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UNUSUAL LIMIT THEOREMS FOR THE TWO-TAILED PARETO DISTRIBUTION

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Abstract. We examine order statistics from a two-sided Pareto distribution. It turns out that the smallest two order statistics and the largest two order statistics have very unusual limits. We obtain strong and weak exact laws for the smallest and the largest order statistics. For such statistics we also study the generalized law of the iterated logarithm. For the second smallest and second largest order statistics we prove the central limit theorem even though their second moment is infinite.

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1. INTRODUCTION

This paper extends the work done in [1]. Since then there have been other papers, [5]–[7], on the same set of random variables, but in different directions. In [1] we obtained unusual strong laws where the random variables have right and left tails with the same thickness. In most other Exact Strong Laws, we only looked at random variables that had a thicker right tail or no left tail at all. But here we consider random variables X with two-sided Pareto distribution for which both $\mathbb{E}X^+ = \infty$ and $\mathbb{E}X^- = \infty$, where $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$. The underlying distribution is

$$f(x) = \begin{cases} \frac{q}{x^2} & \text{if } x \leqslant -1, \\ 0 & \text{if } -1 < x < 1, \\ \frac{p}{x^2} & \text{if } x \geqslant -1, \end{cases}$$

where $p + q = 1$. In this paper we shall study arrays of random variables

$$\begin{array}{c} X_{11}, X_{12}, \ldots, X_{1m} \\ X_{21}, X_{22}, \ldots, X_{2m} \\ \vdots \\ X_{i1}, X_{i2}, \ldots, X_{im} \\ \vdots \end{array}$$

which will be denoted by $\{X_{i1}, X_{i2}, \ldots, X_{im}\}$. We shall assume that these random variables are all independent, i.e., independent in each row and the rows are independent of each other. We look at all kinds of order statistics from a fixed sample of size m. The kth order statistic in row i is denoted by $X_{i(k)}$ or by $X_{(k)}$ if there is no doubt, and has the density

$$f_{X_{(k)}}(x) = \begin{cases} \frac{m!}{(k-1)!(m-k)!}\left(-\frac{q}{x}\right)^{k-1}\frac{q}{x^2}\left(1+\frac{q}{x}\right)^{m-k} & \text{if } x \leqslant -1, \\ 0 & \text{if } -1 < x < 1, \\ \frac{m!}{(k-1)!(m-k)!}\left(1-\frac{p}{x}\right)^{k-1}\frac{p}{x^2}\left(\frac{p}{x}\right)^{m-k} & \text{if } x \geqslant -1. \end{cases} \tag{1.1}$$

The interesting cases are when k is 1, 2, $m-1$ or m.

We mention that the constant C used in the proofs denotes a generic real number that is not necessarily the same in each appearance. It is usually used as an upper bound in order to establish the convergence of various series. And it will also be used as a generic lower bound for divergent series. Also, we define $\lg x = \ln(\max\{e, x\})$ and $\lg_2 x = \lg(\lg x)$, which is *not* logarithm to base 2.

2. STRONG LAWS

We first look at the smallest order statistic, and then at the largest one. Since for the smallest order statistic the left tail is bigger than the right one, the limit will be negative. But the result also depends on the parameters m, q and α. What is fascinating is that we can obtain Exact Strong Laws for these random variables. In Section 4 we show that these strong laws are quite special, in that it is quite difficult to balance partial sums of random variables that possess infinite expectations with constants and achieve an almost sure result.

THEOREM 2.1. *Let $\{X_{i1}, \ldots, X_{im}\}$ be i.i.d. random variables from our two-sided Pareto distribution. Then for $\alpha > -2$ we have*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \frac{(\lg i)^\alpha}{i} X_{i(1)}}{(\lg n)^{\alpha+2}} = \frac{-mq}{\alpha+2} \quad \textit{almost surely}.$$

Proof. Let $a_n = (\lg n)^\alpha/n$, $b_n = (\lg n)^{\alpha+2}$ and $c_n = b_n/a_n = n(\lg n)^2$. We use the decomposition

$$\text{(2.1)}\quad \frac{1}{b_n}\sum_{i=1}^{n} a_i X_{i(1)} = \frac{1}{b_n}\sum_{i=1}^{n} a_i[X_{i(1)}\mathbb{I}(|X_{i(1)}| \leqslant c_i) - \mathbb{E}(X_{(1)}\mathbb{I}(|X_{(1)}| \leqslant c_i))] + \frac{1}{b_n}\sum_{i=1}^{n} a_i X_{i(1)}\mathbb{I}(|X_{i(1)}| > c_i) + \frac{1}{b_n}\sum_{i=1}^{n} a_i \mathbb{E}(X_{(1)}\mathbb{I}(|X_{(1)}| \leqslant c_i)).$$

The first term on the right hand side of (2.1) vanishes almost surely by the Khinchin –Kolmogorov Convergence Theorem (see [3, p. 113]) and Kronecker's lemma. We focus on the left tail since in this situation it dominates the right tail. We have

$$\sum_{n=1}^{\infty}\frac{1}{c_n^2}\mathbb{E}(X_{(1)}^2\mathbb{I}(|X_{(1)}| \leqslant c_n)) = \sum_{n=1}^{\infty}\frac{1}{c_n^2}\Big[\int_{-c_n}^{-1} x^2 f_{X_{(1)}}(x)\,dx + \int_{1}^{c_n} x^2 f_{X_{(1)}}(x)\,dx\Big] < C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{-c_n}^{-1} x^2\Big(\frac{q}{x^2}\Big)\,dx < C\sum_{n=1}^{\infty}\frac{1}{c_n} = C\sum_{n=1}^{\infty}\frac{1}{n(\lg n)^2} < \infty.$$

The second term on the RHS in (2.1) vanishes, with probability 1, by the Borel–Cantelli lemma since

$$\sum_{n=1}^{\infty}\mathbb{P}(|X_{(1)}| > c_n) = \sum_{n=1}^{\infty}\Big[\int_{-\infty}^{-c_n} f_{X_{(1)}}(x)\,dx + \int_{c_n}^{\infty} f_{X_{(1)}}(x)\,dx\Big] < C\sum_{n=1}^{\infty}\int_{-\infty}^{-c_n}\frac{q}{x^2}\,dx < C\sum_{n=1}^{\infty}\frac{1}{c_n} = C\sum_{n=1}^{\infty}\frac{1}{n(\lg n)^2} < \infty.$$

Thus, our almost sure limit follows from the last term in (2.1):

$$\begin{aligned} b_n^{-1}\sum_{i=1}^{n} a_i\mathbb{E}(X_{(1)}\mathbb{I}(|X_{(1)}| \leqslant c_i)) &= b_n^{-1}\sum_{i=1}^{n} a_i\Big[\int_{-c_i}^{-1} x f_{X_{(1)}}(x)\,dx + \int_{1}^{c_i} x f_{X_{(1)}}(x)\,dx\Big] \\ &\sim b_n^{-1}\sum_{i=1}^{n} a_i\int_{-c_i}^{-1} x f_{X_{(1)}}(x)\,dx \sim mqb_n^{-1}\sum_{i=1}^{n} a_i\int_{-c_i}^{-1}\frac{dx}{x} = mqb_n^{-1}\sum_{i=1}^{n} a_i(-\lg c_i) \\ &\sim \frac{-mq}{(\lg n)^{\alpha+2}}\sum_{i=1}^{n}\frac{(\lg i)^{\alpha}}{i}(\lg i) = \frac{-mq}{(\lg n)^{\alpha+2}}\sum_{i=1}^{n}\frac{(\lg i)^{\alpha+1}}{i} \\ &\sim \frac{-mq}{(\lg n)^{\alpha+2}}\cdot\frac{(\lg n)^{\alpha+2}}{\alpha+2} = \frac{-mq}{\alpha+2}, \end{aligned}$$

which concludes this proof. ■

The next result can be found in [1], so we skip the proof.

THEOREM 2.2. *Let $\{X_{i1},\dots,X_{im}\}$ be i.i.d. random variables from our two-sided Pareto distribution. Then for $\alpha > -2$ we have*

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}\frac{(\lg i)^{\alpha}}{i}X_{i(m)}}{(\lg n)^{\alpha+2}} = \frac{mp}{\alpha+2} \quad \textit{almost surely.}$$

Here we have used the weights $a_n = (\lg n)^\alpha/n$, but we can replace $(\lg n)^\alpha$ with any slowly varying function. That will naturally affect the b_n in obtaining an Exact Strong Law. However, if we increase a_n any more than that, say take a higher power than negative one, then we can only obtain a weak law. That is the point of the next two sections.

3. WEAK LAWS

Next we will establish weak laws where strong laws will not hold. What is interesting is that the distribution of our random variables does not change, we just slightly increase the weights a_n. And we do obtain an appropriate norming sequence b_n, in order to obtain a Fair Game and not an Exact Strong Law. We will use these weak laws to establish the almost sure behavior of these partial sums in the next section, which shows why we must select weights as in Section 2.

THEOREM 3.1. *Let $\{X_{i1}, \dots, X_{im}\}$ be i.i.d. random variables from our two-sided Pareto distribution. Then for $\alpha > -1$ and any slowly varying function $L(x)$,*

$$\frac{\sum_{i=1}^n L(i) i^\alpha X_{i(1)}}{L(n)(\lg n) n^{\alpha+1}} \xrightarrow{P} \frac{-mq}{\alpha+1} \quad \text{as } n \to \infty.$$

Proof. Let $a_i = L(i)i^\alpha$ and $b_n = L(n)(\lg n)n^{\alpha+1}$. We will use the Weak Law from [3, p. 356]. Let $\epsilon > 0$. Once again, since the left tail dominates the right tail, we obtain

$$\begin{aligned}
&\sum_{i=1}^n \mathbb{P}(a_i|X_{(1)}|/b_n > \epsilon) \\
&\qquad = m\sum_{i=1}^n \left[\int_{-\infty}^{-\epsilon b_n/a_i} \frac{q}{x^2}\left(1+\frac{q}{x}\right)^{m-1} dx + \int_{\epsilon b_n/a_i}^{\infty} \frac{p}{x^2}\left(\frac{p}{x}\right)^{m-1} dx\right] \\
&\qquad < C\sum_{i=1}^n \int_{-\infty}^{-\epsilon b_n/a_i} \frac{dx}{x^2} < \frac{C\sum_{i=1}^n a_i}{b_n} = \frac{C\sum_{i=1}^n L(i)i^\alpha}{L(n)(\lg n)n^{\alpha+1}} \\
&\qquad < \frac{CL(n)n^{\alpha+1}}{L(n)(\lg n)n^{\alpha+1}} = \frac{C}{\lg n} \to 0.
\end{aligned}$$

As for the variance term, we have

$$\begin{aligned}
&\sum_{i=1}^n \operatorname{Var}\left[\frac{a_i X_{(1)}}{b_n}\mathbb{I}\left(\frac{a_i X_{(1)}}{b_n} < 1\right)\right] \\
&\qquad < b_n^{-2}\sum_{i=1}^n a_i^2 \int_{-b_n/a_i}^{-1} mq\left(1+\frac{q}{x}\right)^{m-1} dx + \int_{1}^{b_n/a_i} mp\left(\frac{p}{x}\right)^{m-1} dx
\end{aligned}$$

$$< Cb_n^{-2} \sum_{i=1}^{n} a_i^2 \int_{-b_n/a_i}^{-1} mq\left(1+\frac{q}{x}\right)^{m-1} dx$$
$$< Cb_n^{-2} \sum_{i=1}^{n} a_i^2 \int_{-b_n/a_i}^{0} dx = \frac{C\sum_{i=1}^{n} a_i}{b_n} \to 0.$$

Next, we must compute the expectation from the Weak Law Theorem of [3, p. 356]. Using $\sum_{i=1}^{n} a_i = o(b_n)$ we see the right tail does not affect our limit at all. We have

$$\sum_{i=1}^{n} \mathbb{E}\left[\frac{a_i X_{(1)}}{b_n} \mathbb{I}\left(\frac{a_i X_{(1)}}{b_n} < 1\right)\right]$$
$$= \frac{m}{b_n} \sum_{i=1}^{n} a_i \left[\int_{-b_n/a_i}^{-1} \frac{q}{x}\left(1+\frac{q}{x}\right)^{m-1} dx + \int_{1}^{b_n/a_i} \frac{p}{x}\left(\frac{p}{x}\right)^{m-1} dx\right]$$
$$\sim \frac{m}{b_n} \sum_{i=1}^{n} a_i \int_{-b_n/a_i}^{-1} \frac{q}{x}\left(1+\frac{q}{x}\right)^{m-1} dx \sim \frac{mq}{b_n} \sum_{i=1}^{n} a_i \int_{-b_n/a_i}^{-1} \frac{dx}{x}$$
$$= \frac{-mq}{b_n} \sum_{i=1}^{n} a_i[\lg(b_n) - \lg(a_i)].$$

It is interesting that both of these terms are equally important. We have

$$b_n^{-1} \sum_{i=1}^{n} a_i \lg(b_n) = \frac{\sum_{i=1}^{n} L(i) i^{\alpha}[\lg(L(n)) + \lg_2 n + (\alpha+1)\lg n]}{L(n)(\lg n) n^{\alpha+1}}$$
$$\sim \frac{(\alpha+1)\sum_{i=1}^{n} L(i) i^{\alpha} \lg n}{L(n)(\lg n) n^{\alpha+1}} \sim \frac{(\alpha+1) L(n) \frac{n^{\alpha+1}}{\alpha+1} \lg n}{L(n)(\lg n) n^{\alpha+1}} = 1,$$

while

$$b_n^{-1} \sum_{i=1}^{n} a_i \lg(a_i) = \frac{\sum_{i=1}^{n} L(i) i^{\alpha}[\lg(L(i)) + \alpha \lg i]}{L(n)(\lg n) n^{\alpha+1}}$$
$$\sim \frac{\alpha \sum_{i=1}^{n} L(i)(\lg i) i^{\alpha}}{L(n)(\lg n) n^{\alpha+1}} \sim \frac{\alpha L(n)(\lg n)\frac{n^{\alpha+1}}{\alpha+1}}{L(n)(\lg n) n^{\alpha+1}} = \frac{\alpha}{\alpha+1}.$$

Combining these two terms, we see that our limit is indeed $-mq\left(1-\frac{\alpha}{\alpha+1}\right) = \frac{-mq}{\alpha+1}$, which concludes this proof. ∎

This next weak law can also be found in [1].

THEOREM 3.2. *Let $\{X_{i1}, \ldots, X_{im}\}$ be i.i.d. random variables from our two-sided Pareto distribution. Then for $\alpha > -1$ and any slowly varying function $L(x)$,*

$$\frac{\sum_{i=1}^{n} L(i) i^{\alpha} X_{i(m)}}{L(n)(\lg n) n^{\alpha+1}} \xrightarrow{P} \frac{mp}{\alpha+1} \quad \text{as } n \to \infty.$$

We will use these two theorems in the next section, to show the odd fluctuations of these partial sums.

4. GENERALIZED LAWS OF THE ITERATED LOGARITHM

This section nicely compares what we accomplished in the previous two sections. It shows us how precise our strong laws from Section 2 are. And it also shows that even when we can obtain a weak law, the almost sure counterpart does not necessarily hold. Furthermore, the decomposition here is very delicate and since these random variables have support on all of the reals, we need to be extra careful in showing which terms are negligible and which are not. The idea of how to decompose our partial sums comes from [4].

THEOREM 4.1. *Let $\{X_{i1}, \ldots, X_{im}\}$ be i.i.d. random variables from our two-sided Pareto distribution. Then for $\alpha > -1$ and any slowly varying function $L(x)$,*

$$\liminf_{n\to\infty} \frac{\sum_{i=1}^{n} L(i) i^{\alpha} X_{i(1)}}{L(n)(\lg n) n^{\alpha+1}} = -\infty \quad \textit{almost surely}$$

and

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} L(i) i^{\alpha} X_{i(1)}}{L(n)(\lg n) n^{\alpha+1}} = \frac{-mq}{\alpha+1} \quad \textit{almost surely.}$$

Proof. Here we set $a_n = L(n)n^{\alpha}$, $b_n = L(n)(\lg n)n^{\alpha+1}$, $c_n = b_n/a_n = n \lg n$, but we also need a fourth sequence $d_n = n$.

To find the lower limit, let $M > 0$; then

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}(a_n X_{(1)}^{-} > M b_n) &= \sum_{n=1}^{\infty} \int_{-\infty}^{-Mc_n} \frac{mq}{x^2}\left(1 + \frac{q}{x}\right)^{m-1} dx \\ &> C \sum_{n=1}^{\infty} \int_{-\infty}^{-Mc_n} \frac{dx}{x^2} > C \sum_{n=1}^{\infty} \frac{1}{c_n} = C \sum_{n=1}^{\infty} \frac{1}{n \lg n} = \infty. \end{aligned}$$

Thus

$$\limsup_{n\to\infty} \frac{a_n X_{n(1)}^{-}}{b_n} = \infty \quad \text{almost surely,}$$

and since

$$\frac{a_n X_{n(1)}^{-}}{b_n} \leqslant \frac{\sum_{i=1}^{n} a_i X_{i(1)}^{-}}{b_n},$$

we have

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} a_i X_{i(1)}^{-}}{b_n} = \infty \quad \text{almost surely.}$$

On the other hand, $\mathbb{E}(X_{(1)}^{+}) < \infty$ and using $\sum_{i=1}^{n} a_i = o(b_n)$ we see that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i X_{i(1)}^{+}}{b_n} = 0 \quad \text{almost surely.}$$

Putting all this together we obtain

$$\liminf_{n\to\infty} \frac{\sum_{i=1}^{n} a_i X_{i(1)}}{b_n} = \liminf_{n\to\infty}\left(\frac{\sum_{i=1}^{n} a_i X_{i(1)}^{+}}{b_n} - \frac{\sum_{i=}^{n} a_i X_{i(1)}^{-}}{b_n}\right)$$
$$= 0 - \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} a_i X_{i(1)}^{-}}{b_n} = -\infty$$

almost surely, of course.

Now the other result is quite difficult. From Theorem 3.1, we can claim that

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} a_i X_{i(1)}}{b_n} \geqslant \frac{-mq}{\alpha+1} \quad \text{almost surely.}$$

Hence we need to prove that

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} a_i X_{i(1)}}{b_n} \leqslant \frac{-mq}{\alpha+1} \quad \text{almost surely.}$$

This is where the sequence $d_n = n$ comes into play. We break our partial sum into five pieces, toss one away and then examine the other four. We have

$$\begin{aligned}
(4.1)\quad & b_n^{-1}\sum_{i=1}^{n} a_i X_{i(1)} \\
&= b_n^{-1}\sum_{i=1}^{n} a_i[X_{i(1)}\mathbb{I}(|X_{i(1)}|\leqslant d_i) - \mathbb{E}(X_{(1)}\mathbb{I}(|X_{(1)}|\leqslant d_i))] \\
&\quad + b_n^{-1}\sum_{i=1}^{n} a_i[X_{i(1)}\mathbb{I}(d_i < X_{i(1)}\leqslant c_i) - \mathbb{E}(X_{(1)}\mathbb{I}(d_i < X_{(1)}\leqslant c_i))] \\
&\quad + b_n^{-1}\sum_{i=1}^{n} a_i X_{i(1)}\mathbb{I}(X_{i(1)} > c_i) + b_n^{-1}\sum_{i=1}^{n} a_i X_{i(1)}\mathbb{I}(X_{i(1)} < -d_i) \\
&\quad + b_n^{-1}\sum_{i=1}^{n} a_i\mathbb{E}(X_{(1)}\mathbb{I}(-d_i\leqslant X_{(1)}\leqslant c_i)).
\end{aligned}$$

Since the fourth term on the right hand side in (4.1) is strictly negative, we have

$$\begin{aligned}
(4.2)\quad & b_n^{-1}\sum_{i=1}^{n} a_i X_{i(1)} \\
&\leqslant b_n^{-1}\sum_{i=1}^{n} a_i[X_{i(1)}\mathbb{I}(|X_{i(1)}|\leqslant d_i) - \mathbb{E}(X_{(1)}\mathbb{I}(|X_{(1)}|\leqslant d_i))] \\
&\quad + b_n^{-1}\sum_{i=1}^{n} a_i[X_{i(1)}\mathbb{I}(d_i < X_{i(1)}\leqslant c_i) - \mathbb{E}(X_{(1)}\mathbb{I}(d_i < X_{(1)}\leqslant c_i))] \\
&\quad + b_n^{-1}\sum_{i=1}^{n} a_i X_{i(1)}\mathbb{I}(X_{i(1)} > c_i) + b_n^{-1}\sum_{i=1}^{n} a_i\mathbb{E}(X_{(1)}\mathbb{I}(-d_i\leqslant X_{(1)}\leqslant c_i)).
\end{aligned}$$

The first term on the RHS in (4.2) vanishes almost surely by the Khinchin–Kolmogorov Convergence Theorem and Kronecker's lemma since

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{-n}^{n}x^2 f_{X_{(1)}}(x)\,dx &< C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{-n}^{-1}x^2\frac{mq}{x^2}\left(1+\frac{q}{x}\right)^{m-1}dx\\ &< C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{-n}^{-1}dx < C\sum_{n=1}^{\infty}\frac{n}{c_n^2} = C\sum_{n=1}^{\infty}\frac{1}{n(\lg n)^2}<\infty.\end{aligned}$$

One can use the same technique for the second term in (4.2) or just notice that the right tail has a finite expectation, so in view of $\sum_{i=1}^{n} a_i = o(b_n)$ that term also vanishes. But to be rigorous,

$$\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{n}^{n\lg n}x^2\frac{mp}{x^2}\left(\frac{p}{x}\right)^{m-1}dx < C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{n}^{n\lg n}x^{-m+1}\,dx<\infty$$

for any $m \geqslant 2$. When $m = 2$, we have

$$\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{n}^{n\lg n}\frac{dx}{x} = \sum_{n=1}^{\infty}\frac{\lg_2 n}{(n\lg n)^2}<\infty$$

and when $m > 2$ we have

$$\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{n}^{n\lg n}x^{-m+1}\,dx < \sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{1}^{\infty}x^{-m+1}\,dx < C\sum_{n=1}^{\infty}\frac{1}{c_n^2}<\infty.$$

Similarly, the third term is on the smaller side, so

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbb{P}(X_{(1)}>c_n) &= \sum_{n=1}^{\infty}\int_{c_n}^{\infty}\frac{mp}{x^2}\left(\frac{p}{x}\right)^{m-1}dx\\ &< C\sum_{n=1}^{\infty}\int_{c_n}^{\infty}x^{-m-1}\,dx < C\sum_{n=1}^{\infty}\frac{1}{c_n^m}<\infty\end{aligned}$$

since $m \geqslant 2$. The final term in (4.2) produces our limit. The right tail is not necessary since it has a finite expectation. Thus

$$\begin{aligned}b_n^{-1}\sum_{i=1}^{n}a_i\int_{-d_i}^{c_i}x f_{X_{(1)}}(x)\,dx &\sim b_n^{-1}\sum_{i=1}^{n}a_i\int_{-i}^{-1}x\frac{mq}{x^2}\left(1+\frac{q}{x}\right)^{m-1}dx\\ &\sim \frac{mq}{b_n}\sum_{i=1}^{n}a_i\int_{-i}^{-1}\frac{dx}{x} = \frac{mq}{L(n)(\lg n)n^{\alpha+1}}\sum_{i=1}^{n}L(i)i^{\alpha}(-\lg i)\\ &= \frac{-mq\sum_{i=1}^{n}L(i)(\lg i)i^{\alpha}}{L(n)(\lg n)n^{\alpha+1}} \to \frac{-mq}{\alpha+1}\end{aligned}$$

as $n \to \infty$. This show that the almost sure upper limit is indeed $-mq/(\alpha+1)$, which concludes this proof. ■

Next we turn to the largest order statistic.

THEOREM 4.2. *Let $\{X_{i1}, \ldots, X_{im}\}$ be i.i.d. random variables from our two-sided Pareto distribution. Then for $\alpha > -1$ and any slowly varying function $L(x)$,*

$$\liminf_{n\to\infty} \frac{\sum_{i=1}^n L(i) i^\alpha X_{i(m)}}{L(n)(\lg n) n^{\alpha+1}} = \frac{mp}{\alpha+1} \quad \textit{almost surely}$$

and

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^n L(i) i^\alpha X_{i(m)}}{L(n)(\lg n) n^{\alpha+1}} = \infty \quad \textit{almost surely.}$$

Proof. As in the previous proof, let $a_n = L(n)n^\alpha$, $b_n = L(n)(\lg n)n^{\alpha+1}$, $c_n = b_n/a_n = n \lg n$ and $d_n = n$. From Theorem 3.2 we have

$$\liminf_{n\to\infty} \frac{\sum_{i=1}^n L(i) i^\alpha X_{i(m)}}{L(n)(\lg n) n^{\alpha+1}} \leqslant \frac{mp}{\alpha+1} \quad \text{almost surely,}$$

so in this proof we need to obtain

$$\liminf_{n\to\infty} \frac{\sum_{i=1}^n L(i) i^\alpha X_{i(m)}}{L(n)(\lg n) n^{\alpha+1}} \geqslant \frac{mp}{\alpha+1} \quad \text{almost surely.}$$

The decomposition we use is similar, but naturally it is flipped over:

$$\begin{aligned}
(4.3) \quad & b_n^{-1} \sum_{i=1}^n a_i X_{i(m)} \\
&= b_n^{-1} \sum_{i=1}^n a_i [X_{i(m)} \mathbb{I}(|X_{i(m)}| \leqslant d_i) - \mathbb{E}(X_{(m)} \mathbb{I}(|X_{(m)}| \leqslant d_i))] \\
&\quad + b_n^{-1} \sum_{i=1}^n a_i [X_{i(m)} \mathbb{I}(-c_i \leqslant X_{i(m)} < -d_i) - \mathbb{E}(X_{(m)} \mathbb{I}(-c_i \leqslant X_{(m)} < -d_i))] \\
&\quad + b_n^{-1} \sum_{i=1}^n a_i X_{i(m)} \mathbb{I}(X_{i(m)} < -c_i) + b_n^{-1} \sum_{i=1}^n a_i X_{i(m)} \mathbb{I}(X_{i(m)} > d_i) \\
&\quad + b_n^{-1} \sum_{i=1}^n a_i \mathbb{E}(X_{(m)} \mathbb{I}(-c_i \leqslant X_{(m)} \leqslant d_i)).
\end{aligned}$$

Since the fourth term on the RHS in (4.3) is strictly positive, we have

$$
\begin{aligned}
(4.4)\quad & b_n^{-1}\sum_{i=1}^{n} a_i X_{i(m)} \\
&\geqslant b_n^{-1}\sum_{i=1}^{n} a_i[X_{i(m)}\mathbb{I}(|X_{i(m)}|\leqslant d_i)-\mathbb{E}(X_{(m)}\mathbb{I}(|X_{(m)}|\leqslant d_i))] \\
&\quad + b_n^{-1}\sum_{i=1}^{n} a_i[X_{i(m)}\mathbb{I}(-c_i\leqslant X_{i(m)}<-d_i)-\mathbb{E}(X_{(m)}\mathbb{I}(-c_i\leqslant X_{(m)}<-d_i))] \\
&\quad + b_n^{-1}\sum_{i=1}^{n} a_i X_{i(m)}\mathbb{I}(X_{i(m)}<-c_i)+b_n^{-1}\sum_{i=1}^{n} a_i\mathbb{E}(X_{(m)}\mathbb{I}(-c_i\leqslant X_{(m)}\leqslant d_i)).
\end{aligned}
$$

By the usual arguments the first term on the RHS in (4.4) vanishes. Note that the right tail is now the thicker one:

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{-n}^{n} x^2 f_{X_{(m)}}(x)\,dx &< C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{1}^{n} x^2\frac{mp}{x^2}\left(1-\frac{p}{x}\right)^{m-1}dx \\
&< C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{1}^{n} dx < C\sum_{n=1}^{\infty}\frac{n}{c_n^2}=C\sum_{n=1}^{\infty}\frac{1}{n(\lg n)^2}<\infty.
\end{aligned}
$$

The next term in (4.4) easily converges to zero, since it is on the smaller tail:

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{-c_n}^{-d_n} x^2 f_{X_{(m)}}(x)\,dx &= \sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{-c_n}^{-d_n} x^2\frac{mq}{x^2}\left(\frac{-q}{x}\right)^{m-1}dx \\
&< C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_{-c_n}^{-d_n} x^{-m+1}\,dx<\infty
\end{aligned}
$$

for any $m\geqslant 2$; just let $u=-x$ and the proof is the same as in the previous theorem. Likewise for all $m\geqslant 2$,

$$
\begin{aligned}
\sum_{n=1}^{\infty}\mathbb{P}(X_{(m)}<-c_n) &= \sum_{n=1}^{\infty}\int_{-\infty}^{-c_n}\frac{mq}{x^2}\left(\frac{-q}{x}\right)^{m-1}dx \\
&< C\sum_{n=1}^{\infty}\int_{-\infty}^{-c_n} x^{-m-1}\,dx < C\sum_{n=1}^{\infty}\frac{1}{c_n^m}<\infty.
\end{aligned}
$$

So, our limit is

$$
\begin{aligned}
b_n^{-1}\sum_{i=1}^{n} a_i\int_{-c_i}^{d_i} x f_{X_{(m)}}(x)\,dx &\sim b_n^{-1}\sum_{i=1}^{n} a_i\int_{1}^{i} x\frac{mp}{x^2}\left(1-\frac{p}{x}\right)^{m-1}dx \\
&\sim \frac{mp}{b_n}\sum_{i=1}^{n} a_i\int_{1}^{i}\frac{dx}{x}=\frac{mp}{L(n)(\lg n)n^{\alpha+1}}\sum_{i=1}^{n} L(i)i^{\alpha}(\lg i)\to\frac{mp}{\alpha+1}
\end{aligned}
$$

as $n \to \infty$, where we threw out the left integral, since $\sum_{i=1}^n a_i = o(b_n)$. Putting all this together we have

$$\liminf_{n\to\infty} \frac{\sum_{i=1}^n L(i) i^\alpha X_{i(m)}}{L(n)(\lg n) n^{\alpha+1}} = \frac{mp}{\alpha+1} \quad \text{almost surely.}$$

Turning to the almost sure upper limit, let $M > 0$. Then

$$\begin{aligned}\sum_{n=1}^\infty \mathbb{P}(a_n X_{(m)}^+ > M b_n) &= \sum_{n=1}^\infty \int_{Mc_n}^\infty \frac{mp}{x^2}\left(1 - \frac{p}{x}\right)^{m-1} dx \\ &> C \sum_{n=1}^\infty \int_{Mc_n}^\infty \frac{dx}{x^2} > C \sum_{n=1}^\infty \frac{1}{c_n} = C \sum_{n=1}^\infty \frac{1}{n \lg n} = \infty.\end{aligned}$$

Hence

$$\limsup_{n\to\infty} \frac{a_n X_{n(m)}^+}{b_n} = \infty \quad \text{almost surely,}$$

and since

$$\frac{a_n X_{n(m)}^+}{b_n} \leqslant \frac{\sum_{i=1}^n a_i X_{i(m)}^+}{b_n},$$

we have

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^n a_i X_{i(m)}^+}{b_n} = \infty \quad \text{almost surely.}$$

Since $\mathbb{E}(X_{(m)}^-) < \infty$ and $\sum_{i=1}^n a_i = o(b_n)$, we have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n a_i X_{i(m)}^-}{b_n} = 0 \quad \text{almost surely.}$$

Thus

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^n L(i) i^\alpha X_{i(m)}}{L(n)(\lg n) n^{\alpha+1}} = \infty \quad \text{almost surely,}$$

which concludes this proof. ■

5. CENTRAL LIMIT THEOREMS

Here we look at the second smallest and second largest order statistics. They both have a finite mean, but an infinite variance. We will apply Theorem 4 from [2]. We will start with the smaller once again. By using (1.1) with $k = 2$ we get the density of the second smallest order statistic:

$$f_{X_{(2)}}(x) = \begin{cases} m(m-1)\left(-\frac{q}{x}\right)\frac{q}{x^2}\left(1 + \frac{q}{x}\right)^{m-2} & \text{if } x \leqslant -1, \\ 0 & \text{if } -1 < x < 1, \\ m(m-1)\left(1 - \frac{p}{x}\right)\frac{p}{x^2}\left(\frac{p}{x}\right)^{m-2} & \text{if } x \geqslant -1. \end{cases}$$

There are three conditions that we need to meet in order to apply that theorem. The first one is that

$$G(x) = \mathbb{E}(X_{(2)}^2 \mathbb{I}(|X_{(2)}| \leqslant x)) \tag{5.1}$$

is slowly varying. The other two are

$$G\left(\frac{B_n}{\min_{1\leqslant i\leqslant n} a_i}\right) \sim G\left(\frac{B_n}{\max_{1\leqslant i\leqslant n} a_i}\right) \tag{5.2}$$

and for all $\epsilon > 0$,

$$\sum_{i=1}^{n} \mathbb{P}(|X_{(2)}| > \epsilon B_n/a_i) = o(1) \tag{5.3}$$

where once again a_i are our weights and now B_n is our norming sequence. Since $m \geqslant 3$,

$$\begin{aligned}
G(x) &= \mathbb{E}(\mathbb{X}_{(2)}^2 \mathbb{I}(|X_{(2)}| \leqslant x)) \\
&= m(m-1) \int_{-x}^{-1} t^2 \left(\frac{-q}{t}\right) \frac{q}{t^2} \left(1 + \frac{q}{t}\right)^{m-2} dt \\
&\quad + m(m-1) \int_{1}^{x} t^2 \left(1 - \frac{p}{t}\right) \frac{p}{t^2} \left(\frac{p}{t}\right)^{m-2} dt \\
&\sim m(m-1) \int_{-x}^{-1} t^2 \left(\frac{-q}{t}\right) \frac{q}{t^2} \left(1 + \frac{q}{t}\right)^{m-2} dt \\
&\sim m(m-1) \int_{-x}^{-1} t^2 \left(\frac{-q}{t}\right) \frac{q}{t^2}\, dt \\
&= m(m-1)(-q^2) \int_{-x}^{-1} \frac{dt}{t} = m(m-1)q^2 \lg x.
\end{aligned}$$

Thus the classic slowly varying function, logarithm, appears once again, yielding (5.1). The formula for B_n is quite restrictive. It is $B_n^2 \sim nG(B_n)$, which for us is

$$B_n^2 \sim m(m-1)q^2 n \lg(B_n),$$

which allows us to choose as our norming sequence

$$B_n = q\sqrt{\binom{m}{2} n \lg n}.$$

For simplicity we will let $a_i = (\lg i)^\alpha$, which makes (5.2) trivial. But in order to satisfy (5.3) we will have to set α to be less than one-half. The real pain in these theorems is the computation of the mean, which is not really necessary but is included. Sadly, they do not simplify into a nice expression, like our sequence B_n.

THEOREM 5.1. *Let $\{X_{i1}, \ldots, X_{im}\}$ be i.i.d. random variables from our two-sided Pareto distribution. If $\alpha < 1/2$, then*

$$\frac{\sum_{i=1}^{n} (\lg i)^{\alpha} \big[X_{i(2)} - \mathbb{E}(X_{(2)})\big]}{q\sqrt{\binom{m}{2} n \lg n}} \xrightarrow{d} N(0,1) \quad \text{as } n \to \infty.$$

Proof. With (5.1) and (5.2) satisfied, we turn our attention to (5.3). Let $\epsilon > 0$. Then

$$\begin{aligned}\sum_{i=1}^{n} \mathbb{P}(|X_{(2)}| > \epsilon B_n/a_i) &< C \sum_{i=1}^{n} \left(\int_{-\infty}^{-\epsilon B_n/a_i} \frac{dx}{x^3} + \int_{\epsilon B_n/a_i}^{\infty} \frac{dx}{x^m} \right) \\ &< \frac{C \sum_{i=1}^{n} a_i^2}{B_n^2} < \frac{C \sum_{i=1}^{n} (\lg i)^{2\alpha}}{n \lg n} \to 0\end{aligned}$$

since $\alpha < 1/2$. In order to compute the mean, we can no longer ignore the smaller tails, nor can we approximate integrands with bounds. The mean is

$$\begin{aligned}\mathbb{E}(X_{(2)}) &= m(m-1) \int_{-\infty}^{-1} x \left(\frac{-q}{x}\right) \frac{q}{x^2} \left(1 + \frac{q}{x}\right)^{m-2} dx \\ &\quad + m(m-1) \int_{1}^{\infty} x \left(1 - \frac{p}{x}\right) \frac{p}{x^2} \left(\frac{p}{x}\right)^{m-2} dx \\ &= m(m-1) q^2 \int_{-\infty}^{-1} \left(\frac{-1}{x^2}\right) \left(1 + \frac{q}{x}\right)^{m-2} dx \\ &\quad + m(m-1) p^{m-1} \int_{1}^{\infty} \left(1 - \frac{p}{x}\right) x^{-m+1} \, dx.\end{aligned}$$

In the first of the two integrals, we let $u = 1 + q/x$, so

$$\begin{aligned}&m(m-1) q^2 \int_{-\infty}^{-1} \left(\frac{-1}{x^2}\right) \left(1 + \frac{q}{x}\right)^{m-2} dx \\ &\qquad = m(m-1) q \int_{1}^{p} u^{m-2} \, du = mq(p^{m-1} - 1),\end{aligned}$$

which is negative, of course. The second integral is

$$\begin{aligned}m(m-1) p^{m-1} \int_{1}^{\infty} (x^{-m+1} - p x^{-m}) \, dx &= m(m-1) p^{m-1} \left(\frac{1}{m-2} - \frac{p}{m-1} \right) \\ &= \frac{m p^{m-1}}{m-2} (q(m-1) + p).\end{aligned}$$

Combining these two we have

$$\mathbb{E}(X_{(2)}) = mq(p^{m-1} - 1) + \frac{mp^{m-1}}{m-2}(q(m-1) + p),$$

concluding this proof. ■

We finish with a central limit theorem for the second largest order statistic. According to (1.1) its density is

$$f_{X_{(m-1)}}(x) = \begin{cases} m(m-1)\left(-\frac{q}{x}\right)^{m-2}\frac{q}{x^2}\left(1+\frac{q}{x}\right) & \text{if } x \leqslant -1, \\ 0 & \text{if } -1 < x < 1, \\ m(m-1)\left(1-\frac{p}{x}\right)^{m-2}\frac{p}{x^2}\left(\frac{p}{x}\right) & \text{if } x \geqslant -1. \end{cases}$$

THEOREM 5.2. *Let $\{X_{i1}, \ldots, X_{im}\}$ be i.i.d. random variables from our two-sided Pareto distribution. If $\alpha < 1/2$, then*

$$\frac{\sum_{i=1}^{n} (\lg i)^{\alpha}[X_{i(m-1)} - \mathbb{E}(X_{(m-1)})]}{p\sqrt{\binom{m}{2} n \lg n}} \xrightarrow{d} N(0,1) \quad \textit{as } n \to \infty.$$

Proof. The proof is similar to the proof of Theorem 5.1. In this setting

$$G(x) = \mathbb{E}(X_{(m-1)}^2 \mathbb{I}(|X_{(m-1)}| \leqslant x)) \sim m(m-1)p^2 \int_1^x \frac{dt}{t} = m(m-1)p^2 \lg x.$$

Our norming sequence is

$$B_n = p\sqrt{\binom{m}{2} n \lg n}.$$

And for all $\epsilon > 0$,

$$\begin{aligned} \sum_{i=1}^{n} \mathbb{P}(|X_{(m-1)}| > \epsilon B_n/a_i) &< C \sum_{i=1}^{n} \left(\int_{-\infty}^{-\epsilon B_n/a_i} \frac{dx}{x^m} + \int_{\epsilon B_n/a_i}^{\infty} \frac{dx}{x^3} \right) \\ &< \frac{C\sum_{i=1}^{n} a_i^2}{B_n^2} < \frac{C\sum_{i=1}^{n} (\lg i)^{2\alpha}}{n \lg n} \to 0 \end{aligned}$$

since $\alpha < 1/2$. Finally, by straightforward computations we get

$$\mathbb{E}(X_{(m-1)}) = mp(1 - q^{m-1}) - \frac{mq^{m-1}}{m-2}(p(m-1) + q),$$

concluding this proof. ■

We could have selected many other coefficients, but $a_i = (\lg i)^{\alpha}$ was simple and it works quite well with this theorem. And notice that when $\alpha = 0$ we have the unweighted case, which is nice. We also have the unweighted case in our weak laws and one-sided strong laws, but not in our Exact Strong Laws.

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