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LARGE DEVIATIONS FOR GENERALIZED CONDITIONED GAUSSIAN PROCESSES AND THEIR BRIDGES

BY

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Abstract. We study the asymptotic behavior of a Gaussian process conditioned to n linear functionals of its paths and of the bridge of such a process. In particular, functional large deviation results are stated for small time. Two examples are considered.

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1. INTRODUCTION

Let $X = (X_t)_{t\geqslant 0}$ be a continuous, real Gaussian process with continuous and positive definite covariance function $k(t,s) = \mathrm{Cov}(X_t, X_s)$, mean function $m(t) = \mathbb{E}(X_t)$ of bounded variation and $X_0 = m(0)$. For fixed $n \in \mathbb{N}$ and $T > 0$, we consider the conditioning of the process X on n linear functionals $\boldsymbol{G}_T(X) = \big(G_T^1(X), \ldots, G_T^n(X)\big)^\intercal$ of its path,

$$\boldsymbol{G}_T(X) = \int_0^T \boldsymbol{g}(t)dX_t = \Big(\int_0^T g_1(t)dX_t, \ldots, \int_0^T g_n(t)dX_t\Big)^\intercal,$$

where $\boldsymbol{g} = (g_1, \ldots, g_n)^\intercal$ is a suitable vectorial function defined on $[0,T]$ and $\int_0^T g_i(t)dX_t$, $i = 1, \ldots, n$, is the Wiener integral (as defined in Subsection 2.2). We assume, without any loss of generality, that the functions g_i, $i = 1, \ldots, n$, are linearly independent. The linearly dependent components of $\boldsymbol{g}$ can be simply removed from the conditioning. Informally, the generalized conditioned process $X^{\boldsymbol{g};\boldsymbol{x}}$, for $\boldsymbol{x} \in \mathbb{R}^n$, is the law of the Gaussian process X conditioned on the set

$$\Big\{\int_0^T \boldsymbol{g}(t)dX_t = \boldsymbol{x}\Big\} = \bigcap_{i=1}^n \Big\{\int_0^T g_i(t)dX_t = x_i\Big\}.$$

We will work on the canonical filtered probability space $\big(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P}\big)$, where $\Omega = C([0,T])$ is the space of (real) continuous functions defined on $[0,T]$, $\mathscr{F}$ is the Borel σ-algebra on $C([0,T])$ with respect to the uniform norm, and $\mathbb{P}$ is the Gaussian measure corresponding to the Gaussian coordinate process $X_t(\omega) = \omega_t$. The filtration $(\mathscr{F}_t)_{t\in[0,T]}$ is the intrinsic filtration of the coordinate process X that is augmented with the null sets and made right-continuous. Within this framework the law $\mathbb{P}^{\boldsymbol{g};\boldsymbol{x}}$ of $X^{\boldsymbol{g};\boldsymbol{x}}$ is the regular conditional distribution

$$\mathbb{P}^{\boldsymbol{g};\boldsymbol{x}}(X \in E) = \mathbb{P}(X^{\boldsymbol{g};\boldsymbol{x}} \in E) = \mathbb{P}\Big(X \in E \Big| \int_0^T \boldsymbol{g}(t)dX_t = \boldsymbol{x}\Big), \quad E \in \mathscr{F}.$$

For more details see [11]. In this paper we consider the asymptotic behavior for small time of the conditioned process and of the bridge of the conditioned process. In particular, functional large deviation results are stated for the family $\{(X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon t} - X^{\boldsymbol{g};\boldsymbol{x}}_T)_{t\in[0,1]}\}_\varepsilon$ as $\varepsilon \to 0$ and for the bridge of such a family of processes. Two examples are considered: fractional Brownian motion and integrated Gaussian processes. In particular, we generalize the results in [2], where functional large deviation results are given for Gaussian processes (and their bridges) conditioned to stay in n fixed points at n fixed past instants. This kind of results improves numerical simulations concerning some processes that have to be killed as soon as a prescribed level is reached. Hence, this paper tries to complete the chain of results about the small time behavior of conditioned Gaussian processes, which was started with the Markovian case in [1] and later developed in [3] and [6].

The paper is organized as follows. After a brief recall of some results related to large deviations for Gaussian process and integration (Section 2), we first obtain a functional large deviation result for the conditioned process (Section 3) and then (Section 4) we obtain a functional large deviation result for the bridge.

2. PRELIMINARY RESULTS

In this section we briefly recall some main facts related to reproducing kernel Hilbert space and integration for Gaussian processes. There are many references in the literature on this topic where all details and proofs can be found; some classical references include, for example, [5] and [4]. Let us note that in Subsection 2.1 we consider the process only in the interval $[0,1]$ (since large deviations are stated for processes defined on this interval).

2.1. Reproducing kernel Hilbert space and large deviations. Let $U = (U_t)_{0\leqslant t\leqslant 1}$ be a continuous centered (for simplicity) real Gaussian process with $U_0 = 0$ defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with covariance function $R(t,s) = \mathrm{Cov}(U_t, U_s)$, $t, s \in [0,1]$. Throughout the paper, $\mathscr{M}[0,1]$ will denote the dual set of $C([0,1])$, i.e., the set of the signed Borel measures on $[0,1]$, and for any $\lambda \in \mathscr{M}[0,1]$, $\langle \lambda, \cdot\rangle$ will stand for the associated linear functional: $\langle \lambda, h\rangle = \int_0^1 h(t)d\lambda(t)$,

$h \in C([0,1])$. Then, for any $\lambda \in \mathscr{M}[0,1]$, $\langle \lambda, U \rangle = \int_0^1 U_t \, d\lambda(t)$ is a centered Gaussian random variable taking values on $\mathbb{R}$. We have

$$\mathrm{Var}(\langle \lambda, U \rangle) = \int_0^1 \int_0^1 R(t,s) d\lambda(t) d\lambda(s) \quad \text{for any } \lambda \in \mathscr{M}[0,1].$$

The associated reproducing kernel Hilbert space $\mathscr{H}$ is a Hilbert space in $C([0,1])$ which is usually defined through the following dense subset:

$$\mathscr{T} = \Big\{ h \in C([0,1]) \,:\, h(t) = \int_0^1 R(t,s) d\lambda(s) \text{ with } \lambda \in \mathscr{M}[0,1] \Big\}.$$

For $h_1, h_2 \in \mathscr{T}$, with

$$h_1(t) = \int_0^1 R(t,s) d\lambda_1(s), \quad h_2(t) = \int_0^1 R(t,s) d\lambda_2(s),$$

the inner product is defined as follows:

$$(h_1, h_2)_{\mathscr{H}} = \int_0^1 \int_0^1 R(t,s) d\lambda_1(t) d\lambda_2(s).$$

In the sequel, we will speak about "the reproducing kernel Hilbert space associated with the covariance function $R(t,s)$". In fact, given a continuous symmetric and positive definite function $R(t,s)$ defined on $[0,1] \times [0,1]$, one can build a centered and continuous Gaussian process $U = (U_t)_{t \in [0,1]}$ having R as its covariance function. For more details see [2].

The main property we are going to use is related to the Cramér transform of a continuous Gaussian process (see Section 3.4 in [5] for details).

THEOREM 2.1 (Cramér transform). *Let I denote the Cramér transform*, *that is*,

$$\begin{aligned} I(x) &= \sup_{\lambda \in \mathscr{M}[0,1]} \big(\langle \lambda, x \rangle - \log \mathbb{E}(e^{\langle \lambda, U \rangle}) \big) \\ &= \sup_{\lambda \in \mathscr{M}[0,1]} \big(\langle \lambda, x \rangle - \tfrac{1}{2} \int_0^1 \int_0^1 R(t,s) \lambda(dt) \lambda(ds) \big). \end{aligned}$$

Then,

$$I(x) = \begin{cases} \dfrac{1}{2} \|x\|_{\mathscr{H}}^2 & \text{if } x \in \mathscr{H}, \\ +\infty & \text{otherwise.} \end{cases}$$

In the following, $\gamma_\varepsilon > 0$ will denote an infinitesimal function ($\gamma_\varepsilon \to 0$ as $\varepsilon \to 0$).

The rate γ_ε^2 will play the role of the inverse speed of the large deviation principles we are going to study.

DEFINITION 2.1. A family of continuous processes $\{U^\varepsilon\}_\varepsilon = \{(U_t^\varepsilon)_{t\in[0,1]}\}_\varepsilon$ is *exponentially tight* with respect to the speed function γ_ε^2 if, for all $b > 0$, there exists a compact $K_b \subset C([0,1])$ such that

$$\limsup_{\varepsilon\to 0} \gamma_\varepsilon^2 \log \mathbb{P}(U^\varepsilon \notin K_b) \leqslant -b.$$

DEFINITION 2.2. A family of processes $\{U^\varepsilon\}_\varepsilon = \{(U_t^\varepsilon)_{t\in[0,1]}\}_\varepsilon$ satisfies a *large deviation principle* on $C([0,1])$ with the inverse speed γ_ε^2 and the (good) rate function I if

- $\lim_{\varepsilon\to 0} \gamma_\varepsilon = 0$;
- the set $\{I \leqslant a\}$ is a compact in $C([0,1])$ for any fixed $a \in \mathbb{R}$;

and the following inequalities hold:

- for any open set G in $C([0,1])$,

$$\liminf_{\varepsilon\to 0} \gamma_\varepsilon^2 \log \mathbb{P}(U^\varepsilon \in G) \geqslant -\inf_{h\in G} I(h);$$

- for any closed set F in $C([0,1])$,

$$\limsup_{\varepsilon\to 0} \gamma_\varepsilon^2 \log \mathbb{P}(U^\varepsilon \in F) \leqslant -\inf_{h\in F} I(h).$$

Suppose $\{U^\varepsilon\}_\varepsilon$ is a family of continuous Gaussian processes. Because of the special form of the Laplace transform for Gaussian measures, by applying the Gärtner–Ellis theorem (see, e.g., [4]) a large deviation principle can be stated if a nice asymptotic behavior holds for the Laplace transforms, as summarized in the following theorem.

THEOREM 2.2. *Let $\{U^\varepsilon\}_\varepsilon$ be an exponentially tight family of continuous Gaussian processes with respect to the speed function γ_ε. Suppose that, for any $\lambda \in \mathscr{M}[0,1]$,*

$$0 = \lim_{\varepsilon\to 0} \mathbb{E}(\langle\lambda, U^\varepsilon\rangle) \ \text{ and } \ \Lambda(\lambda) = \lim_{\varepsilon\to 0} \frac{\operatorname{Var}(\langle\lambda, U^\varepsilon\rangle)}{\gamma_\varepsilon^2} \equiv \int_0^1\int_0^1 \bar{R}(t,s)\lambda(dt)\lambda(ds)$$

for some continuous, symmetric and positive definite function $\bar{R}$. Then, $\{U^\varepsilon\}_\varepsilon$ satisfies a large deviation principle on $C([0,1])$ with the inverse speed γ_ε^2 and the (good) rate function

$$I(h) = \begin{cases} \dfrac{1}{2}\|h\|_{\bar{\mathscr{H}}}^2 & \text{if } h \in \bar{\mathscr{H}}, \\ +\infty & \text{otherwise}, \end{cases} \tag{2.1}$$

where $\bar{\mathscr{H}}$ and $\|\cdot\|_{\bar{\mathscr{H}}}$ denote, respectively, the reproducing kernel Hilbert space and the norm associated with the covariance function $\bar{R}$.

To prove exponential tightness we shall use the following result (see Proposition 2.1 of [8]).

PROPOSITION 2.1. *Let* $\{U^\varepsilon\}_\varepsilon$ *be a family of continuous Gaussian processes, where* $U_0^\varepsilon = 0$ *for all* $\varepsilon > 0$. *Suppose there exist constants* $\alpha, M_1, M_2 > 0$ *such that for* $\varepsilon > 0$

$$\sup_{s,t\in[0,1],s\neq t} \frac{|\mathbb{E}(U_t^\varepsilon) - \mathbb{E}(U_s^\varepsilon)|}{|t-s|^\alpha} \leqslant M_1 \tag{2.2}$$

and

$$\sup_{s,t\in[0,1],s\neq t} \frac{|\mathrm{Cov}(U_t^\varepsilon, U_t^\varepsilon) - 2\mathrm{Cov}(U_s^\varepsilon, U_t^\varepsilon) + \mathrm{Cov}(U_s^\varepsilon, U_s^\varepsilon)|}{\gamma_\varepsilon^2\, |t-s|^{2\alpha}} \leqslant M_2. \tag{2.3}$$

Then $\{U^\varepsilon\}_\varepsilon$ *is exponentially tight with respect to the speed function* γ_ε^2.

2.2. Wiener integrals. In this section we will restate basic facts and definitions related to the Wiener integral. In our approach we will follow [7]. Let $U = (U_t)_{t\geqslant 0}$ be a continuous centered (for simplicity) Gaussian process with $U_0 = 0$, defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and with covariance function $R(t,s) = \mathrm{Cov}(U_t, U_s)$, $t, s \geqslant 0$.

For $T > 0$ we want to define, for suitable functions f, the integral

$$I_T(f) = \int_0^T f(t)dU_t.$$

Consider the set $\mathscr{S}_T$ of all step functions on $[0,T)$, $f(t) = \sum_{n=1}^N f_n 1_{[a_n,b_n)}(t)$ with $0 \leqslant a_n < b_n \leqslant T$, $f_n \in \mathbb{R}$ for $n = 1, \ldots, N$ ($N \in \mathbb{N}$), and for $f \in \mathscr{S}_T$ define

$$\int_0^T f(t)dU_t = \sum_{n=1}^N f_n(U_{b_n} - U_{a_n}).$$

$\mathscr{S}_T$ is clearly a linear space and for $f, g \in \mathscr{S}_T$, we have

$$\mathbb{E}\Big[\int_0^T f(t)dU_t\Big] = 0,$$

and

$$\mathbb{E}\Big[\int_0^T f(t)dU_t \int_0^T g(t)dU_t\Big] = \int_0^T\int_0^T f(t)g(s)d^2R(t,s).$$

If we define, for $f, g \in \mathscr{S}_T$,

$$(f,g) = \int_0^T\int_0^T f(t)g(s)d^2R(t,s),$$

or, equivalently, for $s, t \geqslant 0$

$$(1_{[0,t)}, 1_{[0,s)}) = R(t,s),$$

then $(\cdot,\cdot)$ is an inner product and the completion $\Lambda_T(R)$ of $\mathscr{S}_T$ is a Hilbert space.

Now, let $H_T(U)$ be a closed linear subspace of $\mathbb{L}^2(\Omega, \mathscr{F}, \mathbb{P})$ spanned by $U = (U_t)_{0 \leqslant t \leqslant T}$. We can establish an isomorphism between $H_T(U)$ and $\Lambda_T(R)$ as follows. The map $\mathscr{S}_T \to H_T(U)$, $f \mapsto \int_0^T f(t)dU_t$ preserves inner products and hence it can be extended to an isomorphism from $\Lambda_T(R)$ to a closed subspace of $H_T(U)$. It is possible to prove that the isomorphism is onto $H_T(U)$. We denote this isomorphism by I_T and define the integral of $f \in \Lambda_T(R)$ with respect to U by

$$\int_0^T f(t)dU_t = I_T(f).$$

The integral is defined for functions in $\Lambda_T(R)$ and thus it is of interest to identify usual functions in $\Lambda_T(R)$ besides step functions. Suppose that R is of bounded variation on $[0,T) \times [0,T)$; then it determines uniquely, in the usual way, a finite signed measure denoted again by R. Let $\mathscr{L}_T$ be the set of all measurable functions f on $[0,T)$ such that

$$\int_0^T\int_0^T |f(t)f(s)|d^2|R|(t,s) < +\infty, \quad \int_0^T\int_0^T |f(t)|d^2|R|(t,s) < +\infty,$$

where $|R|$ is the total variation of measure R. We have the following result (Theorem 1.1 in [7]).

THEOREM 2.3. *Let R be of bounded variation on $[0,T)\times[0,T)$. Then $\mathscr{L}_T$ is a dense subset of $\Lambda_T(R)$. Moreover, if $f_1, f_2 \in \mathscr{L}_T$ and $\int_0^T\int_0^T |f_1(t)f_2(s)|d^2R(t,s) < +\infty$, then*

$$(f_1,f_2) = \int_0^T\int_0^T f_1(t)f_2(s)d^2R(t,s) = \mathrm{Cov}\Big(\int_0^T f_1(t)dU_t, \int_0^T f_2(s)dU_s\Big).$$

REMARK 2.1. If f has bounded variation, then the integral $\int_0^T f(t)dU_t$, for $T > 0$, can be pathwise defined (see, e.g., [12] and [10]) by

$$\int_0^T f(t)dU_t = f(T)U_T - \int_0^T U_t df(t). \tag{2.4}$$

3. LARGE DEVIATIONS FOR THE CONDITIONED PROCESS

Let $X = (X_t)_{t \geqslant 0}$ be a continuous Gaussian process with definite positive covariance function k, mean function m of bounded variation and $X_0 = m(0)$. Let $X^{\boldsymbol{g};\boldsymbol{x}}$ be the conditioned process defined in the Introduction.

Denote by $C^{\boldsymbol{g}} = (c^{\boldsymbol{g}}_{ij})_{i,j=1,\dots,n}$ the matrix defined by

$$c^{\boldsymbol{g}}_{ij} = \mathrm{Cov}\Big(\int_0^T g_i(t)dX_t, \int_0^T g_j(t)dX_t\Big). \tag{3.1}$$

Note that $C^{\boldsymbol{g}}$ does not depend on the mean m of X nor on the values of $\boldsymbol{x}$ but only on the conditioning function $\boldsymbol{g}$ and the covariance k. Since g_i's are linearly independent and k is positive definite, the matrix $C^{\boldsymbol{g}}$ is invertible. Let us write

$$(1_{[0,t)}, \boldsymbol{g})^{\intercal} = \big((1_{[0,t)}, g_1), \dots, (1_{[0,t)}, g_n)\big),$$

$(\cdot,\cdot)$ being the inner product in $\Lambda_T(k)$, i.e., for $i = 1, \dots, n$,

$$(1_{[0,t)}, g_i) = \mathrm{Cov}\Big(X_t, \int_0^T g_i(u)dX_u\Big).$$

The following theorem (Theorem 3.1 in [11]) gives mean and covariance function of the generalized conditioned process.

THEOREM 3.1. *The generalized conditioned process $X^{\boldsymbol{g};\boldsymbol{x}}$ can be represented as follows*:

$$X^{\boldsymbol{g};\boldsymbol{x}}_t = X_t - (1_{[0,t)}, \boldsymbol{g})^{\intercal}(C^{\boldsymbol{g}})^{-1}\Big(\int_0^T \boldsymbol{g}(u)dX_u - \boldsymbol{x}\Big). \tag{3.2}$$

Moreover, the generalized conditioned process $X^{\boldsymbol{g};\boldsymbol{x}}$ is a Gaussian process with

$$\begin{aligned} m^{\boldsymbol{g};\boldsymbol{x}}(t) &= \mathbb{E}[X^{\boldsymbol{g};\boldsymbol{x}}_t] \\ &= m(t) - (1_{[0,t)}, \boldsymbol{g})^{\intercal}(C^{\boldsymbol{g}})^{-1}\Big(\int_0^T \boldsymbol{g}(u)dm(u) - \boldsymbol{x}\Big) \\ &= m(t) - \mu^{\boldsymbol{g};\boldsymbol{x}}(t), \end{aligned} \tag{3.3}$$

$$\begin{aligned} k^{\boldsymbol{g}}(t,s) &= \mathrm{Cov}(X^{\boldsymbol{g};\boldsymbol{x}}_t, X^{\boldsymbol{g};\boldsymbol{x}}_s) \\ &= k(t,s) - (1_{[0,t)}, \boldsymbol{g})^{\intercal}(C^{\boldsymbol{g}})^{-1}(1_{[0,s)}, \boldsymbol{g}) \\ &= k(t,s) - \kappa^{\boldsymbol{g}}(t,s). \end{aligned} \tag{3.4}$$

REMARK 3.1. Let us note that the covariance function of the conditioned process depends on the conditioning functions $g_1, \dots, g_n$ and on the time T but not on the vector $\boldsymbol{x}$.

REMARK 3.2. If the conditioning functions g_i are the indicator functions of the interval $[0, T_i)$ for $i = 1, \dots, n$, then the corresponding generalized conditioned process is the process conditioned to be in x_i at time T_i. In this case large deviation results are contained in [2].

Our first aim is to study the behavior of the covariance function and of the mean function of the original process X in order to get first a functional large deviation principle for the family of processes $\{(X_{T+\varepsilon t} - X_T)_{t\in[0,1]}\}_\varepsilon$, then for $\{(X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon t} - X^{\boldsymbol{g};\boldsymbol{x}}_T)_{t\in[0,1]}\}_\varepsilon$ as $\varepsilon \to 0$. The two next assumptions guarantee that Theorem 2.2 is applicable to the family $\{(X_{T+\varepsilon t} - X_T)_{t\in[0,1]}\}_\varepsilon$.

ASSUMPTION 3.1. *For any fixed $T > 0$ there exists an asymptotic covariance function $\bar{k}(t, s)$ defined as*

(3.5)
$$\begin{aligned}\bar{k}(t,s) &= \lim_{\varepsilon\to 0}\frac{\operatorname{Cov}(X_{T+\varepsilon t} - X_T, X_{T+\varepsilon s} - X_T)}{\gamma_\varepsilon^2}\\ &= \lim_{\varepsilon\to 0}\frac{k(T+\varepsilon t, T+\varepsilon s) - k(T+\varepsilon t, T) - k(T, T+\varepsilon s) + k(T,T)}{\gamma_\varepsilon^2},\end{aligned}$$

uniformly in $(t, s) \in [0, 1] \times [0, 1]$.

For any fixed $T > 0$,

$$\lim_{\varepsilon\to 0}\mathbb{E}(X_{T+\varepsilon t} - X_T) = \lim_{\varepsilon\to 0}\big(m(T+\varepsilon t) - m(T)\big) = 0, \tag{3.6}$$

uniformly in $t \in [0, 1]$.

ASSUMPTION 3.2. *For any fixed $T > 0$ there exist $M_1 > 0$, $M_2 > 0$ and $\alpha > 0$ such that for $\varepsilon > 0$,*

$$\sup_{s,t\in[0,1],s\neq t}\frac{|m(T+\varepsilon t) - m(T+\varepsilon s)|}{|t-s|^\alpha} \leqslant M_1 \tag{3.7}$$

and

(3.8)
$$\sup_{s,t\in[0,1],s\neq t}\frac{|k(T+\varepsilon t, T+\varepsilon t) - 2k(T+\varepsilon t, T+\varepsilon s) + k(T+\varepsilon s, T+\varepsilon s)|}{\gamma_\varepsilon^2\,|t-s|^{2\alpha}} \leqslant M_2.$$

Assumption 3.1 intuitively defines "a local process". In fact, it says that as $\varepsilon \to 0$ the process $\{(X_{T+\varepsilon t} - X_T)_{t\in[0,1]}\}_\varepsilon$ behaves like a Gaussian process with covariance given by $\gamma_\varepsilon^2\bar{k}(s, t)$. Assumption 3.2 guarantees exponential tightness for the family $\{(X_{T+\varepsilon t} - X_T)_{t\in[0,1]}\}_\varepsilon$. Let us discuss some simple but useful consequences of the assumptions introduced above.

As an immediate application of Theorem 2.2 (take $U_t^\varepsilon = X_{T+\varepsilon t} - X_T$), Assumptions 3.1 and 3.2 imply that the family $\{(X_{T+\varepsilon t} - X_T)_{t\in[0,1]}\}_\varepsilon$ satisfies a large deviation principle on $C([0, 1])$, with the inverse speed γ_ε^2 and the good rate function given by

$$J_X(h) = \begin{cases}\frac{1}{2}\|h\|^2_{\bar{\mathscr{H}}} & \text{if } h \in \bar{\mathscr{H}},\\ +\infty & \text{otherwise,}\end{cases} \tag{3.9}$$

where $\bar{\mathscr{H}}$ is the reproducing kernel Hilbert space associated with the covariance function $\bar{k}(t,s)$ and the symbol $\|\cdot\|_{\bar{\mathscr{H}}}$ denotes the usual norm defined on $\bar{\mathscr{H}}$.

In fact, Assumption 3.1 immediately implies that

$$0 = \lim_{\varepsilon\to 0} \mathbb{E}(\langle \lambda, X_{T+\varepsilon\cdot} - X_T\rangle)$$

and

$$\Lambda(\lambda) = \lim_{\varepsilon\to 0} \frac{\mathrm{Var}(\langle \lambda, X_{T+\varepsilon\cdot} - X_T\rangle)}{\gamma_\varepsilon^2} = \int_0^1\int_0^1 \bar{k}(t,s)\lambda(dt)\lambda(ds).$$

Furthermore, Assumption 3.2 implies that the family $\{(X_{T+\varepsilon t} - X_T)_{t\in[0,1]}\}_\varepsilon$ is exponentially tight with respect to the speed function γ_ε^2.

ASSUMPTION 3.3. *For any fixed $T>0$, for $g_i \in \Lambda_T(k)$, $i=1,\ldots,n$, the following limits exist*:

(3.10)

$$\bar{\rho}_i(t,T) = \lim_{\varepsilon\to 0} \frac{\mathrm{Cov}\big(X_{T+\varepsilon t} - X_T, \int_0^T g_i(u)dX_u\big)}{\gamma_\varepsilon} = \lim_{\varepsilon\to 0} \frac{\int_T^{T+\varepsilon t}\int_0^T g_i(u) d^2k(u,v)}{\gamma_\varepsilon},$$

uniformly in $t\in[0,1]$.

ASSUMPTION 3.4. *For any fixed $T>0$ there exist $M>0$ and $\alpha>0$ such that for $i=1,\ldots,n$ and $\varepsilon>0$,*

$$\sup_{s,t\in[0,1], s\neq t} \frac{\big|\mathrm{Cov}\big(X_{T+\varepsilon t} - X_{T+\varepsilon s}, \int_0^T g_i(u)dX_u\big)\big|}{\gamma_\varepsilon |t-s|^\alpha} \leqslant M. \tag{3.11}$$

REMARK 3.3. Let us observe that Assumption 3.3 implies that for any fixed $T>0$

$$\lim_{\varepsilon\to 0}\mathrm{Cov}\big(X_{T+\varepsilon t} - X_T, \int_0^T g_i(u)dX_u\big) = \lim_{\varepsilon\to 0}(1_{[T,T+\varepsilon t)}, g_i) = 0, \tag{3.12}$$

uniformly in $t\in[0,1]$. Therefore,

$$\lim_{\varepsilon\to 0}\big(\mu^{\boldsymbol{g};\boldsymbol{x}}(T+\varepsilon t) - \mu^{\boldsymbol{g};\boldsymbol{x}}(T)\big) = 0, \tag{3.13}$$

uniformly in $t\in[0,1]$. In fact, we have

$$\mu^{\boldsymbol{g};\boldsymbol{x}}(T+\varepsilon t) - \mu^{\boldsymbol{g};\boldsymbol{x}}(T) = (1_{[T,T+\varepsilon t)}, \boldsymbol{g})^\intercal (C^{\boldsymbol{g}})^{-1}\big(\int_0^T \boldsymbol{g}(u)dm(u) - \boldsymbol{x}\big),$$

and (3.13) immediately follows from (3.12).

REMARK 3.4. Let us observe that Assumption 3.4 implies that there exist $M_1 > 0$ and $M_2 > 0$ such that the following estimates hold:

$$\sup_{s,t\in[0,1],s\neq t} \frac{|\mu^{\boldsymbol{g};\boldsymbol{x}}(T+\varepsilon t) - \mu^{\boldsymbol{g};\boldsymbol{x}}(T+\varepsilon s)|}{|t-s|^\alpha} \leqslant M_1 \tag{3.14}$$

and

$$\sup_{s,t\in[0,1],s\neq t} \frac{|\kappa^{\boldsymbol{g}}(T+\varepsilon t, T+\varepsilon t) - 2\kappa^{\boldsymbol{g}}(T+\varepsilon t, T+\varepsilon s) + \kappa^{\boldsymbol{g}}(T+\varepsilon s, T+\varepsilon s)|}{\gamma_\varepsilon^2 |t-s|^{2\alpha}} \leqslant M_2. \tag{3.15}$$

In fact, we have

$$\mu^{\boldsymbol{g};\boldsymbol{x}}(T+\varepsilon t) - \mu^{\boldsymbol{g};\boldsymbol{x}}(T+\varepsilon s) = (1_{[T+\varepsilon s, T+\varepsilon t)}, \boldsymbol{g})^\intercal (C^{\boldsymbol{g}})^{-1}\Big(\int_0^T \boldsymbol{g}(u)dm(u) - \boldsymbol{x}\Big),$$

and straightforward computations show that

$$\begin{aligned}\kappa^{\boldsymbol{g}}(T+\varepsilon t, T+\varepsilon t) - 2\kappa^{\boldsymbol{g}}(T+\varepsilon t, T+\varepsilon s) + \kappa^{\boldsymbol{g}}(T+\varepsilon s, T+\varepsilon s)\\ = (1_{[T+\varepsilon s, T+\varepsilon t)}, \boldsymbol{g})^\intercal (C^{\boldsymbol{g}})^{-1} (1_{[T+\varepsilon s, T+\varepsilon t)}, \boldsymbol{g}).\end{aligned}$$

Therefore, (3.14) and (3.15) immediately follow from (3.11).

Now, to achieve a large deviation principle for the generalized conditioned process $X^{\boldsymbol{g};\boldsymbol{x}}$, we have to investigate the behavior of the functions $k^{\boldsymbol{g}}$ and $m^{\boldsymbol{g};\boldsymbol{x}}$ (defined in (3.4) and (3.3), respectively) in a small time interval of length ε.

LEMMA 3.1. *Under Assumptions* 3.1 *and* 3.3 *we have*

$$\lim_{\varepsilon\to 0} \mathbb{E}(X^{\boldsymbol{g};x}_{T+\varepsilon t} - X^{\boldsymbol{g};x}_T) = 0, \quad \textit{uniformly in } t\in[0,1], \tag{3.16}$$

and

$$\lim_{\varepsilon\to 0} \frac{\operatorname{Cov}(X^{\boldsymbol{g};x}_{T+\varepsilon t} - X^{\boldsymbol{g};x}_T, X^{\boldsymbol{g};x}_{T+\varepsilon s} - X^{\boldsymbol{g};x}_T)}{\gamma_\varepsilon^2} = \bar{k}^{\boldsymbol{g}}(t,s), \tag{3.17}$$

uniformly in $(t,s)\in[0,1]\times[0,1]$, *with*

$$\bar{k}^{\boldsymbol{g}}(t,s) = \bar{k}(t,s) - \bar{\boldsymbol{\rho}}(t,T)^\intercal (C^{\boldsymbol{g}})^{-1} \bar{\boldsymbol{\rho}}(s,T), \tag{3.18}$$

where $\bar{\boldsymbol{\rho}}(t,T)^\intercal = \big(\bar{\rho}_1(t,T), \dots, \bar{\rho}_n(t,T)\big)$ *and* $\bar{\rho}_i(t,T)$ *is defined in* (3.10) *for* $i = 1,\dots,n$.

Proof. Since

$$\mathbb{E}(X^{\boldsymbol{g};x}_{T+\varepsilon t} - X^{\boldsymbol{g};x}_T) = m(T+\varepsilon t) - m(T) - \big(\mu^{\boldsymbol{g};\boldsymbol{x}}(T+\varepsilon t) - \mu^{\boldsymbol{g};\boldsymbol{x}}(T)\big),$$

(3.16) easily follows from equations (3.6) and (3.13).

Furthermore,

$$\lim_{\varepsilon\to 0}\frac{\mathrm{Cov}(X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon t}-X^{\boldsymbol{g};\boldsymbol{x}}_{T},X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon s}-X^{\boldsymbol{g};\boldsymbol{x}}_{T})}{\gamma_\varepsilon^2}$$
$$=\lim_{\varepsilon\to 0}\frac{k^{\boldsymbol{g}}(T+\varepsilon t,T+\varepsilon s)-k^{\boldsymbol{g}}(T+\varepsilon t,T)-k^{\boldsymbol{g}}(T,T+\varepsilon s)+k^{\boldsymbol{g}}(T,T)}{\gamma_\varepsilon^2},$$

therefore (3.18) easily follows from Assumptions 3.1 and 3.3 and by the definition of $k^{\boldsymbol{g}}$. ■

LEMMA 3.2. *Under Assumptions* 3.2 *and* 3.4 *it follows that the family of processes* $\{(X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon t}-X^{\boldsymbol{g};\boldsymbol{x}}_{T})_{t\in[0,1]}\}_\varepsilon$ *is exponentially tight with respect to the speed function* γ_ε^2.

P r o o f. Assumption 3.2 and Remark 3.4 imply that conditions (2.2) and (2.3) hold for $\{(X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon t}-X^{\boldsymbol{g};\boldsymbol{x}}_{T})_{t\in[0,1]}\}_\varepsilon$. ■

We are now ready to prove the main large deviation result of this section:

THEOREM 3.2. *Under Assumptions* 3.1–3.4 *it follows that the family of processes* $\{(X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon t}-X^{\boldsymbol{g};\boldsymbol{x}}_{T})_{t\in[0,1]}\}_\varepsilon$ *satisfies a large deviation principle on* $C([0,1])$ *with the inverse speed* γ_ε^2 *and the good rate function*

$$J^{\boldsymbol{g}}_X(h)=\begin{cases}\frac{1}{2}\,\|h\|^2_{\bar{\mathscr{H}}^{\boldsymbol{g}}} & \text{if } h\in\bar{\mathscr{H}}^{\boldsymbol{g}},\\ +\infty & \text{otherwise},\end{cases}\tag{3.19}$$

$\bar{\mathscr{H}}^{\boldsymbol{g}}$ *being the reproducing kernel Hilbert space associated with the covariance function defined in* (3.18).

P r o o f. By (3.16),

$$\lim_{\varepsilon\to 0}\mathbb{E}(\langle\lambda,X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon\cdot}-X^{\boldsymbol{g};\boldsymbol{x}}_{T}\rangle)=\lim_{\varepsilon\to 0}\int_0^1\mathbb{E}(X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon u}-X^{\boldsymbol{g};\boldsymbol{x}}_{T})d\lambda(u)=0$$

and, by (3.17),

$$\lim_{\varepsilon\to 0}\frac{\mathrm{Var}(\langle\lambda,X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon\cdot}-X^{\boldsymbol{g};\boldsymbol{x}}_{T}\rangle)}{\gamma_\varepsilon^2}$$
$$=\lim_{\varepsilon\to 0}\int_0^1 d\lambda(v)\int_0^1 d\lambda(u)\frac{\mathrm{Cov}(X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon v}-X^{\boldsymbol{g};\boldsymbol{x}}_{T},X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon u}-X^{\boldsymbol{g};\boldsymbol{x}}_{T})}{\gamma_\varepsilon^2}$$
$$=\int_0^1 d\lambda(v)\int_0^1 d\lambda(u)\bar{k}^{\boldsymbol{g}}(v,u).$$

The large deviation principle follows from Theorem 2.2. Notice that $\bar{k}^{\boldsymbol{g}}$ is a continuous covariance function, being the (uniform) limit of a continuous, symmetric and positive definite function. Therefore, we can assert that $\{(X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon t} - X^{\boldsymbol{g};\boldsymbol{x}}_{T})_{t\in[0,1]}\}_\varepsilon$ satisfies a large deviation principle on $C([0,1])$, with the inverse speed γ_ε^2 and the good rate function $J^{\boldsymbol{g}}_X$ defined in (3.19). ■

REMARK 3.5. Let $X^{\boldsymbol{g};\boldsymbol{x}}$ be the conditioned process and suppose that one of the conditioning functions g_i, for instance g_1, is the indicator function of the interval $[0,T)$, i.e., $X^{\boldsymbol{g};\boldsymbol{x}}_T = x_1$. Then, by the contraction principle (see Theorem 4.2.1 in [4]), the family $\{(X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ satisfies a standard large deviation principle on $C([0,1])$, with the inverse speed γ_ε^2 and the good rate function

$$\widetilde{J}^{\boldsymbol{g}}_X(h) = J^{\boldsymbol{g}}_X(h - x_1) = \begin{cases} \dfrac{1}{2}\,\|h - x_1\|^2_{\bar{\mathscr{H}}^{\boldsymbol{g}}} & \text{if } h - x_1 \in \bar{\mathscr{H}}^{\boldsymbol{g}}, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.20}$$

Here $\bar{\mathscr{H}}^{\boldsymbol{g}}$ is the reproducing kernel Hilbert space associated with the covariance function defined in (3.18).

EXAMPLE 3.1. In this section we consider two examples to which Theorem 3.2 applies. Let X be a continuous, centered Gaussian process with covariance function k. Suppose $g_1(t) = 1_{[0,T)}(t)$ and $g_2(t) = (T-t)/T$, that is,

$$X_T = \int_0^T g_1(u)dX_u = x_1,$$

and, by (2.4),

$$\frac{1}{T}\int_0^T X_u du = \int_0^T g_2(u)dX_u = x_2.$$

Then the matrix $(C^{\boldsymbol{g}})^{-1}$ (see [11]) is given by

$$(C^{\boldsymbol{g}})^{-1} = \frac{1}{\det(C^{\boldsymbol{g}})}\begin{pmatrix} c^{\boldsymbol{g}}_{22} & -c^{\boldsymbol{g}}_{12} \\ -c^{\boldsymbol{g}}_{12} & c^{\boldsymbol{g}}_{11} \end{pmatrix},$$

where

$$\begin{aligned} c^{\boldsymbol{g}}_{11} &= k(T,T), \\ c^{\boldsymbol{g}}_{12} &= \frac{1}{T}\int_0^T k(T,u)du, \\ c^{\boldsymbol{g}}_{22} &= \frac{1}{T^2}\int_0^T\int_0^T k(v,u)dv\,du, \\ \det(C^{\boldsymbol{g}}) &= \frac{1}{T^2}\int_0^T\int_0^T [k(T,T)k(v,u) - k(T,v)k(T,u)]dv\,du. \end{aligned}$$

Fractional Brownian motion. Let X be the fractional Brownian motion of Hurst index H. Let us recall that a *fractional Brownian motion with Hurst index* $H \in (0,1)$ is a continuous, centered Gaussian process whose covariance function is

$$k(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

By Theorem 3.2 in [2], we know that Assumption 3.1 holds with $\bar{k}(t,s) = k(t,s)$ and that $\bar{\rho}_1(t,T) = 0$ as $t \in [0,1]$. Furthermore, by the Fubini theorem we have

$$\begin{aligned}
\bar{\rho}_2(t,T) &= \lim_{\varepsilon\to 0} \frac{\operatorname{Cov}\big(X_{T+\varepsilon t} - X_T, \int_0^T g_2(u) dX_u\big)}{\gamma_\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\operatorname{Cov}\big(X_{T+\varepsilon t} - X_T, \frac{1}{T}\int_0^T X_u du\big)}{\varepsilon^H} \\
&= \lim_{\varepsilon\to 0} \frac{1}{T}\int_0^T \frac{\operatorname{Cov}(X_{T+\varepsilon t} - X_T, X_u) du}{\varepsilon^H} = 0,
\end{aligned}$$

uniformly in $t \in [0,1]$.

So, we have $\bar{k}^{\boldsymbol{g}}(t,s) = k(t,s)$. Simple calculations show that also Assumptions 3.2 and 3.4 are satisfied with $\alpha = H$, $\gamma_\varepsilon = \varepsilon^H$. Therefore, it follows that the family $\{(X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon t} - X^{\boldsymbol{g};\boldsymbol{x}}_T)_{t\in[0,1]}\}_\varepsilon$ satisfies a large deviation principle with $\gamma_\varepsilon^2 = \varepsilon^{2H}$ as the non-conditioned process. Note that the same result was obtained in [2] for the n-fold conditional fractional Brownian motion.

Integrated Gaussian process. Let Z be a centered Gaussian process with covariance function $\kappa(t,s) = \operatorname{Cov}(Z_t, Z_s)$, and let X be the integrated process, i.e.,

$$X_t = \int_0^t Z_u du.$$

The process X is a continuous, centered Gaussian process whose covariance function is given by

$$k(t,s) = \int_0^t \int_0^s \kappa(u,v) du dv.$$

From Theorem 3.3 in [2] we infer that Assumption 3.1 is satisfied with $\bar{k}(t,s) = ts\kappa(T,T)$. Since

$$\lim_{\varepsilon\to 0} \frac{\operatorname{Cov}(X_{T+\varepsilon t} - X_T, X_T)}{\varepsilon} = t\int_0^T \kappa(T,v) dv,$$

we have

$$\bar{\rho}_1(t,T) = \lim_{\varepsilon\to 0} \frac{\mathrm{Cov}\big(X_{T+\varepsilon t} - X_T, \int_0^T g_1(u)dX_u\big)}{\gamma_\varepsilon}$$
$$= \lim_{\varepsilon\to 0} \frac{\mathrm{Cov}(X_{T+\varepsilon t} - X_T, X_T)}{\varepsilon} = t\int_0^T \kappa(T,v)dv,$$

uniformly in $t \in [0,1]$. Furthermore,

$$\bar{\rho}_2(t,T) = \lim_{\varepsilon\to 0} \frac{\mathrm{Cov}\big(X_{T+\varepsilon t} - X_T, \int_0^T g_2(u)dX_u\big)}{\gamma_\varepsilon}$$
$$= \lim_{\varepsilon\to 0} \frac{\mathrm{Cov}\big(X_{T+\varepsilon t} - X_T, \frac{1}{T}\int_0^T X_u du\big)}{\varepsilon}$$
$$= \lim_{\varepsilon\to 0} \frac{1}{T}\int_0^T \frac{\mathrm{Cov}(X_{T+\varepsilon t} - X_T, X_u)du}{\varepsilon}$$
$$= \frac{t}{T}\int_0^T\int_0^u \kappa(T,v)dv\,du,$$

uniformly in $t \in [0,1]$. Therefore, also Assumption 3.3 is satisfied and

$$\bar{k}^{\boldsymbol{g}}(t,s) = ts\kappa(T,T) - \bar{\boldsymbol{\rho}}(t,T)^{\intercal}(C^{\boldsymbol{g}})^{-1}\bar{\boldsymbol{\rho}}(s,T).$$

Since $\rho_1(t,T) = c_1(T)t$ and $\rho_2(t,T) = c_2(T)t$, setting $\boldsymbol{c}(T)^{\intercal} = \big(c_1(T), c_2(T)\big)$, we obtain

$$\bar{k}^{\boldsymbol{g}}(t,s) = ts\kappa(T,T) - \bar{\boldsymbol{\rho}}(t,T)^{\intercal}(C^{\boldsymbol{g}})^{-1}\bar{\boldsymbol{\rho}}(s,T)$$
$$= ts\big(\kappa(T,T) - \boldsymbol{c}(T)^{\intercal}(C^{\boldsymbol{g}})^{-1}\boldsymbol{c}(T)\big).$$

Simple calculations show that also Assumptions 3.2 and 3.4 are fulfilled with $\alpha = 1$, $\gamma_\varepsilon = \varepsilon$. Therefore, the conditioned integrated Gaussian process satisfies a large deviation principle. Note that the limit covariance is $\mathrm{const}\cdot st$ just as for the n-fold conditional process in [2]. The constant is different since it depends on the functions $g_1, \dots, g_n$.

4. LARGE DEVIATIONS FOR THE BRIDGE OF THE CONDITIONED PROCESS

Let $(X_t^{\boldsymbol{g};\boldsymbol{x}})_{t\geqslant 0}$ be the generalized conditioned process as defined in Remark 3.5, i.e., $X_T^{\boldsymbol{g};\boldsymbol{x}} = x_1$, and let us now consider the process $Y^{\boldsymbol{g};\boldsymbol{x};y}$ defined as the *bridge* of the process $X^{\boldsymbol{g};\boldsymbol{x}}$, i.e., the process $X^{\boldsymbol{g};\boldsymbol{x}}$ conditioned to be in y at the future time $T+\varepsilon$. Then, one has the equality in law

$$Y_{T+\varepsilon t}^{\boldsymbol{g};\boldsymbol{x};y} = X_{T+\varepsilon t}^{\boldsymbol{g};\boldsymbol{x}} - \beta_T^{\varepsilon}(t)(X_{T+\varepsilon}^{\boldsymbol{g};\boldsymbol{x}} - y), \tag{4.1}$$

where

$$\beta_T^\varepsilon(t) = \frac{k^{\boldsymbol{g}}(T+\varepsilon t, T+\varepsilon)}{k^{\boldsymbol{g}}(T+\varepsilon, T+\varepsilon)}. \tag{4.2}$$

ASSUMPTION 4.1. *For any fixed $T>0$ there exist $M_1>0$ and $\alpha>0$ such that for $\varepsilon>0$,*

$$\sup_{s,t\in[0,1],s\neq t} \frac{|\mathrm{Cov}(X_{T+\varepsilon t}^{\boldsymbol{g};\boldsymbol{x}} - X_{T+\varepsilon s}^{\boldsymbol{g};\boldsymbol{x}}, X_{T+\varepsilon}^{\boldsymbol{g};\boldsymbol{x}})|}{\gamma_\varepsilon^2 |t-s|^\alpha} \leqslant M_1. \tag{4.3}$$

Now, to achieve a large deviation principle for $\{(Y_{T+\varepsilon t}^{\boldsymbol{g};\boldsymbol{x};y})_{t\in[0,1]}\}_\varepsilon$, one needs a nice asymptotic behavior for β_T^ε.

LEMMA 4.1. *Let Assumptions* 3.1 *and* 3.3 *be satisfied. Then the following limit exists*:

$$\lim_{\varepsilon\to 0} \beta_T^\varepsilon(t) = \frac{\bar{k}^{\boldsymbol{g}}(t,1)}{\bar{k}^{\boldsymbol{g}}(1,1)} = \bar{\beta}_T(t), \quad \textit{uniformly in } t\in[0,1]. \tag{4.4}$$

P r o o f. We have

$$\begin{aligned}
|\beta_T^\varepsilon(t) - \bar{\beta}_T(t)| &= \left| \frac{k^{\boldsymbol{g}}(T+\varepsilon t, T+\varepsilon)}{k^{\boldsymbol{g}}(T+\varepsilon, T+\varepsilon)} - \frac{\bar{k}^{\boldsymbol{g}}(t,1)}{\bar{k}^{\boldsymbol{g}}(1,1)} \right| \\
&\leqslant \frac{\gamma_\varepsilon^2}{k^{\boldsymbol{g}}(T+\varepsilon, T+\varepsilon)} \left| \frac{k^{\boldsymbol{g}}(T+\varepsilon t, T+\varepsilon)}{\gamma_\varepsilon^2} - \bar{k}^{\boldsymbol{g}}(t,1) \right| \\
&\quad + |\bar{k}^{\boldsymbol{g}}(t,1)| \left| \frac{\gamma_\varepsilon^2}{k^{\boldsymbol{g}}(T+\varepsilon, T+\varepsilon)} - \frac{1}{\bar{k}^{\boldsymbol{g}}(1,1)} \right|.
\end{aligned}$$

From (3.17), since $X_T^{\boldsymbol{g};\boldsymbol{x}} = x_1$, we get

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,1]} \left| \frac{k^{\boldsymbol{g}}(T+\varepsilon t, T+\varepsilon)}{\gamma_\varepsilon^2} - \bar{k}^{\boldsymbol{g}}(t,1) \right| = 0$$

and

$$\lim_{\varepsilon\to 0} \left| \frac{\gamma_\varepsilon^2}{k^{\boldsymbol{g}}(T+\varepsilon, T+\varepsilon)} - \frac{1}{\bar{k}^{\boldsymbol{g}}(1,1)} \right| = 0,$$

so that the statement holds. ■

LEMMA 4.2. *Let $Y^{\boldsymbol{g};\boldsymbol{x};y}$ be the bridge of the conditioned process $X^{\boldsymbol{g};\boldsymbol{x}}$, as defined in* (4.1). *Under Assumption* 4.1 *the family of processes $\{(Y_{T+\varepsilon t}^{\boldsymbol{g};\boldsymbol{x};y})_{t\in[0,1]}\}_\varepsilon$ is exponentially tight with respect to the speed function γ_ε^2.*

P r o o f. We want to prove that Assumption 4.1 implies that conditions (2.2) and (2.3) hold for the family $\{(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$. One has

$$\begin{aligned}&\mathbb{E}(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t})-\mathbb{E}(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon s})\\&=\big(m^{\boldsymbol{g};\boldsymbol{x}}(T+\varepsilon t)-m^{\boldsymbol{g};\boldsymbol{x}}(T+\varepsilon s)\big)-\big(m^{\boldsymbol{g};\boldsymbol{x}}(T+\varepsilon)-y\big)\big(\beta^\varepsilon_T(t)-\beta^\varepsilon_T(s)\big),\end{aligned}$$

therefore,

$$\begin{aligned}&\sup_{s,t\in[0,1],s\neq t}\frac{|\mathbb{E}(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t})-\mathbb{E}(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon s})|}{|t-s|^\alpha}\\&\leqslant\sup_{s,t\in[0,1],s\neq t}\frac{|m^{\boldsymbol{g};\boldsymbol{x}}(T+\varepsilon t)-m^{\boldsymbol{g};\boldsymbol{x}}(T+\varepsilon s)|}{|t-s|^\alpha}\\&\quad+\frac{|m^{\boldsymbol{g};\boldsymbol{x}}(T+\varepsilon)-y|}{k^{\boldsymbol{g}}(T+\varepsilon,T+\varepsilon)}\sup_{s,t\in[0,1],s\neq t}\frac{|k^{\boldsymbol{g}}(T+\varepsilon t,T+\varepsilon)-k^{\boldsymbol{g}}(T+\varepsilon s,T+\varepsilon)|}{|t-s|^\alpha}\\&=\sup_{s,t\in[0,1],s\neq t}\frac{|m^{\boldsymbol{g};\boldsymbol{x}}(T+\varepsilon t)-m^{\boldsymbol{g};\boldsymbol{x}}(T+\varepsilon s)|}{|t-s|^\alpha}\\&\quad+\gamma_\varepsilon^2\frac{|m^{\boldsymbol{g};\boldsymbol{x}}(T+\varepsilon)-y|}{k^{\boldsymbol{g}}(T+\varepsilon,T+\varepsilon)}\sup_{s,t\in[0,1],s\neq t}\frac{|k^{\boldsymbol{g}}(T+\varepsilon t,T+\varepsilon)-k^{\boldsymbol{g}}(T+\varepsilon s,T+\varepsilon)|}{\gamma_\varepsilon^2|t-s|^\alpha}.\end{aligned}$$

We already proved that (2.2) is satisfied for the family $\{(X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon t}-X^{\boldsymbol{g};\boldsymbol{x}}_T)_{t\in[0,1]}\}_\varepsilon$. Since $X^{\boldsymbol{g};\boldsymbol{x}}(T)=x_1$, we have $m^{\boldsymbol{g};\boldsymbol{x}}(T+\varepsilon)\to x_1$, and from (3.17) we obtain

$$\lim_{\varepsilon\to0}\frac{k^{\boldsymbol{g}}(T+\varepsilon,T+\varepsilon)}{\gamma_\varepsilon^2}=\bar{k}^{\boldsymbol{g}}(1,1).$$

Assumption 4.1 implies that for some $M>0$

$$\sup_{s,t\in[0,1],s\neq t}\frac{|\mathbb{E}(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t})-\mathbb{E}(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon s})|}{|t-s|^\alpha}\leqslant M,$$

hence (2.2) holds. Straightforward computations show that

$$\begin{aligned}&\mathrm{Cov}(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t},Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t})-2\mathrm{Cov}(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t},Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon s})+\mathrm{Cov}(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon s},Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon s})\\&\quad=k^{\boldsymbol{g}}(T+\varepsilon t,T+\varepsilon t)-2k^{\boldsymbol{g}}(T+\varepsilon t,T+\varepsilon s)+k^{\boldsymbol{g}}(T+\varepsilon s,T+\varepsilon s)\\&\qquad-\frac{\big(k^{\boldsymbol{g}}(T+\varepsilon t,T+\varepsilon)-k^{\boldsymbol{g}}(T+\varepsilon s,T+\varepsilon)\big)^2}{k^{\boldsymbol{g}}(T+\varepsilon,T+\varepsilon)}.\end{aligned}$$

Therefore,

$$\sup_{s,t\in[0,1],s\neq t} \frac{|\mathrm{Cov}(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t}, Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t}) - 2\,\mathrm{Cov}(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t}, Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon s}) + \mathrm{Cov}(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon s}, Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon s})|}{\gamma_\varepsilon^2 |t-s|^{2\alpha}}$$
$$\leqslant \sup_{s,t\in[0,1],s\neq t} \frac{|k^{\boldsymbol{g}}(T+\varepsilon t, T+\varepsilon t) - 2k^{\boldsymbol{g}}(T+\varepsilon t, T+\varepsilon s) + k^{\boldsymbol{g}}(T+\varepsilon s, T+\varepsilon s)|}{\gamma_\varepsilon^2 |t-s|^{2\alpha}}$$
$$+ \frac{\gamma_\varepsilon^2}{k^{\boldsymbol{g}}(T+\varepsilon, T+\varepsilon)} \sup_{s,t\in[0,1],s\neq t} \left(\frac{k^{\boldsymbol{g}}(T+\varepsilon t, T+\varepsilon) - k^{\boldsymbol{g}}(T+\varepsilon s, T+\varepsilon)}{\gamma_\varepsilon^2 |t-s|^{\alpha}} \right)^2.$$

With the same arguments as in the first part of the proof there exists $M > 0$ such that

$$\sup_{s,t\in[0,1],s\neq t} \frac{|\mathrm{Cov}(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t}, Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t}) - 2\,\mathrm{Cov}(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t}, Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon s}) + \mathrm{Cov}(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon s}, Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon s})|}{\gamma_\varepsilon^2 |t-s|^{2\alpha}} \leqslant M$$

and (2.3) follows. ■

THEOREM 4.1. *Let $Y^{\boldsymbol{g};\boldsymbol{x};y}$ be the bridge of the conditioned process $X^{\boldsymbol{g};\boldsymbol{x}}$, as defined in* (4.1), *satisfying Assumptions* 3.1–3.4. *Under Assumptions* 4.1 *the family of processes $\{(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ satisfies a large deviation principle on $C([0,1])$, with the inverse speed γ_ε^2 and the good rate function*

$$J_Y^{\boldsymbol{g}}(h) = \begin{cases} \dfrac{1}{2}\|h-\bar{m}\|^2_{\bar{\mathscr{H}}_{Y^{\boldsymbol{g}}}} & \text{if } h_0 = x_1, h_1 = y, h-\bar{m} \in \bar{\mathscr{H}}_{Y^{\boldsymbol{g}}}, \\ +\infty & \text{otherwise}, \end{cases} \tag{4.5}$$

where $\bar{m}_t = x_1 + \bar{\beta}_T(t)(y-x_1)$ and $\bar{\mathscr{H}}_{Y^{\boldsymbol{g}}}$ is the reproducing kernel Hilbert space associated with the covariance function

$$\bar{k}_Y^{\boldsymbol{g}}(t,s) = \bar{k}^{\boldsymbol{g}}(t,s) - \bar{\beta}_T(s)\bar{k}^{\boldsymbol{g}}(t,1) = \bar{k}^{\boldsymbol{g}}(t,s) - \frac{\bar{k}^{\boldsymbol{g}}(t,1)\bar{k}^{\boldsymbol{g}}(s,1)}{\bar{k}^{\boldsymbol{g}}(1,1)}.$$

Proof. Let $U^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t} = Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t} - \bar{m}_t$, where

$$\bar{m}_t = x_1 + \bar{\beta}_T(t)(y-x_1) = \lim_{\varepsilon\to 0} \mathbb{E}(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t}).$$

By (4.1) and (4.4), the above limit is uniform in $t \in [0,1]$. We will start by showing a large deviation principle for $\{(U^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$. Again by Theorem 2.2 we have

$$\lim_{\varepsilon\to 0} \mathbb{E}(\langle \lambda, U^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon\cdot}\rangle) = \int_0^1 \mathbb{E}(U^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t})\, d\lambda(t) = 0$$

for any $\lambda \in \mathscr{M}[0,1]$. Moreover, from (3.18) and Lemma 4.1 we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\mathrm{Cov}(U^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t},U^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon s})}{\gamma_\varepsilon^2} &= \lim_{\varepsilon\to 0}\frac{\mathrm{Cov}(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t},Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon s})}{\gamma_\varepsilon^2}\\ &= \lim_{\varepsilon\to 0}\frac{\mathrm{Cov}(X^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t},X^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon s})-\beta^\varepsilon_T(s)\mathrm{Cov}(X^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t},X^n_{T+\varepsilon})}{\gamma_\varepsilon^2}\\ &= \bar{k}^{\boldsymbol{g}}(t,s)-\bar{\beta}_T(s)\bar{k}^{\boldsymbol{g}}(t,1)=\bar{k}^{\boldsymbol{g}}_Y(t,s),\end{aligned}$$

uniformly in $s,t\in[0,1]$. Therefore,

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\mathrm{Var}(\langle\lambda,U^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon\cdot}\rangle)}{\gamma_\varepsilon^2} &= \lim_{\varepsilon\to 0}\frac{1}{\gamma_\varepsilon^2}\int_0^1 d\lambda(t)\int_0^1 d\lambda(s)\,\mathrm{Cov}(U^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t},U^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon s})\\ &= \int_0^1 d\lambda(t)\int_0^1 d\lambda(s)\,\bar{k}^{\boldsymbol{g}}_Y(t,s)\end{aligned}$$

for any $\lambda \in \mathscr{M}[0,1]$. Since the family is exponentially tight (from Lemma 4.2), we can conclude that the family of processes $\{(U^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ satisfies a large deviation principle on $C([0,1])$, with the inverse speed γ_ε^2 and the good rate function

$$J^{\boldsymbol{g}}_U(\varphi)=\begin{cases}\dfrac{1}{2}\,\|\varphi\|^2_{\bar{\mathscr{H}}^{\boldsymbol{g}}_Y} & \text{if } \varphi_0=\varphi_1=0,\varphi\in\bar{\mathscr{H}}^{\boldsymbol{g}}_Y,\\ +\infty & \text{otherwise,}\end{cases}$$

where $\bar{\mathscr{H}}^{\boldsymbol{g}}_Y$ is the reproducing kernel Hilbert space associated with the covariance function

$$\bar{k}^{\boldsymbol{g}}_Y(t,s)=\bar{k}^{\boldsymbol{g}}(t,s)-\bar{\beta}_T(s)\bar{k}^{\boldsymbol{g}}(t,1)=\bar{k}^{\boldsymbol{g}}(t,s)-\frac{\bar{k}^{\boldsymbol{g}}(t,1)\bar{k}^{\boldsymbol{g}}(s,1)}{\bar{k}^{\boldsymbol{g}}(1,1)}.$$

Since $Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t}=U^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t}+\bar{m}_t$, by contraction, we get the large deviation principle for $\{(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ on $C([0,1])$, with the inverse speed γ_ε^2 and the good rate function as in (4.5). ∎

REMARK 4.1. The conditioned processes defined in Example 3.1 (fractional Brownian motion and integrated process) satisfy also Assumption 4.1. In fact, from (3.4) we obtain

$$\begin{aligned}k^{\boldsymbol{g}}(t,s)=&\frac{k(t,s)k(T,T)\int_0^T\int_0^T k(u,v)du\,dv-k(t,s)\big(\int_0^T k(T,v)dv\big)^2}{k(T,T)\int_0^T\int_0^T k(u,v)du\,dv-\big(\int_0^T k(u,T)du\big)^2}\\ &-\frac{k(t,T)k(s,T)\int_0^T\int_0^T k(u,v)du\,dv-k(s,T)\int_0^T k(t,v)dv\int_0^T k(T,v)dv}{k(T,T)\int_0^T\int_0^T k(u,v)du\,dv-\big(\int_0^T k(u,T)du\big)^2}\end{aligned}$$

$$
+\frac{k(t,T)\int\limits_0^T k(s,v)dv\int\limits_0^T k(T,v)dv-k(T,T)\int\limits_0^T k(s,v)dv\int\limits_0^T k(t,v)dv}{k(T,T)\int\limits_0^T\int\limits_0^T k(u,v)du\,dv-\big(\int\limits_0^T k(u,T)du\big)^2},
$$

and therefore, by straightforward computations,

$$
\begin{aligned}
&k^{\boldsymbol{g}}(T+\varepsilon t,T+\varepsilon)-k^{\boldsymbol{g}}(T+\varepsilon s,T+\varepsilon)\\
&=\frac{k(T,T)\big(k(T+\varepsilon t,T+\varepsilon)-k(T+\varepsilon s,T+\varepsilon)\big)\int\limits_0^T\int\limits_0^T k(u,v)du\,dv}{k(T,T)\int\limits_0^T\int\limits_0^T k(u,v)du\,dv-\big(\int\limits_0^T k(u,T)du\big)^2} &(1)\\
&-\frac{\big(k(T+\varepsilon t,T+\varepsilon)-k(T+\varepsilon s,T+\varepsilon)\big)\big(\int\limits_0^T k(u,T)du\big)^2}{k(T,T)\int\limits_0^T\int\limits_0^T k(u,v)du\,dv-\big(\int\limits_0^T k(u,T)du\big)^2} &(2)\\
&-\frac{k(T+\varepsilon,T)\big(k(T+\varepsilon t,T)-k(T+\varepsilon s,T)\big)\int\limits_0^T\int\limits_0^T k(u,v)du\,dv}{k(T,T)\int\limits_0^T\int\limits_0^T k(u,v)du\,dv-\big(\int\limits_0^T k(u,T)du\big)^2} &(3)\\
&+\frac{k(T+\varepsilon,T)\int\limits_0^T\big(k(T+\varepsilon t,u)-k(T+\varepsilon s,u)\big)du\int\limits_0^T k(T,v)dv}{k(T,T)\int\limits_0^T\int\limits_0^T k(u,v)du\,dv-\big(\int\limits_0^T k(u,T)du\big)^2} &(4)\\
&+\frac{\big(k(T+\varepsilon t,T+\varepsilon)-k(T+\varepsilon s,T+\varepsilon)\big)\int\limits_0^T k(T+\varepsilon,u)dv\int\limits_0^T k(T,v)dv}{k(T,T)\int\limits_0^T\int\limits_0^T k(u,v)du\,dv-\big(\int\limits_0^T k(u,T)du\big)^2} &(5)\\
&+\frac{k(T,T)\int\limits_0^T\big(k(T+\varepsilon t,v)-k(T+\varepsilon s,v)\big)dv\int\limits_0^T k(T+\varepsilon,v)dv}{k(T,T)\int\limits_0^T\int\limits_0^T k(u,v)du\,dv-\big(\int\limits_0^T k(u,T)du\big)^2}. &(6)
\end{aligned}
$$

Now, adding a subtracting $\big(k(T+\varepsilon t,T)-k(T+\varepsilon s,T)\big)k(T,T)$, we have

$$\begin{aligned}|(1)+(3)| \leqslant C\Big(&\big|\big(k(T+\varepsilon t,T+\varepsilon)-k(T+\varepsilon s,T+\varepsilon)\big)\\&-\big(k(T+\varepsilon t,T)-k(T+\varepsilon s,T)\big)\big|\\&+|k(T+\varepsilon,T)-k(T,T)||k(T+\varepsilon t,T)-k(T+\varepsilon s,T)|\Big).\end{aligned}$$

Furthermore,

$$|(2)+(5)| \leqslant C\big(|k(T+\varepsilon t,T+\varepsilon)-k(T+\varepsilon s,T+\varepsilon)|\int_0^T|k(T+\varepsilon,u)-k(T,u)|du\big)$$

and

$$\begin{aligned}|(4)+(6)| \leqslant C\big(&|k(T+\varepsilon,T)-k(T,T)|\int_0^T|k(T+\varepsilon t,u)-k(T+\varepsilon s,u)|du\\&+\int_0^T|k(T+\varepsilon,u)-k(T,u)|du\int_0^T|k(T+\varepsilon t,u)-k(T+\varepsilon s,u)|du\big).\end{aligned}$$

In both cases (fractional Brownian motion and integrated Gaussian process) it can be easily shown (some algebra) that the following estimates hold:

(4.6)
$$\frac{\big|\big(k(T+\varepsilon t,T+\varepsilon)-k(T+\varepsilon s,T+\varepsilon)\big)-\big(k(T+\varepsilon t,T)-k(T+\varepsilon s,T)\big)\big|}{\gamma_\varepsilon^2|t-s|^{2\alpha}} \leqslant M,$$

$$\frac{\int_0^T|k(T+\varepsilon,u)-k(T,u)|du}{\gamma_\varepsilon} \leqslant M, \tag{4.7}$$

$$\frac{|k(T+\varepsilon t,T+\varepsilon)-k(T+\varepsilon s,T+\varepsilon)|}{\gamma_\varepsilon|t-s|^\alpha} \leqslant M, \tag{4.8}$$

$$\frac{\int_0^T|k(T+\varepsilon t,u)-k(T+\varepsilon s,u)|du}{\gamma_\varepsilon|t-s|^\alpha} \leqslant M. \tag{4.9}$$

From Assumptions 3.3 and 3.4 (for $g_1(t)=1_{[0,T)}(t)$) and from estimates (4.6)–(4.9) we obtain

$$\sup_{s,t\in[0,1],s\neq t}\frac{|\mathrm{Cov}(X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon t}-X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon s},X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon})|}{\gamma_\varepsilon^2|t-s|^\alpha} \leqslant M_1.$$

Notice that Theorem 4.1 gives an unsatisfactory large deviation result for integrated Gaussian processes. In this case the asymptotic covariance function is $\mathrm{const} \cdot ts$ as well, and a degenerate behavior holds for the rate function of the bridge. Also, for Gaussian processes whose (original) covariance function is smooth enough the same degenerate behavior holds for the rate function.

This motivates next subsection, in which we study some refinements allowing one to state non-trivial large deviation estimates or, more precisely, the right large deviation speed. We only give an assumption for the faster large deviation principle; for further details see Subsection 4.1 in [2].

4.1. Faster large deviations for the bridge. Here, Assumptions 3.1 and 3.2 (on the covariance function, the condition on the mean does not depend on the speed) are strengthened as follows:

ASSUMPTION 4.2. (i) *For some* $\gamma \in (0, 1]$, *there exist a function* $\bar{\varphi}_T(t, s)$, *a constant* $a_T > 0$ *and a remaining term* $\mathscr{R}^1_\varepsilon(t, s)$ (*depending on* T) *such that*

$$\mathrm{Cov}(X_{T+\varepsilon t} - X_T, X_{T+\varepsilon s} - X_T) = \varepsilon^2[a_T\, ts + \bar{\varphi}_T(t, s)\varepsilon^\gamma + \mathscr{R}^1_\varepsilon(t, s)] \tag{4.10}$$

with

$$\lim_{\varepsilon\to 0} \sup_{s,t\in[0,1]} \frac{|\mathscr{R}^1_\varepsilon(t, s)|}{\varepsilon^\gamma} = 0.$$

(ii) *For some* $\gamma \in (0, 1]$, *for any fixed* $T > 0$, $i = 1 \ldots, n$, *there exist functions* $\bar{\psi}_i(t, T)$, *constants* $c_i(T)$ *and remaining terms* $\mathscr{R}^{2,i}_\varepsilon(t; T)$ *such that*

$$\mathrm{Cov}\big(X_{T+\varepsilon t} - X_T, \int_0^T g_i(u) dX_u\big) = \varepsilon[c_i(T)\, t + \bar{\psi}_i(t; T)\varepsilon^\gamma + \mathscr{R}^{2,i}_\varepsilon(t; T)] \tag{4.11}$$

with

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,1]} \frac{|\mathscr{R}^2_\varepsilon(t; T)|}{\varepsilon^\gamma} = 0.$$

As a consequence of Assumption 4.2, by using the same arguments as in Lemma 3.1, we have

LEMMA 4.3. *For* $T > 0$,

$$\mathrm{Cov}(X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon t} - X^{\boldsymbol{g};\boldsymbol{x}}_T, X^{\boldsymbol{g};\boldsymbol{x}}_{T+\varepsilon s} - X^{\boldsymbol{g};\boldsymbol{x}}_T) = \varepsilon^2[a^{\boldsymbol{g}}_T\, ts + \bar{\varphi}^{\boldsymbol{g}}_T(t, s)\varepsilon^\gamma + \mathscr{R}^{1,T}_\varepsilon(t, s)], \tag{4.12}$$

where $\mathscr{R}^{1,T}_\varepsilon(t, s) \to 0$ *as* $\varepsilon \to 0$ *uniformly on* $[0, 1] \times [0, 1]$,

$$a^{\boldsymbol{g}}_T = a_T - \boldsymbol{c}(T)^\intercal (C^{\boldsymbol{g}})^{-1}\boldsymbol{c}(T)$$

and

$$\bar{\varphi}^{\boldsymbol{g}}_T(t, s) = \bar{\varphi}_T(t, s) - \boldsymbol{c}(T)^\intercal (C^{\boldsymbol{g}})^{-1}\bar{\boldsymbol{\psi}}(s, T) - \bar{\boldsymbol{\psi}}(t, T)^\intercal (C^{\boldsymbol{g}})^{-1}\boldsymbol{c}(T). \tag{4.13}$$

Also Assumptions 3.2 and 3.4 can be refined in order to have exponential tightness with a larger speed. They can be easily obtained by using the arguments of the previous section.

THEOREM 4.2. *Let* $Y^{\boldsymbol{g};\boldsymbol{x};y}$ *be the bridge of the conditioned process* $X^{\boldsymbol{g};\boldsymbol{x}}$, *as defined in* (4.1), *satisfying Assumption* 4.2 *and such that* $\{(Y^{\boldsymbol{g};\boldsymbol{x};y}_{T+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ *is exponentially tight with respect to the speed function* $\varepsilon^{2+\gamma}$. *Then it satisfies a large deviation principle on* $C([0,1])$, *with the inverse speed* $\varepsilon^{2+\gamma}$ *and the good rate function*

$$J^{\boldsymbol{g}}_Y(h) = \begin{cases} \dfrac{1}{2}\,\|h-\bar m\|^2_{\bar{\mathscr{H}}_{Y^{\boldsymbol{g}}}} & \text{if } h_0 = x_1, h_1 = y, h-\bar m \in \bar{\mathscr{H}}_{Y^{\boldsymbol{g}}}, \\ +\infty & \text{otherwise}, \end{cases} \tag{4.14}$$

where $\bar m_t = x_1 + \bar\beta_T(t)(y-x_1)$ *and* $\bar{\mathscr{H}}_{Y^{\boldsymbol{g}}}$ *is the reproducing kernel Hilbert space associated with the covariance function*

$$\bar k^{\boldsymbol{g}}_Y(t,s) = \bar\varphi^{\boldsymbol{g}}_T(t,s) + ts\,\bar\varphi^{\boldsymbol{g}}_T(1,1) - t\,\bar\varphi^{\boldsymbol{g}}_T(1,s) - s\,\bar\varphi^{\boldsymbol{g}}_T(t,1),$$

with $\varphi^{\boldsymbol{g}}_T$ *defined as in* (4.13).

Proof. The proof is the same as that of Theorem 4.2 in [2]. ∎

REMARK 4.2. Now, if the function $k(t,s)$ is more regular (if, for example, $\gamma = 1$), then Theorem 4.2 would give again a degenerate behavior. In this case, to obtain a non-trivial large deviation principle we have to refine further the hypothesis. We omit these topics for the sake of simplicity, but all details for refinements can be found in [2].

REMARK 4.3. Let us observe that the results about the asymptotic of the hitting probability in [2] (in particular Proposition 5.1) trivially hold also for the asymptotic of the hitting probability for bridges of generalized Gaussian processes.

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