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A TIME-CHANGED STOCHASTIC CONTROL PROBLEM AND ITS MAXIMUM PRINCIPLE

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Abstract. This paper studies a time-changed stochastic control problem, where the underlying stochastic process is a Lévy noise time-changed by an inverse subordinator. We establish a maximum principle for the time-changed stochastic control problem. We also prove the existence and uniqueness of the corresponding time-changed backward stochastic differential equation involved in the stochastic control problem. Some examples are provided for illustration.

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1. INTRODUCTION

Uncertainty is inherent in the real world and changes over time, putting people's decisions at risk. A decision maker wants to select the best choice among all possible ones. Stochastic control theory serves as a tool for such dynamic optimization problems. The world has witnessed many applications of stochastic control theory in various fields such as biology [16], economics [3], and finance [15].

A well known approach to stochastic control problems is based on the maximum principle. This method for the Itô diffusion case was first studied by Kushner [8] and Bismut [2] and further developed by Bensoussan [1], Peng [14], and others. The jump diffusion case was formulated by Framstad, Øksendal and Sulem [4]. The idea of the maximum principle approach is to define a Hamiltonian function and derive the adjoint equations, which involve a backward stochastic differential equation. Under some sufficient conditions, the optimal control is a solution of a coupled system of forward and backward stochastic differential equations.

Time-changed stochastic differential equations and related fractional Fokker–Plank equations have become an indispensable tool in applied scientific areas. An example is $dX(t) = dB(E_t)$ where $X(0) = 0$ and $\{E_t, t \geqslant 0\}$ is the inverse

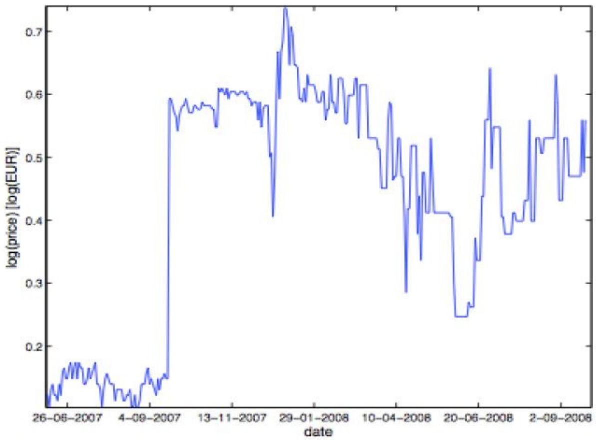

FIGURE 1. Log price of the Kalev stock [5]

of an α-stable subordinator [10]. The subdiffusion $B(E_t)$ is governed by the time-fractional diffusion equation $\partial_t^\alpha q(x,t) = \partial_x^2 q(x,t)$. Some time-changed stochastic differential equations are used to describe real world phenomena. For example, quantitative financial analysts exploit the Black–Scholes framework in derivative pricing, in which the stock price is modeled by Brownian motion. However, some stocks are not actively traded and their prices stay constant for some time periods. Such phenomena can be modeled by time-changed Brownian motion but not by the standard Brownian motion (see Figure 1). Fruitful studies in this area are available [5], [9], [11], [13].

As time-changed stochastic processes have been adopted in more and more areas, the traditional stochastic control problem framework needs updates to fit the time-changed cases. For example, a mutual fund manager, whose investment portfolios consist of stocks whose prices follow time-changed Brownian motions as shown in Figure 1, will find the time-changed stochastic control a better tool to manage the portfolio than the traditional stochastic control. A biologist, who investigates how outside interferences affect the movements of insects, may find the time-changed stochastic control problem better describing the experiment since some insects sometimes move and sometimes stay still. Because time-changed stochastic processes better describe many phenomena and people seek the optimal choice based on them, we believe it is necessary to study stochastic control problems based on time-changed stochastic processes, which will build up a framework to solve potential optimization problems.

In this paper, we investigate time-changed stochastic control problems using the maximum principle method. Specifically, we consider the following time-changed stochastic process [7], [12]:

$$dX(t) = b(t, E_t, X(t-), u(t))\, dE_t + \sigma(t, E_t, X(t-), u(t))\, dB_{E_t} + \int_{|y|<c} \gamma(t, E_t, X(t-), u(t), y)\, \tilde{N}(dE_t, dy), \tag{1.1}$$

with $X(0) = x_0 \neq 0$ and the corresponding performance function

$$J(u) = \mathbb{E}\Big[\int_0^T g(t, E_t, X(t), u(t))\, dE_t + h(X(T))\Big], \quad u \in \mathcal{A}, \tag{1.2}$$

where $u(t) = u(t, w) \in U \subset \mathbb{R}$ is the control and $\mathcal{A}$ denotes the set of *admissible* controls. The compensated Poisson random measure $\tilde{N}$ is defined as $\tilde{N}(dt, dy) = N(dt, dy) - \nu(dy)\, dt$ where ν is a Lévy measure such that $\int_{\mathbb{R}-\{0\}}(|y|^2 \wedge 1)\, \nu(dy) < \infty$, and E_t is the inverse of a subordinator $D(t)$ with Laplace exponent $\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\, \nu(dx)$. In equation (1.1), dE_t describes the drift of the time-changed stochastic process, B_{E_t} is the Brownian motion with time changes incorporated, and the $\tilde{N}(dE_t, dy)$ captures the jumps in addition to the Brownian motion. We establish a maximum principle for the stochastic control problem of finding $u^* \in \mathcal{A}$ such that

$$J(u^*) = \sup_{u \in \mathcal{A}} J(u). \tag{1.3}$$

In (1.2), the performance function can be the utility function, energy consumption function that we care about. For example, the performance function in Example 4.1 is the utility function $\exp(-\delta t)u(t)^2$, where $u(t)$ is the consumption rate. Given the wealth level described by the time-changed process $X(t)$, we seek the optimal consumption rate $u^*(t)$, as indicated in (1.3), that maximizes the overall utility performance $J(u) = \mathbb{E}[\int_0^T \exp(-\delta t)u(t)^2\, dt]$.

Then we extend our result to a more general time-changed stochastic process involving a time drift dt term:

$$\begin{aligned} dX(t) &= \mu(t, E_t, X(t-), u(t))\, dt + b(t, E_t, X(t-), u(t))\, dE_t \\ &\quad + \sigma(t, E_t, X(t-), u(t))\, dB_{E_t} + \int_{|y|<c} \gamma(t, E_t, X(t-), u(t), y)\, \tilde{N}(dE_t, dy) \end{aligned}$$

with $X(0) = x_0 \neq 0$, and the corresponding performance function

$$\begin{aligned} &J(u) \\ &= \mathbb{E}\Big[\int_0^T f(t, E_t, X(t), u(t))\, dt + \int_0^T g(t, E_t, X(t), u(t))\, dE_t + h(X(T))\Big], \quad u \in \mathcal{A}. \end{aligned}$$

The current stochastic problem framework is built upon the combination of a backward stochastic differential equation and Hamiltonian equations [1], [2], [14]. This paper extends the existing literature in two aspects. First, to our best knowledge, we are the first to study the backward stochastic differential equation with inverse subordinator as the time change, which paves the way to further time-changed stochastic control problems. Second, we develop a maximum principle that provides a framework for solving time-changed stochastic control problems. We also provide examples to illustrate the importance of different components in determining the optimal solution.

As for the remaining parts of this paper, some necessary concepts and preliminary results will be given in Section 2. In Sections 3 and 4, we develop a maximum principle for the time-changed stochastic control problems mentioned above and provide some examples for illustration.

2. PRELIMINARIES

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space satisfying the usual hypotheses of completeness and right continuity. Assume that an independent $\mathcal{F}_t$-adapted Poisson random measure N is defined on $\mathbb{R}_+ \times (\mathbb{R} - \{0\})$ with compensator $\tilde{N}$ and intensity measure ν, where ν is a Lévy measure such that $\tilde{N}(dt, dy) = N(dt, dy) - \nu(dy)\,dt$ and $\int_{\mathbb{R}-\{0\}} (|y|^2 \wedge 1)\,\nu(dy) < \infty$.

Let $\{D(t),\, t \geqslant 0\}$ be a right continuous with left limits (RCLL) subordinator starting from 0 with Laplace transform

$$\mathbb{E}e^{-\lambda D(t)} = e^{-t\phi(\lambda)}, \tag{2.1}$$

where the Laplace exponent $\phi(\lambda)$ is $\int_0^\infty (1 - e^{-\lambda x})\,\nu(dx)$, and define its inverse

$$E_t := \inf\{\tau > 0 : D(\tau) > t\}. \tag{2.2}$$

LEMMA 2.1 ([7, Lemma 8]). *Let $\{E_t,\, t \geqslant 0\}$ be the inverse of a subordinator $\{D(t), t \geqslant 0\}$ with Laplace exponent ϕ and infinite Lévy measure. Then $\mathbb{E}[e^{\lambda E_t}] < \infty$ for all $\lambda \in \mathbb{R}$ and $t \geqslant 0$. In particular, for each $t > 0$, the moments of E_t of all orders exist and are given by*

$$\mathbb{E}[E_t^n] = \mathcal{L}_s^{-1}\left[\frac{n!}{s\phi^n(s)}\right](t), \qquad n \in \mathbb{N}, \tag{2.3}$$

where $\mathcal{L}_s^{-1}[g(s)]$ denotes the inverse Laplace transform of a function $g(s)$.

Consider the following time-changed stochastic differential equation:

$$\begin{aligned} dX(t) &= b(t, E_t, X(t-), u(t))\,dE_t + \sigma(t, E_t, X(t-), u(t))\,dB_{E_t} \\ &\quad + \int_{|y|<c} \gamma(t, E_t, X(t-), u(t), y)\,\tilde{N}(dE_t, dy) \end{aligned} \tag{2.4}$$

with $X(0) = x_0 \neq 0$, where b, σ, γ are real-valued functions satisfying Assumptions 2.1 and 2.2 below, so that there exists a unique $\mathcal{G}_t$-adapted process $X(t)$ satisfying time-changed SDE (1.1) (see [6, Lemma 4.1]). The filtration $\{\mathcal{G}_t\}_{t\geqslant 0}$ is defined as

$$\mathcal{G}_t = \bigcap_{u>t} \{[\mathcal{F}_y : 0 \leqslant y \leqslant u] \vee \sigma[E_y : y \geqslant 0]\}. \tag{2.5}$$

ASSUMPTION 2.1 (Lipschitz condition). *There exists a positive constant K such that*

$$|b(t_1,t_2,x,u)-b(t_1,t_2,y,u)|^2+|\sigma(t_1,t_2,x,u)-\sigma(t_1,t_2,y,u)|^2 + \int\limits_{|z|<c} |\gamma(t_1,t_2,x,u,z)-\gamma(t_1,t_2,y,u,z)|^2\,\nu(dz) \leqslant K|x-y|^2 \tag{2.6}$$

for all $t_1, t_2 \in \mathbb{R}_+$ *and* $x, y \in \mathbb{R}$.

ASSUMPTION 2.2. *If* $X(t)$ *is an RCLL and* $\mathcal{G}_t$*-adapted process, then*

$$b(t,E_t,X(t),u(t)), \sigma(t,E_t,X(t),u(t)), \gamma(t,E_t,X(t),u(t),y) \in \mathcal{L}(\mathcal{G}_t), \tag{2.7}$$

where $\mathcal{L}(\mathcal{G}_t)$ *denotes the class of left continuous with right limits* (*LCRL*) *and* $\mathcal{G}_t$*-adapted processes.*

The process $u(t)=u(t,w)\in U\subset\mathbb{R}$ is the control. Assume that u is adapted and RCLL, and that the corresponding equation (1.1) has a unique strong solution $X^{(u)}(t), t \in [0,T]$. Such controls are called *admissible*. The set of admissible controls is denoted by $\mathcal{A}$.

LEMMA 2.2 (Itô formula for time-changed Lévy noise, [12, Lemma 3.1]). *Let* $D(t)$ *be an RCLL subordinator and* E_t *its inverse process as in* (2.2)*. Let* X *be the process defined as follows:*

$$X(t)=x_0+\int_0^t \mu(t,E_t,X(t-))\,dt+\int_0^t b(t,E_t,X(t-))\,dE_t + \int_0^t \sigma(t,E_t,X(t-))\,dB_{E_t}+\int_0^t\int\limits_{|y|<c} \gamma(t,E_t,X(t-),y)\,\tilde{N}(dE_t,dy), \tag{2.8}$$

where μ, b, σ, γ *are measurable functions such that all integrals are defined. Here* c *is the maximum allowable jump size. Then, for all* $F\in C^{1,1,2}(\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R},\mathbb{R})$*, with probability* 1*,*

$$\begin{aligned} &F(t,E_t,X(t))-F(0,0,x_0) \\ &= \int_0^t L_1F(s,E_s,X(s-))\,ds+\int_0^t L_2F(s,E_s,X(s-))\,dE_s \\ &+\int_0^t\int\limits_{|y|<c} \big[F(s,E_s,X(s-)+\gamma(s,E_s,X(s-),y))-F(s,E_s,X(s-))\big]\,\tilde{N}(dE_s,dy) \\ &+\int_0^t F_x(s,E_s,X(s-))\sigma(s,E_s,X(s-))\,dB_{E_s}, \end{aligned} \tag{2.9}$$

where

$$L_1F(t_1,t_2,x) = F_{t_1}(t_1,t_2,x) + F_x(t_1,t_2,x)\mu(t_1,t_2,x),$$
$$\begin{aligned}L_2F(t_1,t_2,x) = {} & F_{t_2}(t_1,t_2,x) + F_x(t_1,t_2,x)b(t_1,t_2,x)\\ & + \tfrac{1}{2}F_{xx}(t_1,t_2,x)\sigma(t_1,t_2,x)^2\\ & + \int\limits_{|y|<c} \big[F(t_1,t_2,x+\gamma(t_1,t_2,x,y)) - F(t_1,t_2,x)\\ & \qquad - F_x(t_1,t_2,x)\gamma(t_1,t_2,x,y)\big]\,\nu(dy).\end{aligned}$$

LEMMA 2.3 (Existence and uniqueness for BSDE). *Consider the following time-changed backward stochastic differential equation:*

$$\begin{aligned}dX(t) = {} & -\mu(t,E_t,X(t-),u(t))\,dE_t + u(t)\,dB_{E_t}\\ & + \int\limits_{\mathbb{R}\setminus\{0\}} h(t,z)\,\tilde{N}(dE_t,dz),\end{aligned} \tag{2.10}$$

with $X(T) = X$*, where* $\mu \in L^2(\mathbb{R}_+,\mathbb{R}_+,\mathbb{R},\mathbb{R})$ *and* $h \in L^2(\mathbb{R}_+,\mathbb{R})$*. If there exists a positive constant* $L_\mu > 0$ *such that* $|\mu(t_1,t_2,x_1,u_1) - \mu(t_1,t_2,x_2,u_2)| \leqslant L_\mu(|x_1-x_2|+|u_1-u_2|)$*, then there exists a unique solution* $(X(t),u(t))$ *of* (2.10).

Proof. To prove uniqueness, suppose $(X_1(t),u_1(t))$ and $(X_2(t),u_2(t))$ are two solutions to (2.10) in $L^2(\Omega\times\mathbb{R}_+)\times L^2(\Omega\times\mathbb{R}_+)$. By the Itô formula,

$$\begin{aligned}|X_1(T)-X_2(T)|^2 - |X_1(t)-X_2(t)|^2 = {} & \int_t^T |u_1(s)-u_2(s)|^2\,dE_s\\ + \int_t^T 2(X_1(s)-X_2(s))\big[-\big(\mu(s,E_s,X_1(s),u_1(s)) & - \mu(s,E_s,X_2(s),u_2(s))\big)\,dE_s\\ & + (u_1(s)-u_2(s))\,dB_{E_s}\big]\end{aligned} \tag{2.11}$$

Thus,

$$\begin{aligned}|X_1(t)-X_2(t)|^2 & + \int_t^T |u_1(s)-u_2(s)|^2\,dE_s\\ & + \int_t^T 2(X_1(s)-X_2(s))\big(u_1(s)-u_2(s)\big)\,dB_{E_s}\\ = {} & \int_t^T 2(X_1(s)-X_2(s))\big(\mu(s,E_s,X_1(s),u_1(s)) - \mu(s,E_s,X_2(s),u_2(s))\big)\,dE_s\\ \leqslant {} & \int_t^T 2L_\mu|X_1(s)-X_2(s)|\big(|X_1(s)-X_2(s)|+|u_1-u_2|\big)\,dE_s\\ \leqslant {} & \int_t^T 2L_\mu\left[|X_1(s)-X_2(s)|^2 + \frac{L_\mu}{2}|X_1(s)-X_2(s)|^2 + \frac{1}{2L_\mu}|u_1(s)-u_2(s)|^2\right]dE_s\\ = {} & (2L_\mu+L_\mu^2)\int_t^T |X_1(s)-X_2(s)|^2\,dE_s + \int_t^T |u_1(s)-u_2(s)|^2\,dE_s.\end{aligned} \tag{2.12}$$

Take expectations on both sides to get

$$\mathbb{E}[|X_1(t)-X_2(t)|^2] \leqslant (2L_\mu + L_\mu^2)\mathbb{E}\Big[\int_t^T |X_1(s)-X_2(s)|^2\,dE_s\Big]. \tag{2.13}$$

Note that we apply the martingale property to derive inequality (2.13) and give some details below.

$$\begin{aligned}
&\int_t^T (X_1(s)-X_2(s))(u_1(s)-u_2(s))\,dB_{E_s}\\
&= \int_0^\infty 1_{\{t\leqslant s\leqslant T\}}(X_1(s)-X_2(s))\big(u_1(s)-u_2(s)\big)\,dB_{E_s}\\
&= \int_0^\infty 1_{\{t\leqslant D(s-)\leqslant T\}}\big(X_1(D(s-))-X_2(D(s-))\big)\big(u_1(D(s-))-u_2(D(s-))\big)\,dB_s,
\end{aligned}$$

since $(X_1(t),u_1(t))$ and $(X_2(t),u_2(t))$ are in $L^2(\Omega\times\mathbb{R}_+)$,

$$\begin{aligned}
&\mathbb{E}\int_0^\infty \big|1_{\{t\leqslant D(s-)\leqslant T\}}\big(X_1(D(s-))-X_2(D(s-))\big)\big(u_1(D(s-))-u_2(D(s-))\big)\big|^2 ds\\
&\qquad\leqslant \mathbb{E}\int_0^\infty \big|(X_1(D(s-))-X_2(D(s-)))\big(u_1(D(s-))-u_2(D(s-))\big)\big|^2 ds < \infty,
\end{aligned}$$

we have

$$\begin{aligned}
&\mathbb{E}\int_t^T (X_1(s)-X_2(s))(u_1(s)-u_2(s))\,dB_{E_s}\\
&= \mathbb{E}\int_0^\infty 1_{\{t\leqslant D(s-)\leqslant T\}}\big(X_1(D(s-))-X_2(D(s-))\big)\big(u_1(D(s-))-u_2(D(s-))\big)\,dB_s\\
&= 0.
\end{aligned}$$

Next we apply time-changed Gronwall's method of [17, Lemma 3.1]. Define $F(t)=\int_t^T |X_1(s)-X_2(s)|^2\,dE_s$. Then $F(T)=0$ and

$$\begin{aligned}
-d(F(t)\exp(kE_t)) &= -\exp(kE_t)\,dF(t) - k\exp(kE_t)F(t)\,dE_t\\
&= \exp(kE_t)\Big(|X_1(t)-X_2(t)|^2 - k\int_t^T |X_1(s)-X_2(s)|^2\,dE_s\Big)\,dE_t,
\end{aligned}$$

thus

$$\begin{aligned}
&-F(T)\exp(kE_T) + F(t)\exp(kE_t)\\
&\qquad= \int_t^T \Big[\exp(kE_s)\Big(|X_1(s)-X_2(s)|^2 - k\int_s^T |X_1(u)-X_2(u)|^2\,dE_u\Big)\Big]\,dE_s.
\end{aligned}$$

Taking expectations and letting $k = 2L_\mu + L_\mu^2$ implies that

$$\begin{aligned}
&\mathbb{E}[F(t)\exp(kE_t)] \\
&= \mathbb{E}\Big[\int_t^T \exp(kE_s)\Big(|X_1(s) - X_2(s)|^2 - k\int_s^T |X_1(u) - X_2(u)|^2\, dE_u\Big)\, dE_s\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\int_t^T \exp(kE_s)\Big(|X_1(s) - X_2(s)|^2 \\
&\qquad\qquad - k\int_s^T |X_1(u) - X_2(u)|^2\, dE_u\Big)\, dE_s\Big] \,\Big|\, \{\sigma(E_s, s \in (t,T))\}\Big] \\
&= \mathbb{E}\Big[\int_t^T \exp(kE_s)\mathbb{E}\Big(|X_1(s) - X_2(s)|^2 \\
&\qquad\qquad\qquad - k\int_s^T |X_1(u) - X_2(u)|^2\, dE_u\Big)\, dE_s \,\Big|\, \{\sigma(E_s, s \in (t,T))\}\Big] \\
&\leqslant 0.
\end{aligned}$$

It follows that

$$\mathbb{E}[F(t)] \leqslant \mathbb{E}[F(t)\exp(kE_t)] \leqslant 0, \tag{2.14}$$

so $X_1(s) = X_2(s)$ a.s. for all $s \in (t,T)$. By (2.11), since $X_1(s) = X_2(s)$ for a.e. $s \in (t,T)$, we have $\int_t^T |u_1(s) - u_2(s)|^2\, dE_s = 0$, thus $u_1(s) = u_2(s)$ for a.e. $s \in (t,T)$. The uniqueness is proved.

To prove the existence, let $u_0(t) = 0$, and $\{(X_n(t), u_n(t)) : 0 \leqslant t \leqslant T\}_{n\geqslant 1}$ be a sequence defined recursively by

$$\begin{aligned}
X_{n-1}(t) - X_n(t) = -\Big[\int_t^T \mu(s, E_s, X_{n-1}(s), u_{n-1}(s))\, dE_s - \int_t^T u_{n-1}(s)\, dB_{E_s} \\
- \int_t^T \int_{\mathbb{R}\setminus\{0\}} h(s,z)\, \tilde{N}(dE_s, dz)\Big].
\end{aligned}$$

Then

$$\begin{cases}
dX_n(t) = -\mu(t, E_t, X_{n-1}(t), u_{n-1}(t))\, dE_t + u_{n-1}(t)\, dB_{E_t} \\
\qquad\qquad + \int_{\mathbb{R}\setminus\{0\}} h(t,z)\, \tilde{N}(dE_t, dz), \\
dX_{n+1}(t) = -\mu(t, E_t, X_n(t), u_n(t))\, dE_t + u_n(t)\, dB_{E_t} \\
\qquad\qquad + \int_{\mathbb{R}\setminus\{0\}} h(t,z)\, \tilde{N}(dE_t, dz), \\
X_n(T) = X_{n+1}(T) = X.
\end{cases}$$

By the Itô formula of Lemma 2.2, there exists $k > 0$ such that

$$
\begin{aligned}
&|X_{n+1}(t) - X_n(t)|^2 + \int_t^T (u_n(s) - u_{n-1}(s))^2 \, dE_s \\
&\qquad\qquad + 2\int_t^T (X_{n+1}(s) - X_n(s))(u_n(s) - u_{n-1}(s)) \, dB_{E_s} \\
&\quad = 2\int_t^T (X_{n+1}(s) - X_n(s)) \\
&\qquad \cdot (\mu(s, E_s, X_n(s), u_n(s)) - \mu(s, E_s, X_{n-1}(s), u_{n-1}(s))) \, dE_s \\
&\quad \leqslant 2L_\mu \int_t^T |X_{n+1}(s) - X_n(s)|(|X_n(s) - X_{n-1}(s)| + |u_n(s) - u_{n-1}(s)|) \, dE_s \\
&\quad \leqslant k\Big[\int_t^T |X_{n+1}(s) - X_n(s)|^2 \, dE_s + \int_t^T |X_n(s) - X_{n-1}(s)|^2 \, dE_s\Big] \\
&\qquad + \frac{1}{2}\int_t^T |u_n(s) - u_{n-1}(s)|^2 \, dE_s.
\end{aligned}
$$

Taking expectations on both sides implies

$$
\begin{aligned}
\text{(2.15)} \quad &\mathbb{E}|X_{n+1}(t) - X_n(t)|^2 + \frac{1}{2}\mathbb{E}\int_t^T |u_n(s) - u_{n-1}(s)|^2 \, dE_s \\
&\leqslant k\mathbb{E}\Big[\int_t^T |X_{n+1}(s) - X_n(s)|^2 \, dE_s + \int_t^T |X_n(s) - X_{n-1}(s)|^2 \, dE_s\Big].
\end{aligned}
$$

Define $F_n(t) = \int_t^T |X_n(s) - X_{n-1}(s)|^2 \, dE_s$ for all $n \geqslant 1$. Then $F_n(T) = 0$ and

$$
\begin{aligned}
-d\big(F_{n+1}(t)\exp(kE_t)\big) &= -\exp(kE_t)\, dF_{n+1}(t) - k\exp(kE_t)F_{n+1}(t)\, dE_t \\
&= \exp(kE_t)\Big[|X_{n+1}(t) - X_n(t)|^2 - k\int_t^T |X_{n+1}(s) - X_n(s)|^2 \, dE_s\Big] \, dE_t.
\end{aligned}
$$

By a similar argument to that for uniqueness and using (2.15),

$$
\begin{aligned}
\mathbb{E}[F_{n+1}(t)\exp(kE_t)] &= \mathbb{E}\Big[\int_t^T \exp(kE_s)\Big[|X_{n+1}(s) - X_n(s)|^2 \\
&\qquad\qquad - k\int_s^T |X_{n+1}(l) - X_n(l)|^2 \, dE_l\Big] \, dE_s\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\int_t^T \exp(kE_s)\Big[|X_{n+1}(s) - X_n(s)|^2 \\
&\qquad - k\int_s^T |X_{n+1}(l) - X_n(l)|^2 \, dE_l\Big] \, dE_s\Big] \,\Big|\, \{\sigma(E_s, s \in (t,T))\}\Big]
\end{aligned}
$$

$$= \mathbb{E}\Big[\int_t^T \exp(kE_s)\mathbb{E}\Big[|X_{n+1}(s) - X_n(s)|^2 - k\int_s^T |X_{n+1}(l) - X_n(l)|^2\, dE_l\Big]\, dE_s \,\Big|\, \{\sigma(E_s, s \in (t,T))\}\Big]$$

$$\leqslant \mathbb{E}\Big[\int_t^T \exp(kE_s)k\mathbb{E}\Big[\int_s^T |X_n(l) - X_{n-1}(l)|^2\, dE_l\Big]\, dE_s \,\Big|\, \{\sigma(E_s, s \in (t,T))\}\Big]$$

$$= \mathbb{E}\Big[\int_t^T k\exp(kE_s)\mathbb{E}[F_n(s)]\, dE_s \,\Big|\, \{\sigma(E_s, s \in (t,T))\}\Big]$$

$$= \mathbb{E}\Big[\int_t^T k\exp(kE_s)F_n(s)\, dE_s\Big],$$

and letting $t = 0$ we get

$$\mathbb{E}F_{n+1}(0) \leqslant \mathbb{E}\int_0^T ke^{kE_s}F_n(s)\, dE_s \leqslant \mathbb{E}\left[(e^{kE_T})^n \frac{F_1(0)}{n!}\right] \to 0$$

as $n \to \infty$. Thus, $\{X_n\}$ is a Cauchy sequence in $L^2(\Omega \times \mathbb{R}_+)$. Taking (2.15) into consideration, $\{u_n\}$ is also a Cauchy sequence in $L^2(\Omega \times \mathbb{R}_+)$. Thus, the existence of solution to (2.10) is proved. ■

3. TIME-CHANGED STOCHASTIC CONTROL PROBLEM

In this section, we solve a time-changed stochastic control problem through the maximum principle approach. An example is provided to illustrate how our method works in a particular case.

We consider a performance criterion $J = J(u)$ of the form

$$J(u) = \mathbb{E}\Big[\int_0^T g(t, E_t, X(t), u(t))\, dE_t + h(X(T))\Big], \qquad u \in \mathcal{A}, \tag{3.1}$$

where $g : [0,T] \times \mathbb{R}_+ \times \mathbb{R} \times U \to \mathbb{R}$ is continuous, $h : \mathbb{R} \to \mathbb{R}$ is C^1, $T < \infty$ is a fixed deterministic time and

$$\mathbb{E}\Big[\int_0^T g(t, E_t, X(t), u(t))\, dE_t + h(X(T))\Big] < \infty, \qquad \forall u \in \mathcal{A}.$$

The stochastic control problem is to find an optimal control $u^* \in \mathcal{A}$ such that

$$J(u^*) = \sup_{u \in \mathcal{A}} J(u). \tag{3.2}$$

Since E_t is right continuous and nondecreasing, $\frac{dE_t}{dt}$ exists for $t \geqslant 0$ a.e.

Define the Hamiltonian $H : [0,T] \times \mathbb{R}_+ \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \times \mathcal{R}$ by

$$H(t_1,t_2,x,u,p,q,r) = g(t_1,t_2,x,u) + pb(t_1,t_2,x,u) + q\sigma(t_1,t_2,x,u) + \int_{\mathbb{R}} \gamma(t_1,t_2,x,u,z)r(t_2,z)\,\nu(dz), \tag{3.3}$$

or

$$\begin{aligned} &H(t,E_t,X(t),u(t),p(t),q(t),r(t,z)) \\ &\quad = g(t,E_t,X(t),u(t)) + p(t)b(t,E_t,X(t),u(t)) \\ &\qquad + q(t)\sigma(t,E_t,X(t),u(t)) + \int_{\mathbb{R}} \gamma(t,E_t,X(t),u(t),z)r(E_t,z)\,\nu(dz), \end{aligned}$$

where $\mathcal{R}$ is the set of functions $r : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that the integral in (3.3) exists.

Define the adjoint equation for the unknown processes $p(t) \in \mathbb{R}$, $q(t) \in \mathbb{R}$, and $r(t,z) \in \mathbb{R}$ to be the backward stochastic differential equation

$$\begin{aligned} dp(t) &= -H_x(t,E_t,X(t),u(t),p(t),q(t),r(t,\cdot))\,dE_t \\ &\qquad + q(t)\,dB_{E_t} + \int_{\mathbb{R}} r(E_t,z)\,\tilde{N}(dE_t,dz), \qquad t < T, \\ p(T) &= h_x(X(T)). \end{aligned} \tag{3.4}$$

THEOREM 3.1 (Time-changed maximum principle). *Let $\hat{u} \in \mathcal{A}$ with corresponding solution $\hat{X} = X^{(\hat{u})}$ of* (1.1) *and suppose there exists a solution $(\hat{p}(t), \hat{q}(t), \hat{r}(t,z))$ of the corresponding adjoint equation* (3.4) *satisfying*

$$\mathbb{E}\Big[\int_0^T (\hat{X}(t) - X^{(u)}(t))^2 \Big(\hat{q}(t)^2 + \int_{\mathbb{R}} \hat{r}(E_t,z)^2\,\nu(dz)\Big)\,dE_t\Big] < \infty$$

and

$$\mathbb{E}\Big[\int_0^T \hat{p}(t)^2 \Big(\sigma(t,E_t,X^{(u)}(t),u(t))^2 + \int_{\mathbb{R}} \gamma(t,E_t,X^{(u)}(t),u(t),z)^2\,\nu(dz)\Big)\,dE_t\Big] < \infty,$$

for $u \in \mathcal{A}$. Moreover, suppose that

$$H(t,E_t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) = \sup_{v\in U} H(t,E_t,\hat{X}(t),v,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))$$

for all t, $h(x)$ in (3.1) *is a concave function of x, and*

$$\hat{H}(x) := \max_{v\in U} H(t_1,t_2,x,v,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))$$

exists and is a concave function of x for all $t \in [0,T]$. Then $\hat{u}$ is an optimal control of the stochastic control problem (3.2).

Proof. Let $u \in \mathcal{A}$ be an admissible control with the corresponding state process $X(t) = X^{(u)}(t)$. We would like to show that

$$J(\hat{u}) - J(u) = \mathbb{E}\Big[\int_0^T [g(t, E_t, \hat{X}(t), \hat{u}(t)) - g(t, E_t, X(t), u(t))]\, dt + h(\hat{X}(T)) - h(X(T))\Big] \geqslant 0. \tag{3.5}$$

Since g is concave, using the Itô formula (2.9) we obtain

$$\begin{aligned}
&\mathbb{E}[h(\hat{X}(T)) - h(X(T))] \\
&\geqslant \mathbb{E}[h_x(\hat{X}(T))(\hat{X}(T) - X(T))] = \mathbb{E}[(\hat{X}(T) - X(T))\hat{P}(T)] \\
&= \mathbb{E}\Big[\int_0^T (\hat{X}(t) - X(t))\, d\hat{p}(t) + \int_0^T \hat{p}(t)\, d(\hat{X}(t) - X(t)) + \int_0^T d\hat{p}(t)\, d(\hat{X}(t) - X(t))\Big] \\
&= \mathbb{E}\Big[\int_0^T (\hat{X}(t) - X(t))\, d\hat{p}(t) + \int_0^T \hat{p}(t)\, d(\hat{X}(t) - X(t)) \\
&\quad + \int_0^T \hat{q}(t)\big(\sigma(t, E_t, \hat{X}(t), \hat{u}(t)) - \sigma(t, E_t, X(t), u(t))\big)\, dE_t \\
&\quad + \int_0^T \int_{\mathbb{R}} \hat{r}(t,z)\big(\gamma(t, E_t, \hat{X}(t), \hat{u}(t)) - \gamma(t, E_t, X(t), u(t))\big)\, \nu(dz)\, dE_t\Big].
\end{aligned}$$

We have

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \hat{p}(t)\, d(\hat{X}(t) - X(t))\Big] \\
&\qquad = \mathbb{E}\Big[\int_0^T \hat{p}(t)\big(b(t, E_t, \hat{X}(t), u(t)) - b(t, E_t, X(t), u(t))\big)\, dE_t\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}
&J(\hat{u}) - J(u) \\
&= \mathbb{E}\Big[\int_0^T (\hat{X}(t) - X(t))\, d\hat{p}(t) + \int_0^T \big(g(t, E_t, \hat{X}(t), \hat{u}(t)) - g(t, E_t, X(t), u(t))\big)\, dE_t \\
&\quad + \int_0^T \hat{p}(t)\big(b(t, E_t, \hat{X}(t), \hat{u}(t)) - b(t, E_t, X(t), u(t))\big)\, dE_t \\
&\quad + \int_0^T \hat{q}(t)\big(\sigma(t, E_t, \hat{X}(t), \hat{u}(t)) - \sigma(t, E_t, X(t), u(t))\big)\, dE_t \\
&\quad + \int_0^T \int_{\mathbb{R}} \hat{r}(t,z)\big(\gamma(t, E_t, \hat{X}(t), \hat{u}(t)) - \gamma(t, E_t, X(t), u(t))\big)\, \nu(dz)\, dE_t\Big].
\end{aligned} \tag{3.6}$$

In addition,

(3.7)
$$\begin{aligned}
&H(t,E_t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t)) - H(t,E_t,X(t),u(t),\hat{p}(t),\hat{q}(t),\hat{r}(t))\\
&\quad= \big(g(t,E_t,\hat{X}(t),\hat{u}(t)) - g(t,E_t,X(t),u(t))\big)\\
&\qquad+ \hat{p}(t)\big(b(t,E_t,\hat{X}(t),\hat{u}(t)) - b(t,E_t,X(t),u(t))\big)\\
&\qquad+ \hat{q}(t)\big(\sigma(t,E_t,\hat{X}(t),\hat{u}(t)) - \sigma(t,E_t,X(t),u(t))\big)\\
&\qquad+ \int_{\mathbb{R}} \hat{r}(t,z)\big(\gamma(t,E_t,\hat{X}(t),\hat{u}(t)) - \gamma(t,E_t,X(t),u(t))\big)\,\nu(dz),
\end{aligned}$$

and by (3.4) we have

$$\begin{aligned}
(\hat{X}(t) - X(t))d\hat{p}(t) &= \hat{X}(t)d\hat{p}(t) - X(t)d\hat{p}(t)\\
&= \hat{X}(t)\Big[-H_x(t,E_t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\,dE_t\\
&\qquad+ \hat{q}(t)\,dB_{E_t} + \int_{\mathbb{R}} \hat{r}(t,z)\,\tilde{N}(dE_t,dz)\Big]\\
&\qquad- X(t)\Big[-H_x(t,E_t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\,dE_t\\
&\qquad+ \hat{q}(t)\,dB_{E_t} + \int_{\mathbb{R}} \hat{r}(t,z)\,\tilde{N}(dE_t,dz)\Big]\\
&= -(\hat{X}(t) - X(t))H_x(t,E_t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\,dE_t\\
&\qquad+ (\hat{X}(t) - X(t))\Big(\hat{q}(t)\,dB_{E_t} + \int_{\mathbb{R}} \hat{r}(t,z)\,\tilde{N}(dE_t,dz)\Big).
\end{aligned} \tag{3.8}$$

Then, since H is concave in x, inserting (3.7) and (3.8) into (3.6) and following the proof in [4], we get

$$\begin{aligned}
J(\hat{u}) - J(u) = \mathbb{E}\Big[&\int_0^T -(\hat{X}(t) - X(t))H_x(t,E_t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\,dE_t\\
&+ \int_0^T \big[H(t,E_t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\\
&- H(t,E_t,X(t),u(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\big]\,dE_t\\
\geqslant 0.\quad \blacksquare
\end{aligned}$$

REMARK 3.1. The maximum principle suggests that the optimal control can be solved using the Hamiltonian framework, which is a boundary value problem and a maximum condition of a function called the Hamiltonian. The advantage of the maximum principle is that maximizing the Hamiltonian is easier and more feasible than directly solving the original stochastic control problem. This leads to closed form solutions for certain classes of optimal control problems.

EXAMPLE 3.1 (The time-changed stochastic linear regulator problem). The linear regulator problem aims to reduce the amount of work or energy consumed by the control system to optimize the controller. In this example, we consider the following time-changed stochastic linear regulator problem:

$$\Phi(x_0) = \inf_{u \in \mathcal{A}} \mathbb{E}\left[\int_0^T \frac{X(t)^2 + u(t)^2}{2}\, dE_t + \lambda X(T)^2\right],$$

where

$$dX(t) = u(t)\, dE_t + \sigma\, dB_{E_t} + \int_{\mathbb{R}} z\, \tilde{N}(dE_t, dz), \qquad X(0) = x_0.$$

Consider the Hamiltonian

$$H(t_1, t_2, x, u, p, q, r) = \frac{x^2 + u^2}{2} + pu + \sigma q + \int_{\mathbb{R}} \gamma z\, \nu(dz).$$

The adjoint equations are

$$\begin{cases} dp(t) = -X(t)\, dE_t + q(t)\, dB_{E_t} + \int_{\mathbb{R}} r(E_t, z)\, \tilde{N}(dE_t, dz), \\ p(T) = 2\lambda X(T). \end{cases} \tag{3.9}$$

The first and second order condition implies that the *Hamiltonian* $H(t_1, t_2, x, u, p, q, r)$ achieves its minimum at $u^*(t) = -p(t)$.

To find an explicit solution of $u^*(t)$, suppose $p(t) = h(E_t)X(t)$, where $h : \mathbb{R}_+ \to \mathbb{R}_+$. Then $u^*(t) = -h(E_t)X(t)$ and

$$\begin{aligned} dp(t) &= h(E_t)dX(t) + h'(E_t)X(t)\, dE_t \\ &= h(E_t)\Big(u(t)\, dE_t + \sigma\, dB_{E_t} + \int_{\mathbb{R}} z\, \tilde{N}(dE_t, dz)\Big) + h'(E_t)X(t)\, dE_t \\ &= X(t)(-h(E_t)^2 + h'(E_t))\, dE_t + h(E_t)\sigma\, dB_{E_t} + h(E_t) \int_{\mathbb{R}} z\, \tilde{N}(dE_t, dz). \end{aligned} \tag{3.10}$$

Comparing (3.9) and (3.10) yields $-h(E_t)^2 + h'(E_t) = -1$ and $h(E_T) = 2\lambda$. The general solution to this ordinary differential equation gives

$$h(E_t) = -\frac{2\lambda - 1 + (2\lambda + 1)e^{2(E_t - E_T)}}{2\lambda - 1 - (2\lambda + 1)e^{2(E_t - E_T)}}. \tag{3.11}$$

Thus, we have an explicit formula for the optimal control, $u^*(t) = -h(E_t)X(t)$. Similarly, $q(t) = h(E_t)\sigma$ and $r(E_t, z) = h(E_t)z$. A simulation of the optimal control $u^*(t)$ with $\lambda = -1/2, \sigma = 1, x_0 = -.01$, standard normal distribution ν, and inverse stable subordinator $E(t)$ having $\alpha = .9$ is displayed in Figure 2.

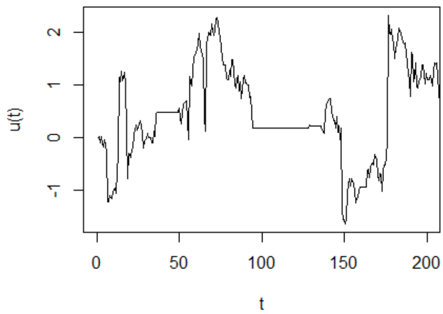

FIGURE 2. Simulation of $u^*(t)$ for Example 1, $\alpha = 0.9$

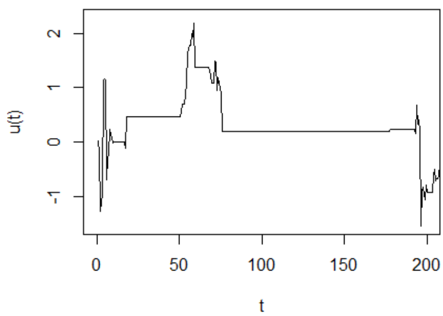

FIGURE 3. Simulation of $u^*(t)$ for Example 1, $\alpha = .7$

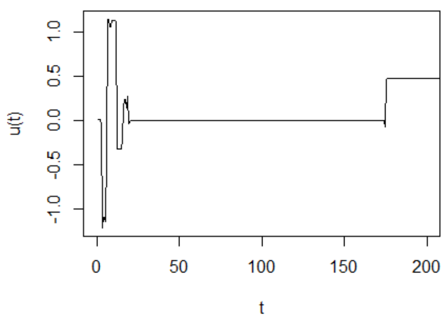

FIGURE 4. Simulation of $u^*(t)$ for Example 1, $\alpha = .5$

Keeping all other parts the same as in Figure 2, we also simulate the optimal control $u^*(t)$ for $\alpha = .7$ and $\alpha = .5$ in Figures 3 and 4, respectively. Overall, replacing t by E_t would only insert some constant periods into the original process. As α gets closer to 1, the constant periods vanish gradually.

REMARK 3.2. To demonstrate the above example in an intuitive way, we simplify the specification by letting $\lambda = 1/2$, $\sigma = 1$, and $z = 0$. The example problem becomes seeking the optimal control of the energy consumption system:

$$\Phi(x_0) = \inf_{u \in \mathcal{A}} \mathbb{E}\left[\int_0^T \frac{X(t)^2 + u(t)^2}{2}\, dE_t + \frac{X(T)^2}{2}\right], \tag{3.12}$$

where

$$dX(t) = u(t)\, dE_t + dB_{E_t}, \qquad X(0) = x_0. \tag{3.13}$$

In this case, $h(E_t) = 1$ and $u^*(t) = -X(t)$. Thus, the optimal control is $du^*(t) = -u^*(t)\, dE_t - dB_{E_t}$, which means that the optimal control keeps the energy consumption constant over time.

4. A MORE GENERAL TIME-CHANGED STOCHASTIC CONTROL PROBLEM

Now we extend the time-changed SDE (1.1) to a more general case by adding a time drift term:

$$\begin{aligned} dX(t) &= \mu(t, E_t, X(t-), u(t))\, dt + b(t, E_t, X(t-), u(t))\, dE_t \\ &\quad + \sigma(t, E_t, X(t-), u(t))\, dB_{E_t} + \int_{|y|<c} \gamma(t, E_t, X(t-), u(t), y)\, \tilde{N}(dE_t, dy) \end{aligned}$$

with $X(0) = x_0 \neq 0$, where μ, b, σ, γ are real-valued functions satisfying Assumptions 2.1 and 2.2.

Suppose the performance function is given by

$$\begin{aligned} J(u) = \mathbb{E}\Big[\int_0^T f(t, E_t, X(t), u(t))\, dt \\ + \int_0^T g(t, E_t, X(t), u(t))\, dE_t + h(X(T))\Big], \qquad u \in \mathcal{A}, \end{aligned} \tag{4.1}$$

where the functions $f, g : [0, T] \times \mathbb{R}_+ \times \mathbb{R} \times U \to \mathbb{R}$ are continuous, $h : \mathbb{R} \to \mathbb{R}$ is C^1, $T < \infty$ is a fixed deterministic time and

$$\begin{aligned} \mathbb{E}\Big[\int_0^T f(t, E_t, X(t), u(t))\, dt \\ + \int_0^T g(t, E_t, X(t), u(t))\, dE_t + h(X(T))\Big] < \infty, \qquad \forall u \in \mathcal{A}. \end{aligned}$$

The stochastic control problem is to find an optimal control $u^* \in \mathcal{A}$ such that

$$J(u^*) = \sup_{u \in \mathcal{A}} J(u). \tag{4.2}$$

REMARK 4.1. The performance functions (3.1) and (4.1) are slightly different in terms of their integral kernels. This difference results in different Hamiltonians and adjoint equations.

Define the Hamiltonian $H : [0,T] \times \mathbb{R}_+ \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \to \mathbb{R}$ by

$$\begin{aligned} H(t_1,t_2,x,u,p,q,r) &= \big(p\mu(t_1,t_2,x,u) + f(t_1,t_2,x,u)\big) \\ &\quad + \big(pb(t_1,t_2,x,u) + q\sigma(t_1,t_2,x,u) + g(t_1,t_2,x,u)\big)\frac{dt_2}{dt_1} \\ &\quad + \int_{\mathbb{R}} \gamma(t_1,t_2,x,u,z)r(t,z)\,\nu(dz)\,\frac{dt_2}{dt}, \end{aligned}$$

or

$$\begin{aligned} &H(t,E_t,X(t),u(t),p(t),q(t),r(t,z)) \\ &\ = \big(p(t)\mu(t,E_t,X(t),u(t)) + f(t,E_t,X(t),u(t))\big) \\ &\quad + \big(p(t)b(t,X(t),u(t)) + q(t)\sigma(t,E_t,X(t),u(t)) + g(t,E_t,X(t),u(t))\big)\frac{dE_t}{dt} \\ &\quad + \int_{\mathbb{R}} \gamma(t,E_t,X(t),u(t),z)r(t,z)\,\nu(dz)\,\frac{dE_t}{dt}. \end{aligned}$$

Define the adjoint equation to be

$$\begin{aligned} dp(t) &= -H_x(t,E_t,X(t),u(t),p(t),q(t),r(t,\cdot))\,dt \\ &\qquad + q(t)\,dB_{E_t} + \int_{\mathbb{R}} r(t,z)\,\tilde{N}(dE_t,dz), \qquad t<T, \\ p(T) &= h_x(X(T)). \end{aligned}$$

THEOREM 4.1 (Time-changed maximum principle). *Let $\hat{u} \in \mathcal{A}$ with corresponding solution $\hat{X} = X^{(\hat{u})}$ and suppose there exists a solution $(\hat{p}(t),\hat{q}(t),\hat{r}(t,z))$ of the corresponding adjoint equation* (3.4) *satisfying*

$$\mathbb{E}\Big[\int_0^T (\hat{X}(t) - X^{(u)}(t))^2\Big(\hat{q}(t)^2 + \int_{\mathbb{R}} \hat{r}(t,z)^2\,\nu(dz)\Big)\,dE_t\Big] < \infty \tag{4.3}$$

and for all $u \in \mathcal{A}$,

$$\begin{aligned} \mathbb{E}\Big[\int_0^T \hat{p}(t)^2\Big(&\sigma(t,E_t,X^{(u)}(t),u(t))^2 \\ &+ \int_{\mathbb{R}} \gamma(t,E_t,X^{(u)}(t),u(t),z)^2\,\nu(dz)\Big)\,dE_t\Big] < \infty. \end{aligned}$$

Moreover, suppose that

$$H(t,E_t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) = \sup_{v\in U} H(t,E_t,\hat{X}(t),v,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))$$

for all $t > 0$, $h(x)$ *in* (4.1) *is a concave function of* x, *and*

$$\hat{H}(x) := \max_{v \in U} H(t_1, t_2, x, v, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) \tag{4.4}$$

exists and is a concave function of x *for all* $t \in [0, T]$. *Then* $\hat{u}$ *is an optimal control of the stochastic control problem* (4.2).

Proof. Let $u \in \mathcal{A}$ be an admissible control with the corresponding state process $X(t) = X^{(u)}(t)$. We would like to show that

$$J(\hat{u}) - J(u) = \mathbb{E}\Big[\int_0^T [f(t, E_t, \hat{X}(t), \hat{u}(t)) - f(t, E_t, X(t), u(t))]\, dt$$
$$+ \int_0^T [g(t, E_t, \hat{X}(t), \hat{u}(t)) - g(t, E_t, X(t), u(t))]\, dE_t + h(\hat{X}(T)) - h(X(T))\Big] \geqslant 0.$$

Since h is concave, using the Itô formula (2.9) we get

$$\begin{aligned}
&\mathbb{E}[h(\hat{X}(T)) - g(X(T))] \geqslant \mathbb{E}[h_x(\hat{X}(T))(\hat{X}(T) - X(T))] \\
&\ = \mathbb{E}[(\hat{X}(T) - X(T))\hat{p}(T)] \\
&\ = \mathbb{E}\Big[\int_0^T (\hat{X}(t) - X(t))\, d\hat{p}(t) + \int_0^T \hat{p}(t)\, d(\hat{X}(t) - X(t)) + \int_0^T d\hat{p}(t)\, d(\hat{X}(t) - X(t))\Big] \\
&\ = \mathbb{E}\Big[\int_0^T (\hat{X}(t) - X(t))\, d\hat{p}(t) + \int_0^T \hat{p}(t)\, d(\hat{X}(t) - X(t)) \\
&\quad + \int_0^T \hat{q}(t)\big(\sigma(t, E_t, \hat{X}(t), \hat{u}(t)) - \sigma(t, E_t, X(t), u(t))\big)\hat{q}(t)\, dE_t \\
&\quad + \int_0^T \int_{\mathbb{R}} \hat{r}(t, z)\big(\gamma(t, E_t, \hat{X}(t), \hat{u}(t)) - \gamma(t, E_t, X(t), u(t))\big)\, \nu(dz)\, dE_t\Big].
\end{aligned}$$

We have

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \hat{p}(t)\, d(\hat{X}(t) - X(t))\Big] \\
&\qquad = \mathbb{E}\Big[\int_0^T \hat{p}(t)\Big(\big(\mu(t, E_t, \hat{X}(t), \hat{u}(t)) - \mu(t, E_t, X(t), u(t))\big)\, dt \\
&\qquad\qquad + \big(b(t, E_t, \hat{X}(t), \hat{u}(t)) - b(t, E_t, X(t), u(t))\big)\, dE_t\Big)\Big].
\end{aligned}$$

Thus,

$$
\begin{aligned}
&J(\hat{u}) - J(u) \\
&\quad = \mathbb{E}\Big[\int_0^T (\hat{X}(t) - X(t))\, d\hat{p}(t) + \int_0^T [f(t, E_t, \hat{X}(t), \hat{u}(t)) - f(t, E_t, X(t), u(t))]\, dt \\
&\qquad + \int_0^T [g(t, E_t, \hat{X}(t), \hat{u}(t)) - g(t, E_t, X(t), u(t))]\, dE_t \\
&\qquad + \int_0^T \hat{p}(t)\big[\big(\mu(t, E_t, \hat{X}(t), \hat{u}(t)) - \mu(t, E_t, X(t), u(t))\big)\, dt \\
&\qquad + \big(b(t, E_t, \hat{X}(t), \hat{u}(t)) - b(t, E_t, X(t), u(t))\big)\, dE_t\big] \\
&\qquad + \int_0^T \hat{q}(t)\big(\sigma(t, E_t, \hat{X}(t), \hat{u}(t)) - \sigma(t, E_t, X(t), u(t))\big)\, dE_t \\
&\qquad + \int_0^T \int_{\mathbb{R}} \hat{r}(t,z)(\gamma(t, E_t, \hat{X}(t), \hat{u}(t)) - \gamma(t, E_t, X(t), u(t)))\, \nu(dz)\, dE_t\Big].
\end{aligned}
$$

In addition,

$$
\begin{aligned}
&\big(H(t, E_t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t)) - H(t, E_t, X(t), u(t), \hat{p}(t), \hat{q}(t), \hat{r}(t))\big)\, dt \\
&\quad = \big[\hat{p}(t)\mu(t, E_t, \hat{X}(t), \hat{u}(t)) - \hat{p}(t)\mu(t, E_t, X(t), u(t)) + f(t, E_t, \hat{X}(t), \hat{u}(t)) \\
&\qquad - f(t, E_t, X(t), u(t))\big]\, dt \\
&\qquad + \big(g(t, E_t, \hat{X}(t), \hat{u}(t)) - g(t, E_t, X(t), u(t))\big)\, dE_t \\
&\qquad + \big(\hat{p}(t)b(t, E_t, \hat{X}(t), \hat{u}(t)) + \hat{q}(t)\sigma(t, E_t, \hat{X}(t), \hat{u}(t))\big)\, dE_t \\
&\qquad - \big(\hat{p}(t)b(t, E_t, X(t), u(t)) + \hat{q}(t)\sigma(t, E_t, X(t), u(t))\big)\, dE_t \\
&\qquad + \int_{\mathbb{R}} \hat{r}(t,z)\big(\gamma(t, E_t, \hat{X}(t), \hat{u}(t)) - \gamma(t, E_t, X(t), u(t))\big)\, \nu(dz)\, dE_t,
\end{aligned}
$$

and

$$
\begin{aligned}
(\hat{X}(t) - X(t))d\hat{p}(t) &= \hat{X}(t)\, d\hat{p}(t) - X(t)d\hat{p}(t) \\
&= \hat{X}(t)\Big[-H_x(t, E_t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))dt + \hat{q}(t)\, dB_{E_t} \\
&\qquad + \int_{\mathbb{R}} r(t,z)\, \tilde{N}(dE_t, dz)\Big] \\
&\qquad - X(t)\Big[-H_x(t, E_t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))\, dt + \hat{q}(t)\, dB_{E_t} \\
&\qquad + \int_{\mathbb{R}} r(t,z)\, \tilde{N}(dE_t, dz)\Big] \\
&= -(\hat{X}(t) - X(t))H_x(t, E_t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))\, dt \\
&\qquad + (\hat{X}(t) - X(t))\Big(\hat{q}(t)\, dB_{E_t} + \int_{\mathbb{R}} \hat{r}(t,z)\, \tilde{N}(dE_t, dz)\Big).
\end{aligned}
$$

Then, by concavity of H and following the proof in [4],

$$
\begin{aligned}
&J(\hat{u}) - J(u) \\
&\quad = \mathbb{E}\Big[\int_0^T -(\hat{X}(t) - X(t))H_x(t, E_t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))\, dt \\
&\qquad + \int_0^T \big[H(t, E_t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)) \\
&\qquad\qquad - H(t, E_t, X(t), u(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))\big]\, dt\Big] \\
&\quad \geqslant 0. \quad \blacksquare
\end{aligned}
$$

EXAMPLE 4.1 (Income and consumption optimization). Consider the stochastic control problem

$$\Phi(x_0) = \sup_{u \in \mathcal{A}} \mathbb{E}\Big[\int_0^\tau \exp(-\delta t) u(t)^2\, dt\Big],$$

where

$$\tau = \inf\{t > 0 : X(t) \leqslant 0\}$$

and

$$
\begin{aligned}
dX(t) &= -u(t)dt + X(t)\Big(b\, dE_t + \sigma\, dB_{E_t} + \theta \int_{\mathbb{R}} z\tilde{N}(dz, dE_t)\Big), \\
X(0) &= x_0 > 0,
\end{aligned}
$$

where $\delta > 0$, σ, and θ are constants and $b = -(\sigma^2 + \theta^2 \int_{\mathbb{R}} z^2\, \nu(dz))/2$.

We can interpret $u(t)$ as the consumption rate, $X(t)$ as the corresponding wealth, and τ as the bankruptcy time. Then Φ represents the maximal expected total quadratic utility of the consumption up to bankruptcy time.

Define the Hamiltonian H by

$$
\begin{aligned}
H(t) &= -p(t)u(t) + \exp(-\delta t)u(t)^2 \\
&\quad + X(t)\Big(p(t)b + q(t)\sigma + \int_{\mathbb{R}} \theta z r(t,z)\, \nu(dz)\Big)\frac{dE_t}{dt},
\end{aligned}
$$

and the adjoint equation

$$
\begin{aligned}
dp(t) &= -\Big(p(t)b + q(t)\sigma + \int_{\mathbb{R}} \theta z r(t,z)\, \nu(dz)\Big)\, dE_t \\
&\quad + q(t)\, dB_{E_t} + \int_{\mathbb{R}} r(t,z)\, \tilde{N}(dE_t, dz), \qquad t < \tau, \\
p(T) &= 0.
\end{aligned}
\tag{4.5}
$$

Let $\frac{\partial H}{\partial u} = (-p(t) + 2u(t)\exp(-\delta t)) = 0$. Then $u^*(t) = \frac{p(t)}{2}\exp(\delta t)$. Suppose that $p(t) = h(t)X(t)$. Then $u^*(t) = \frac{h(t)X(t)}{2}\exp(\delta t)$, thus

$$
\begin{aligned}
dp(t) &= X(t)h'(t)\,dt + h(t)\,dX(t) \\
&= X(t)h'(t)\,dt + (-u(t)h(t))\,dt \\
&\quad + h(t)X(t)\Big(b\,dE_t + \sigma\,dB_{E_t} + \theta\int_{\mathbb{R}} z\,\tilde{N}(dz, dE_t)\Big) \\
&= X(t)\left(h'(t) - \frac{h(t)}{2}\exp(\delta t)\right)dt \\
&\quad + h(t)X(t)\Big(b\,dE_t + \sigma\,dB_{E_t} + \theta\int_{\mathbb{R}} z\,\tilde{N}(dz, dE_t)\Big).
\end{aligned} \tag{4.6}
$$

Comparing (4.5) and (4.6), we derive that $h'(t) = \frac{h(t)}{2}e^{\delta t}$, or equivalently $h(t) = \exp\big(\frac{1}{2\delta}e^{\delta t}\big)$, thus

$$u^*(t) = \exp\left(\frac{1}{2\delta}e^{\delta t} + \delta t\right)\frac{X(t)}{2}.$$

Moreover,

$$
\begin{aligned}
h(t)X(t)\sigma &= q(t), \\
h(t)X(t)\theta z &= r(t, z).
\end{aligned}
$$

Some algebra implies that

$$
\begin{aligned}
q(t) &= 2\exp(-\delta t)u(t)\sigma, \\
r(t, z) &= 2\exp(-\delta t)u(t)\theta z.
\end{aligned}
$$

A simulation of the optimal control $u^*(t)$ with $\delta = -.001$, $\sigma = 1$, $\theta = 1$, $x_0 = 1$, standard normal distribution ν, and inverse stable subordinator $E(t)$ having $\alpha = .9$ is displayed in Figure 5.

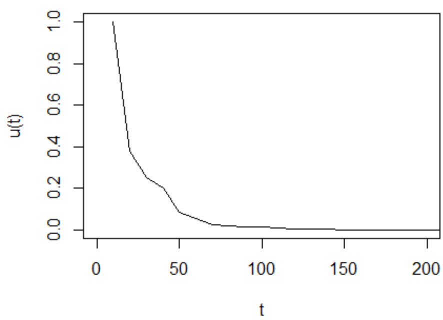

FIGURE 5. Simulation of $u^*(t)$ for Example 2

Because of the existence of the dt term in the underlying process $X(t)$, the simulated process $u^*(t)$ has no periods of constant value. Compared with dE_t

terms, the dt term plays the dominating role in the evolution of the corresponding wealth $X(t)$; see [11] for a detailed discussion. More specifically, the increasing trend $bX(t)dE_t$ is dominated by the consumption rate $-u(t)dt$. Consequently, the optional consumption rate declines as the wealth shrinks in the long run.

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