-1}\,dy\int_0^1 u^{-\alpha(t)-h}(1-u)^{\alpha(t)-1}\,du \\ &\leqslant C\beta(1-h-\alpha_{\sup}, 1/2)\beta(1-h-\alpha(t), \alpha(t)) < \infty, \end{aligned}$$

since $0 < h < 1-\alpha_{\sup}$. $\blacksquare$

We are now interested in the asymptotic behavior of the process Y_α^λ when $\lambda \to \infty$.

PROPOSITION 3.8. *Let* $\{Y^\lambda_{\alpha(t)}(t)\,, t \geqslant 0\}$ *be a GWmOU process restricted to* $t \geqslant 0$ *and set* $\alpha(t) = H(t) + 1/2$ *with* $0 < h < 3/2 - \alpha_{\sup}$. *Then for fixed* t *in* $\mathbb{R}^+$,

$$Y^\lambda_{\alpha(t)}(t) - Y^\lambda_{\alpha(t)}(0) \xrightarrow[\lambda\to\infty]{} B_{H(t)}(t) \qquad \text{in } L^2(\Omega_B).$$

Proof. For each $s \leqslant t$ set $c_\lambda(t-s) = (\lambda/(\lambda+t-s))^{1-h}$, for each $t \geqslant 0$ let $X^\lambda_{\alpha(t)}(t) = Y^\lambda_{\alpha(t)}(t) - Y^\lambda_{\alpha(t)}(0)$, and denote

$$\begin{aligned} A^\lambda_{H(t)}(t) &= \int_{-\infty}^{0} \big([c_\lambda(t-s)(t-s)^{H(t)-1/2} - c_\lambda(-s)(-s)^{H(t)-1/2}] \\ &\quad - [(t-s)^{H(t)-1/2} - (-s)^{H(t)-1/2}]\big)\, dB_s \\ &=: \int_{-\infty}^{0} [A^\lambda_{1,H(t)}(t,s) - A_{1,H(t)}(t,s)]\, dB_s, \\ D^\lambda_{H(t)}(t) &= \int_0^t (t-s)^{H(t)-1/2}(c_\lambda(t-s) - 1)\, dB_s. \end{aligned}$$

By substituting $\alpha(t) = H(t) + 1/2$ we get

$$\begin{aligned} X^\lambda_{\alpha(t)}(t) - B_{H(t)}(t) &= X^\lambda_{H(t)+1/2}(t) - B_{H(t)}(t) \\ &= \frac{1}{\Gamma(H(t)+1/2)}[A^\lambda_{H(t)}(t) + D^\lambda_{H(t)}(t)]. \end{aligned}$$

Thus,

$$\begin{aligned} E_B[(\,X^\lambda_{\alpha(t)}(t) - B_{H(t)}(t)\,)^2] &= \frac{1}{\Gamma(H(t)+1/2)^2} E_B[(A^\lambda_{H(t)}(t) + D^\lambda_{H(t)}(t))^2] \\ &\leqslant \frac{2}{\Gamma(H(t)+1/2)^2}\big(E_B[A^\lambda_{H(t)}(t)^2] + E_B[D^\lambda_{H(t)}(t)^2]\big). \end{aligned}$$

Let us first evaluate the asymptotic behavior of $E_B[A^\lambda_{H(t)}(t)^2]$ when $\lambda \to \infty$.

For fixed $t \geqslant 0$, it is easily seen that

$$A^\lambda_{1,H(t)}(t,s) \xrightarrow[\lambda\to\infty]{} A_{1,H(t)}(t,s). \tag{3.31}$$

Using the elementary inequality, for any $p \geqslant 0$ and $x, y \in \mathbb{R}$,

$$\big||x|^p - |y|^p\big| \leqslant (p \vee 1)2^{(p-2)^+}[|x-y|^p + |y|^{(p-1)^+}|x-y|^{p\wedge 1}]$$

and the fact that $c_\lambda(x) \leqslant 1$ for all $x > -\lambda$, we have, for $s < 0$,

$$\begin{aligned} &|c_\lambda(t-s)(t-s)^{H(t)-1/2} - c_\lambda(-s)(-s)^{H(t)-1/2}| \\ &\quad \leqslant c_\lambda(t-s)|(t-s)^{H(t)-1/2} - (-s)^{H(t)-1/2}| + (-s)^{H(t)-1/2}|c_\lambda(t-s) - c_\lambda(-s)| \\ &\quad \leqslant |(t-s)^{H(t)-1/2} - (-s)^{H(t)-1/2}| + 2t^{1-h}(-s)^{H(t)-1/2}(t-s)^{h-1}. \end{aligned}$$

Moreover, for fixed $t > 0$ such that $H(t) \neq 1/2$, when $s \to -\infty$ we get

$$\big((t-s)^{H(t)-1/2} - (-s)^{H(t)-1/2}\big)^2 \sim (H(t)-1/2)^2 t^2 (-s)^{2H(t)-3}.$$

As a result, $s \mapsto ((t-s)^{H(t)-1/2} - (-s)^{H(t)-1/2})^2$ is integrable at $-\infty$, because $2H(t)-3 < -1$, and as $s \to 0^-$ as well, since $2H(t)-1 > -1$. Consequently,

$$\int_{-\infty}^{0} \big((t-s)^{H(t)-1/2} - (-s)^{H(t)-1/2}\big)^2 \, ds < \infty.$$

Also, by the hypothesis $2 - 2H(t) - 2h > 0$,

$$\int_{-\infty}^{0} (-s)^{2H(t)-1}(t-s)^{2h-2} \, ds = t^{2H(t)+2h-2} \beta(2 - 2H(t) - 2h, 2H(t)) < \infty.$$

The dominated convergence theorem shows that for fixed $t \geqslant 0$,

$$\lim_{\lambda \to \infty} E_B[A^\lambda_{H(t)}(t)^2] = 0.$$

Similarly, one shows that for fixed $t \geqslant 0$, $\lim_{\lambda \to \infty} E_B[D^\lambda_{H(t)}(t)^2] = 0$, which proves the desired result. ■

On the other hand, we now consider the asymptotic behavior of Y^λ_α when $\lambda \to 0$. In the following result, it is assumed that $\alpha(t) = \alpha$ for all t, $1 - \alpha < h < 3/2 - \alpha$ and $1/2 < \alpha < 1$.

PROPOSITION 3.9. *Let $\{\hat{Y}^\lambda_\alpha, t \geqslant 0\}$ be the process defined by*

$$\hat{Y}^\lambda_\alpha(t) = \lambda^{h-1} \int_0^t Y^\lambda_\alpha(s) \, ds, \qquad t \geqslant 0.$$

Then

$$\hat{Y}^\lambda_\alpha(t) \xrightarrow[\lambda \to 0]{} Y_\alpha(t) \quad \text{in } L^2(\Omega_B),$$

where

$$Y_\alpha(t) := \frac{1}{\Gamma(\alpha)(h+\alpha-1)} \Big[\int_{-\infty}^{0} (t-u)^{h+\alpha-1} - (-u)^{h+\alpha-1} \, dB_u + \int_0^t (t-u)^{h+\alpha-1} \, dB_u \Big].$$

Moreover, the process $(Y_\alpha(t))_{t \geqslant 0}$ is (modulo a constant) a fractional Brownian motion with Hurst index $h + \alpha - 1/2$.

Proof. For each $t \geqslant 0$, we have

$$\lambda^{h-1}\int_0^t Y_\alpha^\lambda(s)\,ds = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t} dB_u \int_{u\vee 0}^{t} (\lambda+s-u)^{h-1}(s-u)^{\alpha-1}\,ds$$
$$= \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{0} dB_u \int_0^t (\lambda+s-u)^{h-1}(s-u)^{\alpha-1}\,ds$$
$$+ \frac{1}{\Gamma(\alpha)}\int_0^t dB_u \int_u^t (\lambda+s-u)^{h-1}(s-u)^{\alpha-1}\,ds.$$

Using the same computations as in the proof of Proposition 3.8, it is easily checked that for every $t \geqslant 0$,

$$\hat{Y}_\alpha^\lambda(t) \xrightarrow[\lambda\to 0]{} Y_\alpha(t) :=$$
$$\frac{1}{\Gamma(\alpha)(h+\alpha-1)}\Big[\int_{-\infty}^{0} (t-u)^{h+\alpha-1} - (-u)^{h+\alpha-1}\,dB_u + \int_0^t (t-u)^{h+\alpha-1}\,dB_u\Big]$$

in $L^2(\Omega_B)$. Moreover, it is obvious that the process $(Y_\alpha(t))_{t\geqslant 0}$ is (modulo a constant) a fractional Brownian motion (with moving average definition) with Hurst index $h+\alpha-1/2$. ■

4. APPENDIX

PROPOSITION 4.1. *For all* $0 < p < 1$ *and* $k \geqslant 2$,

$$\sum_{i=1}^{k} x_i^p \leqslant 2^{(k-1)(1-p)}\Big(\sum_{i=1}^{k} x_i\Big)^p \quad \text{if } x_i \geqslant 0 \text{ for } i = 1,\ldots,k. \tag{4.1}$$

Proof. For $k \geqslant 2$, $0 < p < 1$ and $x_i \geqslant 0$ for all $i = 1,\ldots,k$, we will denote by $A(k)$ the inequality

$$A(k):\quad \sum_{i=1}^{k} x_i^p \leqslant 2^{(k-1)(1-p)}\Big(\sum_{i=1}^{k} x_i\Big)^p.$$

Let $k = 2$. Since the function $x \mapsto x^p$, $x \geqslant 0$, is concave for every $0 < p < 1$, we get

$$x^p + y^p \leqslant 2^{1-p}(x+y)^p,$$

so $A(2)$ holds true.

Let us assume that $A(n-1)$ holds. Using $A(2)$, $A(n-1)$, by easy computations we get $A(n)$. Thus by induction, the proof is complete. ■

Throughout the appendix, it is supposed that $0 < h < 3/2 - M_\alpha[a,b]$, $M_\alpha[a,b] < 1$ and $m_\alpha[a,b] > 1/2$ for any compact interval $[a,b] \subset \mathbb{R}$.

LEMMA 4.1. *Fix a compact interval $[a,b] \subset \mathbb{R}$. There exists a constant C depending only on $[a,b]$, λ and h such that*

$$E_B[(Y^\lambda_{\alpha(t)}(t+\tau) - Y^\lambda_{\alpha(t)}(t))^2] \leqslant C|\tau|^{2\alpha(t)-1}$$

for all $t, t+\tau \in [a,b]$ with $|\tau| < 1$.

Proof. Set $\eta = 3 - 2h - 2\alpha(t), \nu = 2\alpha(t) - 1$ and $y = |\tau|/\lambda$. Using (3.21), we get

$$\begin{aligned} (4.2)\qquad & E_B[(Y^\lambda_{\alpha(t)}(t+\tau) - Y^\lambda_{\alpha(t)}(t))^2] \\ &= E_B[(Y^\lambda_{\alpha(t)}(t+\tau))^2] + E_B[Y^\lambda_{\alpha(t)}(t)^2] - 2E_B[Y^\lambda_{\alpha(t)}(t+\tau)Y^\lambda_{\alpha(t)}(t)] \\ &= \frac{2\lambda^\nu}{\Gamma(\alpha(t))^2}\beta(\eta,\nu) - 2E_B[Y^\lambda_{\alpha(t)}(t+\tau)Y^\lambda_{\alpha(t)}(t)]. \end{aligned}$$

Let us first evaluate the second term on the right hand side. By (3.22) we have

$$\begin{aligned} (4.3)\qquad & -2E_B[Y^\lambda_{\alpha(t)}(t+\tau)Y^\lambda_{\alpha(t)}(t)] \\ &= \frac{-2\lambda^\nu}{\Gamma(\alpha(t))^2}\int_0^1 u^{\eta-1}(1-u)^{\alpha(t)-1}(1+yu)^{h-1}(yu+1-u)^{\alpha(t)-1}\,du. \end{aligned}$$

By applying the mean value theorem to the function $t \mapsto (1+yt)^{h-1}$ for $t \in [0,u]$, we obtain

$$\begin{aligned} (4.4)\qquad & -2E_B[Y^\lambda_{\alpha(t)}(t+\tau)Y^\lambda_{\alpha(t)}(t)] \\ &= \frac{-2\lambda^\nu}{\Gamma(\alpha(t))^2}\int_0^1 u^{\eta-1}(1-u)^{\nu-1}\left(1+y\frac{u}{1-u}\right)^{\alpha(t)-1} du \\ &\quad + \frac{2\lambda^\nu(1-h)}{\Gamma(\alpha(t))^2}y\int_0^1 (1+yC_u)^{h-2}u^\eta\,(1-u)^{\alpha(t)-1}(yu+1-u)^{\alpha(t)-1}\,du \\ &=: A_{\lambda,h}(\alpha(t),y) + B_{\lambda,h}(\alpha(t),y). \end{aligned}$$

Let us begin by providing an upper bound for $A_{\lambda,h}$. Using the inequality

$$1 - \frac{yu}{1-(1-y)u} \leqslant \left(1+y\frac{u}{1-u}\right)^{\alpha(t)-1},$$

for $y \neq 0$ we have

$$
\begin{aligned}
(4.5)\quad & A_{\lambda,h}(\alpha(t), y) \\
& \leqslant \frac{-2\lambda^\nu}{\Gamma(\alpha(t))^2} \int_0^1 u^{\eta-1}(1-u)^{\nu-1}\, du + \frac{2\lambda^\nu y}{\Gamma(\alpha(t))^2} \int_0^1 \frac{u^\eta (1-u)^{\nu-1}}{1-(1-y)u}\, du \\
& = \frac{-2\lambda^\nu}{\Gamma(\alpha(t))^2} \beta(\eta,\nu) + \frac{2\lambda^\nu y}{\Gamma(\alpha(t))^2} \beta(\nu, \eta+1)\, {}_2F_1(1, \eta+1, 3-2h, 1-y),
\end{aligned}
$$

where ${}_2F_1$ is called the hypergeometric function, and the last equality is due to Euler's representation integral of ${}_2F_1$ (see [1, Theorem 2.2.1]).

Using Euler's transformation formula (see [1, Theorem 2.2.5]), we get

$$
(4.6)\quad {}_2F_1(1, \eta+1, 3-2h, 1-y) = y^{2\alpha(t)-2}\, {}_2F_1(2-2h, \nu, 3-2h, 1-y).
$$

Set $a = 2-2h$, $b = 2m_\alpha[a,b] - 1$ and $c = 3 - 2h - 2M_\alpha[a,b] + 2m_\alpha[a,b]$. For $y \neq 0$, we have

$$
\begin{aligned}
(4.7)\quad & {}_2F_1(a, \nu, a+1, 1-y) \\
& \qquad = \frac{\Gamma(a+1)}{\Gamma(\nu)\Gamma(\eta+1)} \int_0^1 x^{\nu-1}(1-x)^\eta (1-(1-y)x)^{-a}\, dx \\
& \qquad \leqslant C \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-(1-y)x)^{-a}\, dx = CF(y) \leqslant C,
\end{aligned}
$$

the last inequality coming from the fact that the function F is continuous on $[1, 1/\lambda]$. By plugging (4.6) in (4.5) and using (4.7), we infer that

$$
(4.8)\quad A_{\lambda,h}(\alpha(t), |\tau|) \leqslant \frac{-2\lambda^\nu}{\Gamma(\alpha(t))^2} \beta(\eta,\nu) + C|\tau|^\nu.
$$

On the other hand, since $\eta, \lambda, C_u > 0$, $0 < h < 1$, and $\alpha(t) < 1$, we have

$$
(4.9)\quad B_{\lambda,h}(\alpha(t), |\tau|) \leqslant \frac{2\lambda^{\alpha(t)-1}(1-h)}{\Gamma(\alpha(t))^2} \beta(\alpha(t), \alpha(t))|\tau|^{\alpha(t)} \leqslant M|\tau|^\nu,
$$

where M is the maximum of the continuous function

$$
z \mapsto (2\lambda^{z-1}(1-h)/\Gamma(z)^2)\beta(z,z)
$$

on $[m_\alpha[a,b], M_\alpha[a,b]]$. Thus, by plugging (4.8) and (4.9) in (4.4), we get

$$
-2E_B[Y^\lambda_{\alpha(t)}(t+\tau) Y^\lambda_{\alpha(t)}(t)] \leqslant \frac{-2\lambda^\nu}{\Gamma(\alpha(t))^2} \beta(\eta,\nu) + (M+C)|\tau|^\nu.
$$

Then

$$
E_B[(Y^\lambda_{\alpha(t)}(t+\tau) - Y^\lambda_{\alpha(t)}(t))^2] \leqslant C_1|\tau|^\nu,
$$

where $C_1 = M + C$, which establishes the desired result. ■

LEMMA 4.2. *Fix a compact interval* $[a,b] \subset \mathbb{R}$.

(1) *There exists a constant* C_2 *depending only on* $[a,b]$, λ *and* h *such that*

$$E_B[(Y^\lambda_{\alpha(t)}(t+\tau) - Y^\lambda_{\alpha(t)}(t))^2] \geqslant C_2|\tau|^{2\alpha(t)-1} \tag{4.10}$$

for all $t, t+\tau \in [a,b]$ *with* $|\tau| < 1$.

(2) *As* $\tau \to 0$,

$$E_B[(Y^\lambda_{\alpha(t)}(t+\tau) - Y^\lambda_{\alpha(t)}(t))^2] = C_2|\tau|^{\alpha(t)-1/2} + O(|\tau|^{2\alpha(t)-1}). \tag{4.11}$$

Proof. With the notation of Lemma 4.1, since $\alpha(t) < 1$ for all t, we have

$$B_{\lambda,h}(\alpha(t), y) \leqslant E_B[(Y^\lambda_{\alpha(t)}(t+\tau) - Y^\lambda_{\alpha(t)}(t))^2].$$

For $\tau \neq 0$, using $C_u \in \,]0,1[$ we get

$$\begin{aligned}\left(\frac{1}{|\tau|} + \frac{1}{\lambda}\right)^{\alpha(t)+h-3} |\tau|^{2\alpha(t)-2} &\leqslant (1+y)^{h+\alpha(t)-3} \\ &\leqslant (1-u+yu)^{\alpha(t)-1}(1+yC_u)^{h-2}.\end{aligned} \tag{4.12}$$

Set

$$h(z,x) = (\lambda x)^{3-z-h}(x+\lambda)^{z+h-3},$$

a continuous function on $[m_\alpha[a,b], M_\alpha[a,b]] \times [0,1]$, and let C_2 be the minimum of the function

$$(z,x) \mapsto \frac{2\lambda^{2z-2}(1-h)}{\Gamma(z)^2}\beta(4-2h-2z, z)h(z,x)$$

for $(z,x) \in [m_\alpha[a,b], M_\alpha[a,b]] \times [0,1]$. Then

$$E_B[(Y^\lambda_{\alpha(t)}(t+\tau) - Y^\lambda_{\alpha(t)}(t))^2] \geqslant C_2|\tau|^{\nu},$$

which gives (4.10).

Let us now prove (4.11). If instead of (4.12) we use the following inequality for $y \neq 0$:

$$\left(\frac{1}{|\tau|} + \frac{1}{\lambda}\right)^{\alpha(t)+h-3} |\tau|^{\alpha(t)-3/2} \leqslant (1+y)^{\alpha(t)+h-3},$$

then (4.10) becomes

$$E_B[(Y^\lambda_{\alpha(t)}(t+\tau) - Y^\lambda_{\alpha(t)}(t))^2] \geqslant C_2|\tau|^{\alpha(t)-1/2}.$$

Combining Lemma 4.1 and the last inequality, we get

$$C_2|\tau|^{\alpha(t)-1/2} \leqslant E_B[(Y^\lambda_{\alpha(t)}(t+\tau) - Y^\lambda_{\alpha(t)}(t))^2] \leqslant C_1|\tau|^{\alpha(t)-1/2}.$$

To prove (4.11), it remains to show that $C_2 \leqslant C_1 = M+C$. Since $0 < h < 3/2-z$ and $z < 1$, we obtain

$$\beta(4-2h-2z,z) \leqslant \beta(z,z) \quad \text{and} \quad h(z,x) \leqslant \lambda^{1-z}.$$

Therefore,

$$C_2 \leqslant \frac{2\lambda^{2z-2}(1-h)}{\Gamma(z)^2}\,\beta(4-2h-2z,z)h(z,x) \leqslant M \leqslant C_1,$$

which completes the proof. ■

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