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COMPLETE f-MOMENT CONVERGENCE OF MOVING AVERAGE PROCESSES AND ITS APPLICATION TO NONPARAMETRIC REGRESSION MODELS*

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Abstract. In this paper, we establish a general result on complete f-moment convergence of the moving average process based on widely orthant dependent random variables, which generalizes some results in the literature. In addition, an application of complete consistency to nonparametric regression models is provided. Finally, we provide a numerical simulation to verify the validity of our theoretical results.

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1. INTRODUCTION

Widely orthant dependence is one of the most important dependence structures, and the convergence of moving average processes based on widely orthant dependent (WOD, for short) random variables is of great interest. In probability theory and mathematical statistics, as a more general type of convergence, complete f-moment convergence is much stronger than complete convergence and complete moment convergence. In this paper, we give some results on complete f-moment convergence of moving average processes based on WOD random variables, and present an application to nonparametric regression models.

Firstly, let us introduce some concepts which will be used in this paper.

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Let $\{Y_i, -\infty < i < \infty\}$ be a doubly infinite sequence of random variables defined on the same probability space $\{\Omega, \mathcal{F}, P\}$, and $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers, that is,

$$\sum_{i=-\infty}^{\infty} |a_i| < \infty.$$

Then $\{X_n,\ n \geqslant 1\}$ is a *moving average process* of the sequence $\{Y_i,\ -\infty < i < \infty\}$ if

$$X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, \qquad n \geqslant 1.$$

When the sequence $\{Y_i, -\infty < i < \infty\}$ is independent and identically distributed, many results about the moving average process $\{X_n, n \geqslant 1\}$ have been obtained. For example, stationary stochastic processes in the strong sense were investigated by Ibragimov [10]; Burton and Dehling [4] established a large deviation result; Li et al. [12] got the complete convergence property. Recently, several results have been obtained under the assumption that the sequence $\{Y_n, -\infty < n < \infty\}$ is dependent. Baek et al. [3] established complete convergence under NA assumptions. Kim et al. [11] discussed the complete moment convergence under φ-mixing random variables. Qiu and Chen [15] obtained a result on complete convergence of the moving average process for extended negatively dependent (END) sequences.

We aim to study complete f-moment convergence property for a moving average process based on WOD random variables, and give an application to nonparametric regression models.

Let us introduce the concept of WOD random variables, introduced by Wang et al. [22].

DEFINITION 1.1. An infinite sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be *widely upper orthant dependent* (WUOD, for short) if there exists a finite real sequence $\{g_U(n), n \geqslant 1\}$ such that for each $n \geqslant 1$ and all $x_i \in (-\infty, \infty)$, $1 \leqslant i \leqslant n$,

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leqslant g_U(n) \prod_{i=1}^{n} P(X_i > x_i);$$

and $\{X_n, n \geqslant 1\}$ is *widely lower orthant dependent* (WLOD) if there exists a finite real sequence $\{g_L(n), n \geqslant 1\}$ such that for each $n \geqslant 1$ and all $x_i \in (-\infty, \infty)$, $1 \leqslant i \leqslant n$,

$$P(X_1 \leqslant x_1, X_2 \leqslant x_2, \ldots, X_n \leqslant x_n) \leqslant g_L(n) \prod_{i=1}^{n} P(X_i \leqslant x_i).$$

Finally, $\{X_n, n \geqslant 1\}$ is WOD if it is both WUOD and WLOD, and $g_U(n)$, $g_L(n)$, $n \geqslant 1$, are called the *dominating coefficients*.

Denote $g(n) = \max\{g_U(n), g_L(n)\}$. Clearly, $g(n) \geqslant 1$. It is easily seen that the class of WOD random variables includes END sequences, negatively orthant dependent (NOD) sequences, negatively superadditive dependent (NSD) sequences, negatively associated (NA) sequences and independent sequences as special cases. Thus, the limiting probability behavior of WOD random variables and its applications are of great interest. Many scholars studied the limit probability behavior and applications based on WOD random variables. For example, Wang et al. [25] studied complete convergence for WOD random variables and gave an application to nonparametric regression models. Qiu and Chen [14] obtained some results on complete convergence and complete moment convergence for weighted sums of WOD random variables. The consistency of the nearest neighbor estimator of the density function based on widely orthant dependent samples was investigated by Wang and Hu [24]. Wu et al. [28] investigated complete moment convergence for widely orthant dependent random variables under some mild conditions. Deng and Wang [6] provided an exponential inequality for WOD random variables and presented an application to M estimators in multiple linear models.

Now, let us recall some concepts of convergence. The first one is complete convergence, introduced by Hsu and Robbins [9].

DEFINITION 1.2. A sequence $\{X_n, n \geqslant 1\}$ of random variables *converges completely* to the constant C if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty.$$

By the Borel–Cantelli lemma, this implies that $X_n \to C$ almost surely, so complete convergence is stronger than almost sure convergence.

Chow [5] presented the following more general concept of complete moment convergence, which is much stronger than complete convergence.

DEFINITION 1.3. Let $\{Z_n, n \geqslant 1\}$ be a sequence of random variables, and $a_n > 0$, $b_n > 0$, and $q > 0$. We say that $\{Z_n, n \geqslant 1\}$ *converges moment completely* if

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty \quad \text{for some or all } \varepsilon > 0,$$

where $a_+ = \max\{0, a\}$.

Recently, Wu et al. [26] introduced the concept of complete f-moment convergence, which is more general than complete moment convergence.

DEFINITION 1.4. Let $\{S_n, n \geqslant 1\}$ be a sequence of random variables, $\{c_n, n \geqslant 1\}$ be a sequence of positive constants and $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing continuous function with $f(0) = 0$. We say that $\{S_n, n \geqslant 1\}$ *converges f-moment completely* if

$$\sum_{n=1}^{\infty} c_n E f(\{|S_n| - \varepsilon\}_+) < \infty \quad \text{for all } \varepsilon > 0.$$

It is easy to check that complete f-moment convergence implies complete moment convergence if $f(t) = t^q$, and complete convergence if $c_n = 1$, $n \geqslant 1$, and $f(t) = t$. Thus, complete f-moment convergence is more general than complete moment convergence and complete convergence.

The following concept of stochastic domination will be used.

DEFINITION 1.5. A sequence $\{X_i, -\infty < i < \infty\}$ of random variables is said to be *stochastically dominated* by a random variable X if there exists a positive constant C such that

$$P(|X_i| > x) \leqslant CP(|X| > x)$$

for all $x \geqslant 0$ and $-\infty < i < \infty$.

The main purpose of this paper is to study complete f-moment convergence of moving average processes based on WOD random variables, and give an application to nonparametric regression models. In Section 2, we provide some preliminary lemmas which will be used to prove the main results. The main results and their proofs are stated in Section 3. In Sections 4 and 5, an application and a numerical simulation are presented.

2. PRELIMINARY LEMMAS

The following lemmas are useful for the proofs of the main results. In Lemma 2.1, the first inequality is due to Adler and Rosalsky [1, Lemma 1] and the second to Adler et al. [2, Lemma 3].

LEMMA 2.1. *Let $\{Y_i, i \geqslant 1\}$ be a sequence of random variables, stochastically dominated by a random variable Y. For any $\alpha, b > 0$,*

$$E|Y_i|^\alpha I(|Y_i| \leqslant b) \leqslant C_1[E|Y|^\alpha I(|Y| \leqslant b) + b^\alpha P(|Y| > b)],$$
$$E|Y_i|^\alpha I(|Y_i| > b) \leqslant C_2 E|Y|^\alpha I(|Y| > b),$$

where C_1 and C_2 are positive constants. Thus, $E|Y_i|^\alpha \leqslant CE|Y|^\alpha$, where C is a positive constant.

The following two lemmas are the basic properties of WOD random variables, which can be found in Wang et al. [25].

LEMMA 2.2. *Let $\{Y_n, n \geqslant 1\}$ be a sequence of WOD random variables. If $\{f_n(\cdot), n \geqslant 1\}$ are all nondecreasing (or all nonincreasing), then $\{f_n(Y_n), n \geqslant 1\}$ are still WOD.*

LEMMA 2.3. *Let $\tau > 1$ and $\{Y_n, n \geqslant 1\}$ be a sequence of WOD random variables with $EY_n = 0$ and $E|Y_n|^\tau < \infty$ for each $n \geqslant 1$. Then there exists a positive constant $C(\tau)$ depending only on τ such that for each j and $1 < \tau \leqslant 2$,*

$$E\Bigl|\sum_{i=j+1}^{j+n} Y_i\Bigr|^\tau \leqslant C(\tau)g(n)\sum_{i=j+1}^{j+n} E|Y_i|^\tau,$$

and for each j and $\tau \geqslant 2$,

$$E\Big|\sum_{i=j+1}^{j+n} Y_i\Big|^\tau \leqslant C(\tau)g(n)\Big[\sum_{i=j+1}^{j+n} E|Y_i|^\tau + \Big(\sum_{i=j+1}^{j+n} E|Y_i|^2\Big)^{\tau/2}\Big].$$

By Lemma 2.3, we can easily obtain the following result by using the method of Stout [20, Theorem 2.3.1].

LEMMA 2.4. *Let $\tau > 1$ and $\{Y_n, n \geqslant 1\}$ be a sequence of WOD random variables with $EY_n = 0$ and $E|Y_n|^\tau < \infty$ for each $n \geqslant 1$. Then there exists a positive constant $C(\tau)$ depending only on τ such that for each j and $1 < \tau \leqslant 2$,*

$$E\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{i=j+1}^{j+m} Y_i\Big|^\tau\Big) \leqslant C(\tau)(\log n)^\tau g(n)\sum_{i=j+1}^{j+n} E|Y_i|^\tau,$$

and for each j and $\tau \geqslant 2$,

$$E\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{i=j+1}^{j+m} Y_i\Big|^\tau\Big) \leqslant C(\tau)(\log n)^\tau g(n)\Big[\sum_{i=j+1}^{j+n} E|Y_i|^\tau + \Big(\sum_{i=j+1}^{j+n} E|Y_i|^2\Big)^{\tau/2}\Big].$$

The next lemma can be found in Wu et al. [27].

LEMMA 2.5. *Let $\{Y_i, 1 \leqslant i \leqslant n\}$ and $\{Z_i, 1 \leqslant i \leqslant n\}$ be sequences of random variables. Then for any $q > r > 0$, $\varepsilon > 0$, and $a > 0$,*

$$E\Big(\Big|\sum_{i=1}^{n}(Y_i+Z_i)\Big|-\varepsilon a\Big)_+^r \leqslant C(r)\left(\varepsilon^{-q}+\frac{r}{q-r}\right)a^{r-q}E\Big|\sum_{i=1}^{n} Y_i\Big|^q+C(r)E\Big|\sum_{i=1}^{n} Z_i\Big|^r,$$

and

$$\begin{aligned}E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k}(Y_i+Z_i)\Big|-\varepsilon a\Big)_+^r \leqslant{}& C(r)\left(\varepsilon^{-q}+\frac{r}{q-r}\right)a^{r-q}E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k} Y_i\Big|^q\Big)\\ &+C(r)E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k} Z_i\Big|^r\Big),\end{aligned}$$

where $C(r) = 1$ for $0 < r \leqslant 1$, and $C(r) = 2^{r-1}$ for $r > 1$.

The following lemma is indispensable in proving our main results in Section 3.

LEMMA 2.6. *Let $\alpha, v, p > 0$, and let Y be a random variable. Assume that D is a positive constant and $l(\cdot)$ is a slowly varying function. Then*

$$\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-1}l(n)E|Y|^v I(|Y| > Dn^\alpha) \leqslant \begin{cases} CE|Y|^p l(|Y|^{1/\alpha}), & v < p,\\ CE|Y|^p l(|Y|^{1/\alpha})\log|Y|, & v = p,\\ CE|Y|^v, & v > p,\end{cases}$$

$$\sum_{n=1}^{\infty} n^{\alpha p-1} l(n) P(|Y| > Dn^{\alpha}) \leqslant CE|Y|^p l(|Y|^{1/\alpha}),$$

and for any $\tau > p$,

$$\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-1} l(n) E|Y|^{\tau} I(|Y| \leqslant Dn^{\alpha}) \leqslant CE|Y|^p l(|Y|^{1/\alpha}).$$

Proof. By standard computation, for $\alpha, v, p, D > 0$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-1} l(n) E|Y|^v I(|Y| > Dn^{\alpha}) \\
&\quad = \sum_{n=1}^{\infty} n^{\alpha p-\alpha v-1} l(n) \sum_{j=n}^{\infty} E|Y|^v I(Dj^{\alpha} < |Y| \leqslant D(j+1)^{\alpha}) \\
&\quad = \sum_{j=1}^{\infty} E|Y|^v I(Dj^{\alpha} < |Y| \leqslant D(j+1)^{\alpha}) \sum_{n=1}^{j} n^{\alpha p-\alpha v-1} l(n) \\
&\quad \leqslant \begin{cases} C\sum_{j=1}^{\infty} j^{\alpha p-\alpha v} l(j) E|Y|^v I(Dj^{\alpha} < |Y| \leqslant D(j+1)^{\alpha}) & \text{if } v < p, \\ C\sum_{j=1}^{\infty} \log j\, l(j) E|Y|^p I(Dj^{\alpha} < |Y| \leqslant D(j+1)^{\alpha}) & \text{if } v = p, \\ C\sum_{j=1}^{\infty} E|Y|^v I(Dj^{\alpha} < |Y| \leqslant D(j+1)^{\alpha}) & \text{if } v > p, \end{cases} \\
&\quad \leqslant \begin{cases} C\sum_{j=1}^{\infty} E|Y|^p l(|Y|^{1/\alpha}) I(Dj^{\alpha} < |Y| \leqslant D(j+1)^{\alpha}) & \text{if } v < p, \\ C\sum_{j=1}^{\infty} E|Y|^p l(|Y|^{1/\alpha}) \log|Y| I(Dj^{\alpha} < |Y| \leqslant D(j+1)^{\alpha}) & \text{if } v = p, \\ C\sum_{j=1}^{\infty} E|Y|^v I(Dj^{\alpha} < |Y| \leqslant D(j+1)^{\alpha}) & \text{if } v > p, \end{cases} \\
&\quad \leqslant \begin{cases} CE|Y|^p l(|Y|^{1/\alpha}) & \text{if } v < p, \\ CE|Y|^p l(|Y|^{1/\alpha}) \log|Y| & \text{if } v = p, \\ CE|Y|^v, & \text{if } v > p, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-1} l(n) P(|Y| > Dn^{\alpha}) \\
&\quad = \sum_{n=1}^{\infty} n^{\alpha p-1} l(n) \sum_{j=n}^{\infty} P(Dj^{\alpha} < |Y| \leqslant D(j+1)^{\alpha}) \\
&\quad \leqslant C\sum_{j=1}^{\infty} j^{\alpha p} l(j) P(Dj^{\alpha} < |Y| \leqslant D(j+1)^{\alpha}) \\
&\quad \leqslant C\sum_{j=1}^{\infty} E|Y|^p l(|Y|^{1/\alpha}) I(Dj^{\alpha} < |Y| \leqslant D(j+1)^{\alpha}) \leqslant CE|Y|^p l(|Y|^{1/\alpha}).
\end{aligned}$$

For any $\tau > p$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-1} l(n) E|Y|^{\tau} I(|Y| \leqslant Dn^{\alpha}) \\
&\quad = \sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-1} l(n) \sum_{j=1}^{n} E|Y|^{\tau} I(D(j-1)^{\alpha} < |Y| \leqslant Dj^{\alpha}) \\
&\quad \leqslant C \sum_{j=1}^{\infty} j^{\alpha p-\alpha\tau} l(j) E|Y|^{\tau} I(D(j-1)^{\alpha} < |Y| \leqslant Dj^{\alpha}) \\
&\quad \leqslant C \sum_{j=1}^{\infty} E|Y|^{p} l(|Y|^{1/\alpha}) I(D(j-1)^{\alpha} < |Y| \leqslant Dj^{\alpha}) \leqslant CE|Y|^{p} l(|Y|^{1/\alpha}).
\end{aligned}$$

The proof is complete. ∎

3. MAIN RESULTS

Let $f : \mathbb{R}^{+} \to \mathbb{R}^{+}$ be an increasing and continuous function with $f(0) = 0$, and let $h : \mathbb{R}^{+} \to \mathbb{R}^{+}$ be the inverse function of $f(t)$, that is, $h(f(t)) = t$ for $t \geqslant 0$. Assume that for some positive constants v and δ,

$$\int_{f(\delta)}^{\infty} h^{-v}(t)\, dt < \infty. \tag{3.1}$$

Let $l(\cdot)$ be a slowly varying function.

Using the above functions f, h and l, we present our main results.

THEOREM 3.1. *Let $v > 0$, $\alpha > 1/2$, $p > 0$ and $\alpha(p \vee v) > 1$. Let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers, and $\{X_n, n \geqslant 1\}$ be the corresponding moving average process of a doubly infinite WOD sequence $\{Y_i, -\infty < i < \infty\}$ with $EY_i = 0$. Assume further that $\sum_{i=-\infty}^{\infty} |a_i|^v < \infty$ when $0 < v < 1$, and $\{Y_i, -\infty < i < \infty\}$ is stochastically dominated by a random variable Y such that*

$$\begin{cases} E|Y|^{p} l(|Y|^{1/\alpha}) < \infty & \text{if } v < p, \\ E|Y|^{p} l(|Y|^{1/\alpha}) \log|Y| < \infty & \text{if } v = p, \\ E|Y|^{v} < \infty & \text{if } v > p. \end{cases} \tag{3.2}$$

For $n \geqslant 1$, denote

$$S_n = \max_{1 \leqslant m \leqslant n} \Big| \sum_{k=1}^{m} X_k \Big| / (g^{1/v}(n) n^{\alpha}).$$

Then the sequence $\{S_n, n \geqslant 1\}$ converges f-moment completely with $c_n = n^{\alpha p-2} l(n)$, that is, for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) E f(\{|S_n| - \varepsilon\}_{+}) < \infty. \tag{3.3}$$

Proof. It is easily checked that

$$(3.4)\qquad \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) Ef(\{|S_n|-\varepsilon\}_+)$$
$$= \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \int_0^{\infty} P(|S_n| > \varepsilon + h(t))\,dt$$
$$= \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \int_0^{f(\delta)} P(|S_n| > \varepsilon + h(t))\,dt$$
$$+ \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \int_{f(\delta)}^{\infty} P(|S_n| > \varepsilon + h(t))\,dt$$
$$\triangleq I' + I''.$$

To prove (3.3), we only need to show $I' < \infty$ and $I'' < \infty$.

For I', because $g(n) \geqslant 1$, we can easily see $g^{1/v}(n) \geqslant 1$ for $v > 0$. By Markov's inequality, we have

$$\begin{aligned} I' &\leqslant f(\delta) \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P(|S_n| > \varepsilon) \\ &\leqslant f(\delta) \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\Big(\max_{1\leqslant m\leqslant n} \Big|\sum_{k=1}^{m} X_k\Big| > \varepsilon g^{1/v}(n) n^{\alpha}/2 + \varepsilon n^{\alpha}/2\Big) \\ &\leqslant C f(\delta) \sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2} l(n) g^{-1}(n) E\Big(\max_{1\leqslant m\leqslant n} \Big|\sum_{k=1}^{m} X_k\Big| - \varepsilon n^{\alpha}/2\Big)_+^{v}. \end{aligned}$$

Hence, to prove $I' < \infty$, we only need to prove that

$$(3.5)\qquad I^* \triangleq \sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2} l(n) g^{-1}(n) E\Big(\max_{1\leqslant m\leqslant n} \Big|\sum_{k=1}^{m} X_k\Big| - \varepsilon n^{\alpha}/2\Big)_+^{v} < \infty.$$

In order to prove (3.5), we consider the following two cases.

CASE 1: $0 < p \vee v < 1$. For $1 \leqslant i \leqslant n$, set

$$Y_i^{(n,1)} = Y_i I(|Y_i| \leqslant n^{\alpha}), \qquad Y_i^{(n,2)} = Y_i - Y_i^{(n,1)} = Y_i I(|Y_i| > n^{\alpha}).$$

Then, for each positive integer m,

$$\begin{aligned} \sum_{k=1}^{m} X_k &= \sum_{k=1}^{m} \sum_{i=-\infty}^{\infty} a_i Y_{i+k} = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m} Y_j \\ &= \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m} (Y_j^{(n,1)} + Y_j^{(n,2)}). \end{aligned}$$

Noting that $v < 1$, by (3.2), Lemmas 2.1, 2.5 and 2.6, the C_r-inequality, $\sum_{i=-\infty}^{\infty}|a_i|^v < \infty$, and $g^{-1}(n) \leqslant 1$, we find that

$$\begin{aligned}
I^* = & \sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2} l(n) g^{-1}(n) \\
& \times E\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m}(Y_j^{(n,1)}+Y_j^{(n,2)})\Big| - \varepsilon n^{\alpha}/2\Big)_+^v \\
\leqslant & \; C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} l(n) g^{-1}(n) E\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m} Y_j^{(n,1)}\Big|\Big) \\
& + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2} l(n) g^{-1}(n) E\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m} Y_j^{(n,2)}\Big|\Big)^v \\
\leqslant & \; C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} l(n) g^{-1}(n) \sum_{i=-\infty}^{\infty}|a_i| \sum_{j=i+1}^{i+n} E|Y_j| I(|Y_j| \leqslant n^{\alpha}) \\
& + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2} l(n) g^{-1}(n) \sum_{i=-\infty}^{\infty}|a_i|^v \sum_{j=i+1}^{i+n} E|Y_j|^v I(|Y_j| > n^{\alpha}) \\
\leqslant & \; C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1} l(n)[E|Y| I(|Y| \leqslant n^{\alpha}) + n^{\alpha} P(|Y| > n^{\alpha})] \\
& + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-1} l(n) E|Y|^v I(|Y| > n^{\alpha}) \\
\leqslant & \; CE|Y|^p l(|Y|^{1/\alpha}) + \begin{cases} CE|Y|^p l(|Y|^{1/\alpha}), & v < p, \\ CE|Y|^p l(|Y|^{1/\alpha}) \log|Y|, & v = p, \\ CE|Y|^v, & v > p, \end{cases} \\
< & \; \infty,
\end{aligned}$$

which implies (3.5).

CASE 2: $p \vee v \geqslant 1$. Take $\frac{1}{\alpha(p\vee v)} < q < 1$, and for each $n \geqslant 1$ and $-\infty < i < \infty$, set

$$\begin{aligned}
Y_i^{(n,1)} &= -n^{\alpha q} I(Y_i < -n^{\alpha q}) + Y_i I(|Y_i| \leqslant n^{\alpha q}) + n^{\alpha q} I(Y_i > n^{\alpha q}), \\
Y_i^{(n,2)} &= (Y_i - n^{\alpha q}) I(n^{\alpha q} < Y_i \leqslant n^{\alpha} + n^{\alpha q}) + n^{\alpha} I(Y_i > n^{\alpha} + n^{\alpha q}), \\
Y_i^{(n,3)} &= (Y_i + n^{\alpha q}) I(-n^{\alpha} - n^{\alpha q} \leqslant Y_i < -n^{\alpha q}) - n^{\alpha} I(Y_i < -n^{\alpha} - n^{\alpha q}), \\
Y_i^{(n,4)} &= (Y_i - n^{\alpha q} - n^{\alpha}) I(Y_i > n^{\alpha} + n^{\alpha q}), \\
Y_i^{(n,5)} &= (Y_i + n^{\alpha q} + n^{\alpha}) I(Y_i < -n^{\alpha} - n^{\alpha q}).
\end{aligned}$$

Then, for each positive integer m,

$$\sum_{k=1}^{m} X_k = \sum_{k=1}^{m}\sum_{i=-\infty}^{\infty} a_i Y_{i+k} = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m} Y_j = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m}\sum_{k=1}^{5} Y_j^{(n,k)}.$$

Since $\sum_{j=-\infty}^{\infty}|a_j| < \infty, EY_j = 0$, and $1/\alpha(p \vee v) < q < 1$, by (3.2) and Lemma 2.1 we have

$$\begin{aligned}
n^{-\alpha} \max_{1\leqslant m\leqslant n}\Big|E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m} Y_j^{(n,1)}\Big| &\leqslant n^{-\alpha}\sum_{i=-\infty}^{\infty}|a_i| \max_{1\leqslant m\leqslant n}\Big|\sum_{j=i+1}^{i+m} EY_j^{(n,1)}\Big| \\
&\leqslant n^{-\alpha}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}[E|Y_j|I(|Y_j|>n^{\alpha q}) + n^{\alpha q}P(|Y_j|>n^{\alpha q})] \\
&\leqslant Cn^{1-\alpha}E|Y|I(|Y|>n^{\alpha q}) \leqslant Cn^{1-\alpha}\cdot n^{\alpha q(1-(p\vee v))}E|Y|^{(p\vee v)} \\
&\leqslant Cn^{1-\alpha q(p\vee v)-\alpha(1-q)} \to 0, \quad n\to\infty.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
n^{-\alpha} \max_{1\leqslant m\leqslant n}\Big|E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m} Y_j^{(n,2)}\Big| &\leqslant n^{-\alpha}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n} E|Y_j|I(|Y_j|>n^{\alpha q}) \\
&\leqslant Cn^{1-\alpha}E|Y|I(|Y|>n^{\alpha q}) \\
&\leqslant Cn^{1-\alpha q(p\vee v)-\alpha(1-q)} \to 0, \quad n\to\infty,
\end{aligned}$$

and

$$\begin{aligned}
n^{-\alpha} \max_{1\leqslant m\leqslant n}\Big|E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m} Y_j^{(n,4)}\Big| &\leqslant n^{-\alpha}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n} E|Y_j|I(|Y_j|>n^{\alpha}) \\
&\leqslant Cn^{1-\alpha}E|Y|I(|Y|>n^{\alpha}) \\
&\leqslant Cn^{1-\alpha(p\vee v)} \to 0, \quad n\to\infty.
\end{aligned}$$

We can also deduce that

$$\begin{aligned}
n^{-\alpha} \max_{1\leqslant m\leqslant n}\Big|E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m} Y_j^{(n,3)}\Big| \to 0, \quad n\to\infty, \\
n^{-\alpha} \max_{1\leqslant m\leqslant n}\Big|E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m} Y_j^{(n,5)}\Big| \to 0, \quad n\to\infty.
\end{aligned}$$

From $EY_j = 0$, Lemma 2.5 and the C_r-inequality, when $\tau > v$ and $0 < v < 1$, it follows that

$$\begin{aligned}
I^* &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2}l(n)g^{-1}(n)E\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m}(Y_j^{(n,1)} - EY_j^{(n,1)})\Big| \\
&\quad + \max_{1\leqslant m\leqslant n}\Big|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m}\sum_{k=2}^{5} Y_j^{(n,k)}\Big| - \varepsilon n^{\alpha}/3\Big)_+^{v} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2}l(n)g^{-1}(n)E\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m}(Y_j^{(n,1)} - EY_j^{(n,1)})\Big| \\
&\quad + \sum_{k=2}^{5}\sum_{i=-\infty}^{\infty}|a_i|\Big|\sum_{j=i+1}^{i+n} Y_j^{(n,k)}\Big| - \varepsilon n^{\alpha}/3\Big)_+^{v}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2} l(n) g^{-1}(n) E\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m}(Y_j^{(n,1)}-EY_j^{(n,1)})\Big| \\
&\quad+\sum_{k=2}^{3}\sum_{i=-\infty}^{\infty}|a_i|\Big|\sum_{j=i+1}^{i+n}(Y_j^{(n,k)}-EY_j^{(n,k)})\Big| \\
&\quad+\sum_{k=4}^{5}\sum_{i=-\infty}^{\infty}|a_i|\Big|\sum_{j=i+1}^{i+n}Y_j^{(n,k)}\Big|-\varepsilon n^{\alpha}/4\Big)_+^{v} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2} l(n) g^{-1}(n) E\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m}(Y_j^{(n,1)}-EY_j^{(n,1)})\Big|^{\tau}\Big) \\
&\quad+C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2} l(n) g^{-1}(n)\sum_{k=2}^{3} E\Big(\sum_{i=-\infty}^{\infty}|a_i|\Big|\sum_{j=i+1}^{i+n}(Y_j^{(n,k)}-EY_j^{(n,k)})\Big|\Big)^{\tau} \\
&\quad+C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2} l(n) g^{-1}(n)\sum_{k=4}^{5} E\Big(\sum_{i=-\infty}^{\infty}|a_i|\Big|\sum_{j=i+1}^{i+n}Y_j^{(n,k)}\Big|\Big)^{v} \\
&\triangleq I_1+I_2+I_3+I_4+I_5;
\end{aligned}
$$

when $\tau > v \geqslant 1$, we similarly obtain

$$
\begin{aligned}
I^* &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2} l(n) g^{-1}(n) E\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m}(Y_j^{(n,1)}-EY_j^{(n,1)})\Big| \\
&\quad+\sum_{k=2}^{5}\sum_{i=-\infty}^{\infty}|a_i|\Big|\sum_{j=i+1}^{i+n}Y_j^{(n,k)}\Big|-\varepsilon n^{\alpha}/3\Big)_+^{v} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2} l(n) g^{-1}(n) E\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+m}(Y_j^{(n,1)}-EY_j^{(n,1)})\Big|^{\tau}\Big) \\
&\quad+C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2} l(n) g^{-1}(n)\sum_{k=2}^{3} E\Big(\sum_{i=-\infty}^{\infty}|a_i|\Big|\sum_{j=i+1}^{i+n}(Y_j^{(n,k)}-EY_j^{(n,k)})\Big|\Big)^{\tau} \\
&\quad+C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2} l(n) g^{-1}(n)\sum_{k=4}^{5} E\Big(\sum_{i=-\infty}^{\infty}|a_i|\Big|\sum_{j=i+1}^{i+n}(Y_j^{(n,k)}-EY_j^{(n,k)})\Big|\Big)^{v} \\
&\triangleq I_1+I_2+I_3+I_4'+I_5'.
\end{aligned}
$$

For fixed $n \geqslant 1$, Lemma 2.2 shows that $\{Y_j^{(n,k)} - EY_j^{(n,k)}, -\infty < j < \infty\}$ is still a sequence of zero mean WOD random variables for each $k = 1, \ldots, 5$. In order to prove our main result, we consider the following two situations.

CASE 2.1: $1 \leqslant p \vee v < 2$. Let $1 \leqslant p \vee v < \tau \leqslant 2$. Noting that $q < 1$, we have $\alpha(p-\tau)(1-q) - 1 < -1$, and thus by (3.2), Lemmas 2.1 and 2.4, Hölder's inequality, the C_r-inequality and $\sum_{i=-\infty}^{\infty} |a_i| < \infty$ we obtain

$$
\begin{aligned}
I_1 &\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2} l(n) g^{-1}(n) \\
&\quad \times E\Big\{ \sum_{i=-\infty}^{\infty} |a_i| \max_{1\leqslant m\leqslant n} \Big| \sum_{j=i+1}^{i+m} (Y_j^{(n,1)} - EY_j^{(n,1)}) \Big| \Big\}^{\tau} \\
&= C \sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2} l(n) g^{-1}(n) \\
&\quad \times E\Big\{ \sum_{i=-\infty}^{\infty} |a_i|^{1-1/\tau} \Big(|a_i|^{1/\tau} \max_{1\leqslant m\leqslant n} \Big| \sum_{j=i+1}^{i+m} (Y_j^{(n,1)} - EY_j^{(n,1)}) \Big| \Big) \Big\}^{\tau} \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2} l(n) g^{-1}(n) \Big(\sum_{i=-\infty}^{\infty} |a_i| \Big)^{\tau-1} \\
&\quad \times \sum_{i=-\infty}^{\infty} |a_i| E\Big(\max_{1\leqslant m\leqslant n} \Big| \sum_{j=i+1}^{i+m} (Y_j^{(n,1)} - EY_j^{(n,1)}) \Big|^{\tau} \Big) \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2} l(n) (\log n)^{\tau} \sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n} E|Y_j^{(n,1)}|^{\tau} \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2} l(n) (\log n)^{\tau} \sum_{i=-\infty}^{\infty} |a_i| \\
&\quad \times \sum_{j=i+1}^{i+n} [E|Y_j|^{\tau} I(|Y_j| \leqslant n^{\alpha q}) + n^{\alpha q\tau} P(|Y_j| > n^{\alpha q})] \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-1} l(n) (\log n)^{\tau} E|Y|^p n^{(\tau-p)\alpha q} \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha(p-\tau)(1-q)-1} l(n) (\log n)^{\tau} < \infty.
\end{aligned}
$$

For I_2, it follows from (3.2), Lemmas 2.1, 2.3 and 2.6 and the C_r-inequality that

$$
\begin{aligned}
I_2 &= C \sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2} l(n) g^{-1}(n) E\Big\{ \sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n} (Y_j^{(n,2)} - EY_j^{(n,2)}) \Big\}^{\tau} \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2} l(n) \sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n} E|Y_j^{(n,2)} - EY_j^{(n,2)}|^{\tau} \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2} l(n) \sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n} [E|Y_j|^{\tau} I(|Y_j| \leqslant 2n^{\alpha}) + n^{\alpha\tau} P(|Y_j| > n^{\alpha})] \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-1} l(n) E|Y|^{\tau} I(|Y| \leqslant 2n^{\alpha}) + C \sum_{n=1}^{\infty} n^{\alpha p-1} l(n) P(|Y| > n^{\alpha}) \\
&\leqslant CE|Y|^p l(|Y|^{1/\alpha}) < \infty.
\end{aligned}
$$

Similarly, we can also get $I_3 < \infty$.

For I_4, noting that $0 < v < 1$, by the C_r-inequality, Lemma 2.6 and $\sum_{i=-\infty}^{\infty}|a_i|^v < \infty$, we have

$$
\begin{aligned}
I_4 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2}l(n)g^{-1}(n)\sum_{i=-\infty}^{\infty}|a_i|^v E\Big|\sum_{j=i+1}^{i+n} Y_j^{(n,4)}\Big|^v \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2}l(n)\sum_{i=-\infty}^{\infty}|a_i|^v\sum_{j=i+1}^{i+n} E|Y_j|^v I(|Y_j|>n^\alpha) \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-1}l(n)E|Y|^v I(|Y|>n^\alpha) \\
&\leqslant \begin{cases} CE|Y|^p l(|Y|^{1/\alpha}), & v<p, \\ CE|Y|^p l(|Y|^{1/\alpha})\log|Y|, & v=p, \\ CE|Y|^v, & v>p, \end{cases} \\
&< \infty.
\end{aligned} \tag{3.6}
$$

Similar to the proofs of $I_2 < \infty$ and $I_4 < \infty$, we can derive that $I_4' < \infty, I_5 < \infty$ and $I_5' < \infty$. Then $I^* < \infty$ follows immediately.

CASE 2.2: $p \vee v \geqslant 2$. Let $\tau > \max\left\{\frac{\alpha p-1}{\alpha-1/2}, v, p\right\}$. Noting that $q < 1$, we have $\alpha(p-\tau)(1-q)-1 < -1$ and $(1/2-\alpha)\tau+\alpha p-2 < -1$. In this case, it is easy to see that $E|Y|^2 < \infty$. By (3.2), Lemmas 2.1 and 2.4 and $\sum_{i=-\infty}^{\infty}|a_i| < \infty$, we get

$$
\begin{aligned}
I_1 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2}l(n)g^{-1}(n)\sum_{i=-\infty}^{\infty}|a_i| \\
&\quad\times E\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{j=i+1}^{i+m}(Y_j^{(n,1)}-EY_j^{(n,1)})\Big|^\tau\Big) \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2}l(n)\sum_{i=-\infty}^{\infty}|a_i|(\log n)^\tau \\
&\quad\times\Big\{\sum_{j=i+1}^{i+n}E|Y_j^{(n,1)}|^\tau+\Big[\sum_{j=i+1}^{i+n}E(Y_j^{(n,1)})^2\Big]^{\tau/2}\Big\} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2}l(n)\sum_{i=-\infty}^{\infty}|a_i|(\log n)^\tau \\
&\quad\times\sum_{j=i+1}^{i+n}[E|Y_j|^\tau I(|Y_j|\leqslant n^{\alpha q})+n^{\alpha q\tau}P(|Y_j|>n^{\alpha q})] \\
&\quad+C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2}l(n)\sum_{i=-\infty}^{\infty}|a_i|(\log n)^\tau \\
&\quad\times\Big\{\sum_{j=i+1}^{i+n}[EY_j^2 I(|Y_j|\leqslant n^{\alpha q})+n^{2\alpha q}P(|Y_j|>n^{\alpha q})]\Big\}^{\tau/2}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-1} l(n)(\log n)^{\tau}\big[E|Y|^{\tau} I(|Y|\leqslant n^{\alpha q})+n^{\alpha q\tau} P(|Y|>n^{\alpha q})\big] \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau+\tau/2-2}(\log n)^{\tau} l(n) \\
&\quad \times \big[EY^2 I(|Y|\leqslant n^{\alpha q})+n^{2\alpha q} P(|Y|>n^{\alpha q})\big]^{\tau/2} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-1} l(n)(\log n)^{\tau} E|Y|^{p} n^{(\tau-p)\alpha q} \\
&\quad + C\sum_{n=1}^{\infty} n^{(1/2-\alpha)\tau+\alpha p-2} l(n)(\log n)^{\tau}(EY^2)^{\tau/2} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha(p-\tau)(1-q)-1} l(n)(\log n)^{\tau} + C\sum_{n=1}^{\infty} n^{(1/2-\alpha)\tau+\alpha p-2} l(n)(\log n)^{\tau} \\
&< \infty.
\end{aligned}
$$

For I_2, it follows from Lemmas 2.1, 2.3 and 2.6 that

$$
\begin{aligned}
I_2 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2} l(n) g^{-1}(n) \sum_{i=-\infty}^{\infty} |a_i| E\Big|\sum_{j=i+1}^{i+n}\big(Y_j^{(n,2)}-EY_j^{(n,2)}\big)\Big|^{\tau} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2} l(n) \sum_{i=-\infty}^{\infty} |a_i| \Big\{\sum_{j=i+1}^{i+n} E|Y_j^{(n,2)}|^{\tau} + \Big[\sum_{j=i+1}^{i+n} E(Y_j^{(n,2)})^2\Big]^{\tau/2}\Big\} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2} l(n) \sum_{i=-\infty}^{\infty} |a_i| \\
&\quad \times \sum_{j=i+1}^{i+n}\big[E|Y_j|^{\tau} I(|Y_j|\leqslant 2n^{\alpha})+n^{\alpha\tau} P(|Y_j|>n^{\alpha})\big] \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-2} l(n) \sum_{i=-\infty}^{\infty} |a_i| \Big(\sum_{j=i+1}^{i+n} EY_j^2\Big)^{\tau/2} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau-1} l(n) E|Y|^{\tau} I(|Y|\leqslant 2n^{\alpha}) + C\sum_{n=1}^{\infty} n^{\alpha p-1} l(n) P(|Y|>n^{\alpha}) \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\tau+\tau/2-2} l(n)(EY^2)^{\tau/2} \\
&\leqslant CE|Y|^{p} l(|Y|^{1/\alpha}) + C\sum_{n=1}^{\infty} n^{(1/2-\alpha)\tau+\alpha p-2} l(n)(EY^2)^{\tau/2} < \infty.
\end{aligned}
$$

Similarly, we get $I_3 < \infty$.

In addition, similar to the proof of (3.6), we have $I_4 < \infty, I_5 < \infty, I_4' < \infty$ and $I_5' < \infty$ if $v \leqslant 2$; and if $v > 2$, we have $(\alpha(p\vee v)-1)(1-v/2)-1 < -1$. By (3.2), and Lemmas 2.1, 2.3 and 2.6, we derive

$$
I_4' \leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2} l(n) g^{-1}(n) \sum_{i=-\infty}^{\infty} |a_i| E\Big|\sum_{j=i+1}^{i+n}\big(Y_j^{(n,4)}-EY_j^{(n,4)}\big)\Big|^{v}
$$

$$\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2}l(n)\sum_{i=-\infty}^{\infty}|a_i|\Big\{\sum_{j=i+1}^{i+n} E|Y_j^{(n,4)}|^v+\Big[\sum_{j=i+1}^{i+n} E(Y_j^{(n,4)})^2\Big]^{v/2}\Big\}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2}l(n)\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n} E|Y_j|^v I(|Y_j|>n^\alpha)\\
&\quad+C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2}l(n)\sum_{i=-\infty}^{\infty}|a_i|\Big[\sum_{j=i+1}^{i+n} E|Y_j|^2 I(|Y_j|>n^\alpha)\Big]^{v/2}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-1}l(n)E|Y|^v I(|Y|>n^\alpha)\\
&\quad+C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2+v/2}l(n)(E|Y|^{(p\vee v)}\cdot n^{(2-(p\vee v))\alpha})^{v/2}\\
&\leqslant C+C\sum_{n=1}^{\infty} n^{(\alpha(p\vee v)-1)(1-v/2)-1}l(n)<\infty.
\end{aligned}$$

Similarly, we find that $I_5'<\infty$.

Now, $I^*<\infty$ follows immediately from the statements above.

For I'', by (3.1), Markov's inequality and $g^{1/v}(n)\geqslant 1$ for $v>0$, we have

$$\begin{aligned}
I''&\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2}l(n)\int_{f(\delta)}^{\infty} P\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{k=1}^{m} X_k\Big|>\varepsilon n^\alpha+h(t)g^{1/v}(n)\cdot n^\alpha\Big)\,dt\\
&=\sum_{n=1}^{\infty} n^{\alpha p-2}l(n)\int_{f(\delta)}^{\infty} P\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{k=1}^{m} X_k\Big|-\varepsilon n^\alpha>h(t)g^{1/v}(n)\cdot n^\alpha\Big)\,dt\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2}l(n)g^{-1}(n)E\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{k=1}^{m} X_k\Big|-\varepsilon n^\alpha\Big)_+^v\int_{f(\delta)}^{\infty} h^{-v}(t)\,dt\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2}l(n)g^{-1}(n)E\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{k=1}^{m} X_k\Big|-\varepsilon n^\alpha\Big)_+^v.
\end{aligned}$$

Hence, to prove $I''<\infty$, we only need to show that

$$I^{**}\triangleq\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2}l(n)g^{-1}(n)E\Big(\max_{1\leqslant m\leqslant n}\Big|\sum_{k=1}^{m} X_k\Big|-\varepsilon n^\alpha\Big)_+^v<\infty. \tag{3.7}$$

The proof of (3.7) is similar to that of (3.5), one also has to consider two cases; we omit the details.

Now, $I'<\infty$ and $I''<\infty$ follow immediately from the statements above. This completes the proof of the theorem. ■

REMARK 3.1. Theorem 3.1 generalizes Theorem 1.2 of Qiu and Xiao [16] for END random variables to the case of the moving average process of WOD random variables, and strengthens the result of complete moment convergence to complete f-moment convergence. What is more, we also consider the case $0<p\vee v<1$, which was not investigated by Qiu and Xiao [16], and the condition of identical distribution is replaced by the weaker condition of stochastic domination.

REMARK 3.2. Compared with Theorem 3.1 of Lu et al. [13], we use different assumptions to prove complete f-moment convergence for WOD random variables in Theorem 3.1. Meanwhile, when $\{a_i, -\infty < i < \infty\}$ take appropriate values, we can see that partial sums of the moving average process based on WOD random variables reduce to partial sums of sequences of WOD random variables, so our result extends the corresponding one of Lu et al. [13].

REMARK 3.3. Concerning the condition on $f(t)$, we point out that (3.1) holds trivially if we take $f(t) = t^s$ for $t \geqslant 0$ and some $0 < s < v$. Hence, we can get the following corollary from Theorem 3.1.

COROLLARY 3.1. *Let* $v > 0$, $\alpha > 1/2$, $p > 0$ *and* $\alpha(p \vee v) > 1$. *Let* $\{a_i, -\infty < i < \infty\}$ *be an absolutely summable sequence of real numbers, and* $\{X_n, n \geqslant 1\}$ *be the corresponding moving average process of a doubly infinite WOD sequence* $\{Y_i, -\infty < i < \infty\}$ *with* $EY_i = 0$. *Assume further that* $\sum_{i=-\infty}^{\infty} |a_i|^v < \infty$ *when* $0 < v < 1$, *and* $\{Y_i, -\infty < i < \infty\}$ *is stochastically dominated by a random variable* Y *such that*

$$\begin{cases} E|Y|^p l(|Y|^{1/\alpha}) < \infty & \text{if } v < p, \\ E|Y|^p l(|Y|^{1/\alpha}) \log|Y| < \infty & \text{if } v = p, \\ E|Y|^v < \infty & \text{if } v > p. \end{cases}$$

Then for all $\varepsilon > 0$ *and any* $0 < s < v$,

$$\sum_{n=1}^{\infty} n^{\alpha p - \alpha s - 2} l(n) E\Big(\max_{1 \leqslant m \leqslant n} \Big| \sum_{k=1}^{m} X_k \Big| / g^{1/v}(n) - \varepsilon n^{\alpha} \Big)_+^s < \infty.$$

Furthermore, for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) P\Big(\max_{1 \leqslant m \leqslant n} \Big| \sum_{k=1}^{m} X_k \Big| > \varepsilon n^{\alpha} g^{1/v}(n) \Big) < \infty. \tag{3.8}$$

REMARK 3.4. Let $\alpha p - 2 = 0$ and $l(n) \equiv 1$. It can be deduced from (3.8) that

$$n^{-\alpha} \max_{1 \leqslant m \leqslant n} \Big| \sum_{k=1}^{n} X_k \Big| / g^{1/v}(n) \to 0 \text{ completely as } n \to \infty. \tag{3.9}$$

REMARK 3.5. Compared with Corollaries 4.1 and 4.2 of Shen and Wu [18], we relax the restriction on the dominating coefficients $g(n)$ for $n \geqslant 1$ in Corollary 3.1. What is more, if we choose $s = q, \alpha = 1/p, a_0 = 1$, and $a_i \equiv 0$ if $-\infty < i < \infty$ and $i \neq 0$, then we can apply Corollary 3.1 to obtain Corollary 4.2 of Shen and Wu [18].

4. AN APPLICATION TO NONPARAMETRIC REGRESSION MODELS

In this section, we present an application of complete convergence to nonparametric regression models based on moving average processes of WOD errors.

Consider the following nonparametric regression model:

$$Y_k = f(x_k) + \epsilon_k, \quad k = 1, \dots, n, \ n \geqslant 1, \tag{4.1}$$

where x_k are known fixed design points from $A \subset \mathbb{R}^d$, a given compact set for some positive integer $d \geqslant 1$, $f(\cdot)$ is an unknown regression function defined on A, and ϵ_k are random errors. We will consider the following weighted linear regression estimator of $f(\cdot)$:

$$f_n(x) = \sum_{k=1}^{n} W_{nk}(x) Y_k, \quad x \in A \subset \mathbb{R}^d, \tag{4.2}$$

where $W_{nk}(x) = W_{nk}(x; x_1, \dots, x_n)$, $k = 1, \dots, n$, $n \geqslant 1$, are weight functions.

Such an estimator with constant weight was first proposed by Stone [19], adapted by Georgiev [7] to the fixed design case, and then studied by many authors. Roussas [17] considered the fixed regression model with general weights, and supposed that the error random variables come from a strictly stationary stochastic process satisfying the strong mixing condition. Tran et al. [21] investigated a regression function on the basis of noisy observations taken at uniformly spaced design points. Horowitz and Lee [8] considered nonparametric estimation of a regression function that is identified by requiring a specified quantile of the regression error conditional on an instrumental variable being zero. Wang et al. [23] investigated complete consistency for a weighted linear regression estimator under negatively superadditive-dependent errors by using complete convergence. Xi et al. [29] obtained some convergence properties for partial sums of WOD random variables and gave applications to nonparametric regression models. Yang et al. [30] established complete consistency and convergence rate for weighted estimators in nonparametric regression models.

In this subsection, let $c(f)$ denote the set of continuity points of the function f on A. The symbol $\|x\|$ denotes the Euclidean norm. For any fixed design point $x \in A$, the following assumptions on the weight functions $W_{nk}(x)$ will be used:

(H_1) $\sum_{k=1}^{n} W_{nk}(x) \to 1$ as $n \to \infty$;

(H_2) $\sum_{k=1}^{n} |W_{nk}(x)| \leqslant C < \infty$ for all n;

(H_3) $\sum_{k=1}^{n} |W_{nk}(x)| \cdot |f(x_k) - f(x)| I(\|x_k - x\| > a) \to 0$ as $n \to \infty$ for all $a > 0$.

Based on the assumptions above, we will further study complete consistency for the nonparametric regression estimator $f_n(x)$.

THEOREM 4.1. *Let $1/2 < \alpha \leqslant 1, \alpha p = 2$ and $1 \leqslant v < p$. Let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers, and $\{\epsilon_n, n \geqslant 1\}$ be the corresponding moving average process of a doubly infinite WOD sequence $\{\delta_i, -\infty < i < \infty\}$ with $E\delta_i = 0$, that is,*

$$\epsilon_n = \sum_{i=-\infty}^{\infty} a_i \delta_{i+n}.$$

Suppose that conditions (H_1)–(H_3) hold true, and

$$\max_{1 \leqslant k \leqslant n} |W_{nk}(x)| = O(n^{-\alpha} g^{-1/v}(n)).$$

Assume further that $\{\delta_i, -\infty < i < \infty\}$ is stochastically dominated by a random variable δ with $E|\delta|^p < \infty$. Then for all $x \in c(f)$,

$$f_n(x) \to f(x) \quad \textit{completely as } n \to \infty. \tag{4.3}$$

Proof. For $a > 0$ and $x \in c(f)$, we infer from (4.1) and (4.2) that

$$\begin{aligned} |Ef_n(x) - f(x)| \leqslant & \sum_{k=1}^{n} |W_{nk}(x)| \cdot |f(x_k) - f(x)| I(\|x_k - x\| \leqslant a) \\ & + \sum_{k=1}^{n} |W_{nk}(x)| \cdot |f(x_k) - f(x)| I(\|x_k - x\| > a) \\ & + |f(x)| \cdot \Big| \sum_{k=1}^{n} W_{nk}(x) - 1 \Big|. \end{aligned} \tag{4.4}$$

Since $x \in c(f)$, it follows that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x' satisfying $\|x' - x\| < \delta$, we have $|f(x') - f(x)| < \varepsilon$. Taking $a \in (0, \delta)$ in (4.4), we have

$$\begin{aligned} |Ef_n(x) - f(x)| \leqslant & \, \varepsilon \sum_{k=1}^{n} |W_{nk}(x)| \\ & + \sum_{k=1}^{n} |W_{nk}(x)| \cdot |f(x_k) - f(x)| I(\|x_k - x\| > a) \\ & + |f(x)| \cdot \Big| \sum_{k=1}^{n} W_{nk}(x) - 1 \Big|. \end{aligned}$$

By assumptions (H_1)–(H_3) and since $\varepsilon > 0$ is arbitrary, we see that for all $x \in c(f)$,

$$\lim_{n \to \infty} Ef_n(x) = f(x).$$

Hence, to prove (4.3), it suffices to show

$$f_n(x) - Ef_n(x) = \sum_{k=1}^{n} W_{nk}(x) \epsilon_k \to 0 \quad \text{completely as } n \to \infty. \tag{4.5}$$

Noting that $\max_{1\leqslant k\leqslant n}|W_{nk}(x)| = O(n^{-\alpha}g^{-1/v}(n))$ and $v < p$, we apply (3.9) with $l(n) = 1, X_k = \epsilon_k$ and $Y_i = \delta_i$. Then we have

$$\begin{aligned}\Big|\sum_{k=1}^{n} W_{nk}(x)\epsilon_k\Big| &\leqslant \max_{1\leqslant k\leqslant n}|W_{nk}(x)| \max_{1\leqslant m\leqslant n}\Big|\sum_{k=1}^{m}\epsilon_k\Big| \\ &\leqslant Cn^{-\alpha}g^{-1/v}(n)\max_{1\leqslant m\leqslant n}\Big|\sum_{k=1}^{m}\epsilon_k\Big| \to 0 \quad \text{completely as } n\to\infty,\end{aligned}$$

which implies (4.5). Hence, we immediately get (4.3). ■

5. NUMERICAL SIMULATION

In this section, we will present a simulation to study the convergence behavior of (4.3) and the numerical performance of the consistency for the nonparametric regression estimator $f_n(x)$ by using R software.

Let $a_0 = 1$ and $a_i \equiv 0$ if $-\infty < i < \infty$ and $i \neq 0$. Then for each $n \geqslant 1$, we have

$$\epsilon_n = \sum_{i=-\infty}^{\infty} a_i\delta_{i+n} = \delta_n.$$

The data are generated from model (4.1). The marginal distributions of $\delta_1, \ldots, \delta_n$ are such that $(\delta_1, \delta_2), (\delta_3, \delta_4), \ldots, (\delta_{2m-1}, \delta_{2m}), \ldots$ follow the joint distribution of the FGM copula

$$C_{\theta_n}(u, v) = uv + \theta_n uv(1-u)(1-v), \qquad (u, v) \in [0,1]\times[0,1],$$

with $\theta_n = n^{-1}$. We find that $\{\delta_n, n \geqslant 1\}$ is a sequence of WOD random variables with $g(n) = O(n)$; we refer to Wang et al. [22].

We consider the nearest neighbor weight function estimator $f_n(x)$. Let $A = [0,1]$, and $x_i = i/n$, $i = 1,\ldots,n$. For any $x \in A$, we rewrite $|x_1 - x|, |x_2 - x|, \ldots, |x_n - x|$ as follows:

$$|x_{R_1(x)} - x| \leqslant |x_{R_2(x)} - x| \leqslant \cdots \leqslant |x_{R_n(x)} - x|.$$

If $|x_i - x| = |x_j - x|$, then $|x_i - x|$ is permuted before $|x_j - x|$ when $x_i < x_j$.

Let $1 \leqslant k_n \leqslant n$. We define the nearest neighbor weight function estimator as follows:

$$f_n(x) = \sum_{i=1}^{n} \widetilde{W}_{ni}(x)Y_i,$$

where

$$\widetilde{W}_{ni}(x) = \begin{cases} 1/k_n & \text{if } |x_i - x| \leqslant |x_{R_{k_n}(x)} - x|, \\ 0 & \text{otherwise.}\end{cases}$$

TABLE 1. MSE and MAE of the estimator $f_n(x)$ for $f(x) = \ln(x+1)$

$f(x) = \ln(x+1)$		$n = 500$	$n = 1000$	$n = 5000$
$x = 0.25$	MSE	0.00055281	0.00032644	0.00009642
	MAE	0.01878854	0.01446009	0.00783691
$x = 0.5$	MSE	0.00049672	0.00030170	0.00009410
	MAE	0.01767296	0.01370186	0.00768290
$x = 0.75$	MSE	0.00052272	0.00032653	0.00009884
	MAE	0.01819816	0.01441898	0.00780354

TABLE 2. MSE and MAE of the estimator $f_n(x)$ for $f(x) = \sin(2\pi x)$

$f(x) = \sin(2\pi x)$		$n = 500$	$n = 1000$	$n = 5000$
$x = 0.25$	MSE	0.00061419	0.00038020	0.00012965
	MAE	0.01964917	0.01577989	0.00897997
$x = 0.5$	MSE	0.00051962	0.00030750	0.00010701
	MAE	0.01788127	0.01415661	0.00822577
$x = 0.75$	MSE	0.00085179	0.00047776	0.00012971
	MAE	0.02388512	0.01744242	0.00910759

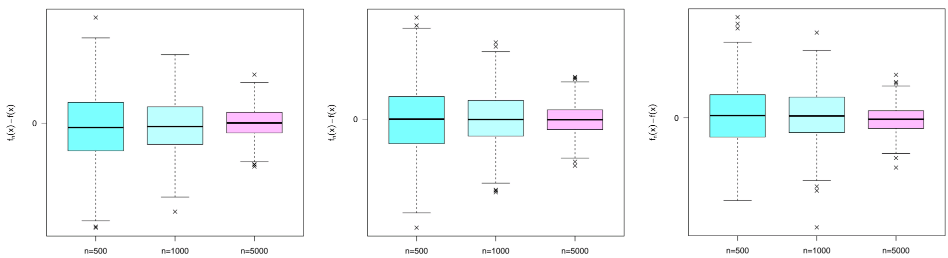

FIGURE 1. Boxplots of $f_n(x)$ with $x = 0.25, 0.5$ and 0.75 when $f(x) = \ln(x+1)$

We take $k_n = \lfloor n^{0.77} \rfloor, \alpha = \frac{51}{100}, p = \frac{200}{51}$ and $v = \frac{50}{13}$ in Theorem 4.1. As stated in Wang et al. [22], conditions (H_1)–(H_3) are satisfied for $\widetilde{W}_{ni}(x)$. We choose $f(x) = \ln(x+1)$ and $f(x) = \sin(2\pi x)$, take the points $x = 0.25, 0.5, 0.75$ and the sample sizes $n = 500, 1000, 5000$, respectively, and compute the values of $f_n(x) - f(x)$ for 500 times. Tables 1 and 2 show the mean squared error (MSE) and the mean absolute error (MAE) of $f_n(x)$. From the tables, we can see that both MSE and MAE decrease markedly as n increases.

Figures 1 and 2 show the boxplots of $f_n(x) - f(x)$ for $f(x) = \ln(x+1)$ and $f(x) = \sin(2\pi x)$, respectively. With the increase of n, the boxes become

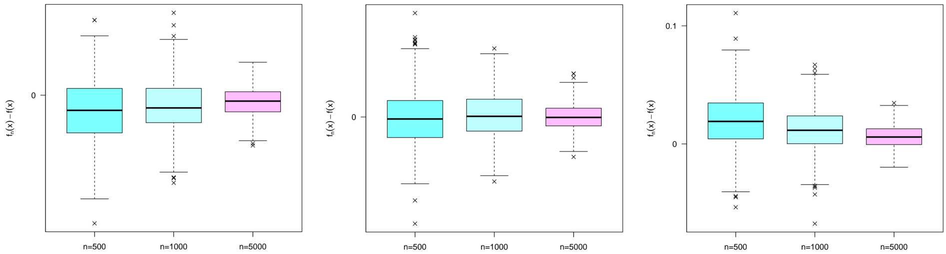

FIGURE 2. Boxplots of $f_n(x)$ with $x = 0.25, 0.5$ and 0.75 when $f(x) = \sin(2\pi x)$

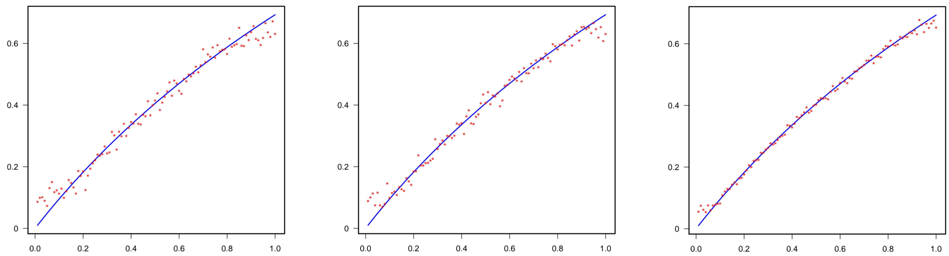

FIGURE 3. Fitting points of $f_n(x)$ with $n = 500, 1000$ and 5000 when $f(x) = \ln(x + 1)$

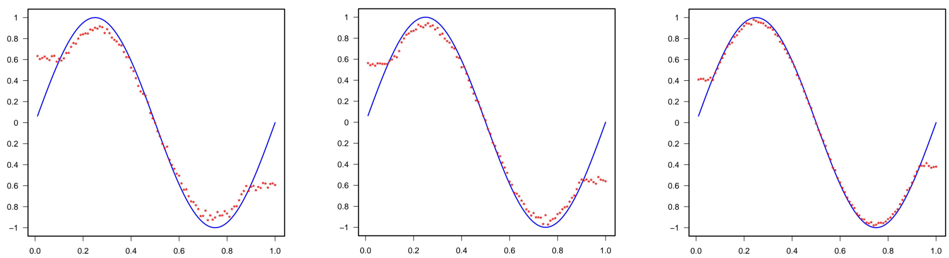

FIGURE 4. Fitting points of $f_n(x)$ with $n = 500, 1000$ and 5000 when $f(x) = \sin(2\pi x)$

narrower and the values of the mean are closer to 0. This shows that the simulation is consistent with the conclusion.

Figures 3 and 4 show the fitting curves of $f_n(x)$ for $f(x) = \ln(x + 1)$ and $f(x) = \sin(2\pi x)$ with sample sizes $n = 500, 1000, 5000$, respectively. From these curves, we can see that the larger the n-value is, the closer the points of $f_n(n)$ are to the line of $f(x)$. This shows that our results are quite effective.

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