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REFLECTED BSDEs WITH GENERAL FILTRATION AND TWO COMPLETELY SEPARATED BARRIERS*

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Abstract. We consider reflected backward stochastic differential equations, with two barriers, defined on probability spaces equipped with filtration satisfying only the usual assumptions of right-continuity and completeness. As for barriers, we assume that there are *càdlàg* processes of class D that are completely separated. We prove the existence and uniqueness of solutions for an integrable final condition and an integrable monotone generator. An application to the zero-sum Dynkin game is given.

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1. INTRODUCTION AND NOTATION

In this paper we study the problem of existence and uniqueness of solutions of backward stochastic differential equations (BSDEs) with two reflecting *càdlàg* barriers L, U. The main new feature is that we deal with equations on probability spaces with general filtration $\mathbb{F} = \{\mathcal{F}_t; t \in [0,T]\}$ satisfying only the usual conditions of right-continuity and completeness and we do not assume that the barriers satisfy the so-called Mokobodzki condition. Instead, we assume that the lower barrier L and the upper barrier U are completely separated in the sense that $L_t < U_t$ and $L_{t-} < U_{t-}$ for $t \in [0,T]$. Moreover, we consider equations with L^p data, where $p \in [1,2]$. Our motivation for considering such a general setting comes from PDEs theory (equations involving nonlocal operators, see [9], [11]) and from the theory of optimal stopping (Dynkin games, see [8], [12], [14], [15]).

Let $T > 0$. Suppose we are given an $\mathcal{F}_T$-measurable random variable ξ, a progressively measurable function $f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ and two adapted *càdlàg* processes L, U such that $L_t \leqslant U_t$, $t \in [0,T]$. Roughly speaking, by a *solution of the reflected BSDE* with terminal condition ξ, generator f and barriers L, U we

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mean a quadruple (Y,K,A,M) of *càdlàg* adapted processes such that Y is of Doob's class D, K,A are increasing processes such that $K_0=A_0=0$, M is a local martingale with $M_0=0$, and a.s. we have

$$
\left\{
\begin{array}{l}
Y_t=\xi+\int_t^T f(s,Y_s)\,ds+\int_t^T dK_s-\int_t^T dA_s-\int_t^T dM_s,\quad t\in[0,T],\\
L_t\leqslant Y_t\leqslant U_t,\quad t\in[0,T],\\
\int_0^T (Y_{t-}-L_{t-})\,dK_t=\int_0^T (U_{t-}-Y_{t-})\,dA_t=0.
\end{array}
\right.
\tag{1.1}
$$

In most papers devoted to reflected BSDEs with two barriers it is assumed that L,U satisfy one of the following conditions:

(a) between L and U one can find a process X such that X is a difference of nonnegative *càdlàg* supermartinagles (the so-called Mokobodzki condition); or

(b) $L_t<U_t$ and $L_{t-}<U_{t-}$ for $t\in[0,T]$ (i.e. the barriers are completely separated).

Problem (1.1) under assumption (a) is studied thoroughly in Klimsiak [8]. Among other things, in [8] it is proved that if f is continuous and monotone with respect to y and satisfies mild integrability conditions (see hypotheses (H1)–(H4) in Section 2), then there exists a unique solution of (1.1).

A drawback to assumption (a), and one of the main reasons why more explicit condition (b) is considered, is that (a) can sometimes be difficult to check. Unfortunately, equations with barriers satisfying (b) are more difficult to deal with. At present, all the existing results on equations with barriers satisfying (b) concern the case where the underlying filtration is Brownian (see Hamadène and Hassani [4], Hamadène et al. [5]) or is generated by a Brownian motion and an independent Poisson random measure (see Hamadène and Wang [6]). Moreover, in [4]–[6] it is assumed that f is Lipschitz continuous and the data (including barriers) are L^2-integrable. Recently, in [7], in the case of Brownian filtration, an existence and uniqueness result was proved for equations with separated continuous barriers, L^1 data and Lipschitz continuous generator.

Our main theorem says that under the assumptions on ξ,f from [8] and *càdlàg* barriers L,U satisfying (b) and such that L^+,U^- are of class D there exists a unique solution of (1.1). Thus we extend the results from [8] to barriers satisfying (b) and at the same time we generalize the results of [4]–[7] to equations with general filtration and less regular data. It is worth pointing out that as a simple corollary to our existence result (it suffices to consider the generator $f\equiv 0$) one gets the following result from the general theory of stochastic processes: if two *càdlàg* processes L,U are completely separated and L^+,U^- are of class D, then there exists a semimartingale of class D between L and U.

The main idea of the proof of our main result is to reduce the problem with completely separated barriers to the problem with barriers satisfying the Mokobodzki condition, and then apply the results of [8]. Such a reduction is possible locally (we use here some modification of a construction from [3]) and enables us to obtain solutions of (1.1) on stochastic intervals of the form $[0,\tau_n]$, where $\{\tau_n\}$ is some stationary sequence of stopping times. These local solutions can be put together to get the solution of (1.1) on $[0,T]$. The last step involves some technicalities, but in general our proof is short and rather simple. In our opinion, it is much simpler than the proof for equations with the underlying Brownian–Poisson filtration and L^2 data given in [6].

The paper is organized as follows. In Section 2 we review some results from [8] concerning reflected BSDEs with one barrier. The proof of the main result is given in Section 3. Finally, in Section 4 we give an application of the results of Section 3 to the zero-sum Dynkin game with payoff function determined by ξ, f and L, U.

Notation. Let $T>0$ and let $(\Omega,\mathcal{F},\mathbb{F}=\{\mathcal{F}_t\}_{t\in[0,T]},P)$ be a filtered probability space with filtration satisfying the usual assumptions of completeness and right-continuity. By $\mathcal{T}$ we denote the set of all $\mathbb{F}$-stopping times such that $\tau\leqslant T$, and by $\mathcal{T}_t$, $t\in[0,T]$, the set of $\tau\in\mathcal{T}$ such that $P(\tau\geqslant t)=1$.

By $\mathcal{V}$ we denote the set of all $\mathbb{F}$-progressively measurable processes of finite variation, and by $\mathcal{V}^1$ the subset of $\mathcal{V}$ consisting of all processes V such that $E|V|_T<\infty$, where $|V|_T$ stands for the variation of V on $[0,T]$. $\mathcal{V}_0$ is the subset of $\mathcal{V}$ consisting of all processes V such that $V_0=0$, $\mathcal{V}_0^+$ (resp. ${}^p\mathcal{V}_0^+$) is the subset of $\mathcal{V}_0$ of all increasing processes (resp. predictable increasing processes). $\mathcal{M}$ (resp. $\mathcal{M}_{loc}$) denotes the set of all $\mathbb{F}$-martingales (resp. local martingales). By $L^1(\mathbb{F})$ we denote the space of all $\mathbb{F}$-progressively measurable processes X such that $E\int_0^T|X_t|dt<\infty$, and by $L^1(\mathcal{F}_T)$ the space of all $\mathcal{F}_T$-measurable random variables ξ such that $E|\xi|<\infty$.

For a stochastic process X we set $X^+=X\vee 0$, $X^-=-(X\wedge 0)$ and $X_{t-}=\lim_{s\nearrow t}X_s$ with the convention that $X_{0-}=X_0$. We also adopt the convention that $\int_a^b=\int_{(a,b]}$.

2. BSDEs WITH ONE REFLECTING BARRIER

In what follows ξ is an $\mathcal{F}_T$-measurable random variable, and L,U are $\mathbb{F}$-progressively measurable *càdlàg* processes, $V\in\mathcal{V}_0$ and $f:\Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}$ is a measurable function such that $f(\cdot,y)$ is an $\mathbb{F}$-progressively measurable process for every $y\in\mathbb{R}$ (for the sake of brevity, in our notation we omit the dependence of f on $\omega\in\Omega$).

We will need the following assumptions on ξ and f:

(H1) There exists a constant $\mu\in\mathbb{R}$ such that for almost every $t\in[0,T]$ and all $y,y'\in\mathbb{R}$,

$$\big(f(t,y)-f(t,y')\big)(y-y')\leqslant\mu|y-y'|^2.$$

(H2) $[0,T]\ni t\mapsto f(t,y)\in L^1(0,T)$ for every $y\in\mathbb{R}$.

(H3) The function $\mathbb{R} \ni y \mapsto f(t,y)$ is continuous for almost every $t \in [0,T]$.

(H4) $\xi \in L^1(\mathcal{F}_T)$, $V \in \mathcal{V}_0 \cap \mathcal{V}^1$, $f(\cdot,0) \in L^1(\mathcal{F})$.

Recall that a stochastic process X on $[0,T]$ is said to be of *class* D if $\{X_\tau : \tau \in \mathcal{T}\}$ is a uniformly integrable family of random variables.

DEFINITION 2.1. We say that a triple (Y,K,M) of *càdlàg* processes is a *solution of the reflected BSDE* with terminal condition ξ, generator $f + dV$ and lower barrier L ($\underline{RBSDE}(\xi, f+dV, L)$ for short) if

(a) Y is a process of class D, $K \in {}^p\mathcal{V}_0^+$, $M \in \mathcal{M}_{loc}$ with $M_0 = 0$,

(b) $L_t \leqslant Y_t$, $t \in [0,T]$, P-a.s.,

(c) $\int_0^T (Y_{t-} - L_{t-})\, dK_t = 0$,

(d) $Y_t = \xi + \int_t^T f(s,Y_s)\, ds + \int_t^T dV_s + \int_t^T dK_s - \int_t^T dM_s$, $t \in [0,T]$, P-a.s.

DEFINITION 2.2. We say that a triple (Y,K,M) of *càdlàg* processes is a *solution of the reflected BSDE* with terminal condition ξ, generator $f + dV$ and upper barrier U ($\overline{RBSDE}(\xi, f+dV, U)$ for short) if

(a) Y is a process of class D, $A \in {}^p\mathcal{V}_0^+$, $M \in \mathcal{M}_{loc}$ with $M_0 = 0$,

(b) $Y_t \leqslant U_t$, $t \in [0,T]$, P-a.s.,

(c) $\int_0^T (U_{t-} - Y_{t-})\, dA_t = 0$,

(d) $Y_t = \xi + \int_t^T f(s,Y_s)\, ds + \int_t^T dV_s - \int_t^T dA_s - \int_t^T dM_s$, $t \in [0,T]$, P-a.s.

Our motivations for considering reflected equations involving a finite variation process V comes from the theory of partial differential equations with measure data. In these applications, V is an additive functional of a Markov process in the Revuz correspondence with some smooth measure, see [9]–[11].

In the theorem below we recall some results on reflecting BSDEs with one barrier proved in [8]. They will play an important role in the proof of our main result in Section 3.

THEOREM 2.1. *Assume that* L^+, U^- *are of class* D *and* (H1)–(H4) *are satisfied.*

(i) *There exists a unique solution* $(\underset{\sim}{Y}, \underset{\sim}{K}, \underset{\sim}{M})$ *of* $\underline{RBSDE}(\xi, f+dV, L)$. *Moreover, if* $(\underset{\sim}{Y}^n, \underset{\sim}{M}^n)$, $n \in \mathbb{N}$, *are solutions of BSDEs of the form*

$$\underset{\sim}{Y}_t^n = \xi + \int_t^T f(s, \underset{\sim}{Y}_s^n)\, ds + \int_t^T dV_s + \int_t^T n(L_s - \underset{\sim}{Y}_s^n)^+\, ds - \int_t^T d\underset{\sim}{M}_s^n,$$

then $\underset{\sim}{Y}_t^n \nearrow \underset{\sim}{Y}_t$, $t \in [0,T]$, *P-a.s.*

(ii) *There exists a unique solution* $(\widetilde{Y}, \widetilde{A}, \widetilde{M})$ *of* $\overline{RBSDE}(\xi, f+dV, U)$. *Moreover, if* $(\widetilde{Y}^n, \widetilde{M}^n)$, $n \in \mathbb{N}$, *are solutions of BSDEs of the form*

$$\widetilde{Y}_t^n = \xi + \int_t^T f(s, \widetilde{Y}_s^n)\, ds + \int_t^T dV_s - \int_t^T n(\widetilde{Y}_s^n - U_s)^+\, ds - \int_t^T d\widetilde{M}_s^n,$$

then $\widetilde{Y}_t^n \nearrow \widetilde{Y}_t$, $t \in [0,T]$, *P-a.s.*

P r o o f. Part (i) is proved in [8], Theorem 4.1, under the assumption that L is of class D. The following argument shows that in fact it suffices to assume that L^+ is of class D. Let (Y^0, M^0) be a solution of the BSDE

$$Y_t^0 = \xi + \int_t^T f(s, Y_s^0)\, ds + \int_t^T dV_s - \int_t^T dM_s^0, \quad t \in [0,T], \tag{2.1}$$

and let $L^\varepsilon = L \vee (Y^0 - \varepsilon)$ for some $\varepsilon > 0$. If L^+ is of class D, then L^ε is of class D, because Y^0 is of class D. Therefore, by Theorem 4.1 in [8], there exists a solution $(Y^\varepsilon, K^\varepsilon, M^\varepsilon)$ of $\underline{RBSDE}(\xi, f + dV, L^\varepsilon)$ such that $K^\varepsilon \in \mathcal{V}_0^+$. In particular,

$$Y_t^\varepsilon = \xi + \int_t^T f(s, Y_s^\varepsilon)\, ds + \int_t^T dV_s + \int_t^T dK_s^\varepsilon - \int_t^T dM_s^\varepsilon, \quad t \in [0,T], \tag{2.2}$$

and

$$Y^\varepsilon \geqslant L^\varepsilon \geqslant L. \tag{2.3}$$

By (2.1), (2.2) and Proposition 2.1 in [9], $Y^\varepsilon \geqslant Y^0$. Hence we have $\mathbb{1}_{\{Y_{t-}^\varepsilon > L_{t-}\}} = \mathbb{1}_{\{Y_{t-}^\varepsilon > L_{t-}^\varepsilon\}}$ for $t \in [0,T]$, and consequently

$$\begin{aligned} \int_0^T (Y_{t-}^\varepsilon - L_{t-})\, dK_t^\varepsilon &= \int_0^T (Y_{t-}^\varepsilon - L_{t-}) \mathbb{1}_{\{Y_{t-}^\varepsilon > L_{t-}\}}(t)\, dK_t^\varepsilon \\ &= \int_0^T (Y_{t-}^\varepsilon - L_{t-}) \mathbb{1}_{\{Y_{t-}^\varepsilon > L_{t-}^\varepsilon\}}(t)\, dK_t^\varepsilon = 0, \end{aligned} \tag{2.4}$$

the last equality being a consequence of the fact that $\int_0^T \mathbb{1}_{\{Y_{t-}^\varepsilon > L_{t-}^\varepsilon\}}(t)\, dK_t^\varepsilon = 0$. By (2.2)–(2.4) the triple $(\underset{\sim}{Y}, \underset{\sim}{K}, \underset{\sim}{M}) = (Y^\varepsilon, K^\varepsilon, M^\varepsilon)$ is a solution of the equation $\underline{RBSDE}(\xi, f + dV, L)$. Uniqueness follows from Corollary 2.2 in [8]. This proves the first part of (i). Observe now that the first component of the solution of $\underline{RBSDE}(\xi, 0, L)$ is a supermartingale of class D dominating L. Therefore, to prove that $\underset{\sim}{Y}_t^n \nearrow \underset{\sim}{Y}_t$, $t \in [0,T]$, it suffices to repeat step by step the proof of Theorem 4.1 in [8]. Since the proof of (ii) is analogous to that of (i), we omit it. ■

3. BSDEs WITH TWO REFLECTING BARRIERS

In this section ξ, f, V and U, L are as in Section 2. We also assume that $L_t \leqslant U_t$ for $t \in [0,T]$, P-a.s.

DEFINITION 3.1. We say that a quadruple (Y, K, A, M) of *càdlàg* processes is a *solution of the reflected BSDE* with terminal condition ξ, generator $f + dV$, lower barrier L and upper barrier U ($RBSDE(\xi, f + dV, L, U)$ for short) if

(LU1) Y is a process of class D, $A, K \in {}^p\mathcal{V}_0^+$, $M \in \mathcal{M}_{loc}$ with $M_0 = 0$;

(LU2) $L_t \leqslant Y_t \leqslant U_t$, $t \in [0,T]$, P-a.s.;

(LU3) $\int_0^T (Y_{t-} - L_{t-})\, dK_t = \int_0^T (U_{t-} - Y_{t-})\, dA_t = 0$;

(LU4) $Y_t = \xi + \int_t^T f(s, Y_s)\, ds + \int_t^T dV_s + \int_t^T d(K_s - A_s) - \int_t^T dM_s$, $t \in [0,T]$, P-a.s.

We will need the following conditions for the barriers L, U:

(B1) $L_t < U_t$ and $L_{t-} < U_{t-}$ for $t \in [0,T]$.

(B2) L^+, U^- are processes of class D.

A sequence $\{\tau_n\} \subset \mathcal{T}$ is said to be of *stationary type* if

$$P(\liminf_{n\to\infty} \{\tau_n = T\}) = 1.$$

The following lemma is an extension of Remark 3.4 in [3].

LEMMA 3.1. *Assume that* L, U *are of class* D *and satisfy* (B1). *Then there exists a process* $H \in \mathcal{V}$ *such that* $L_t \leqslant H_t \leqslant U_t$, $t \in [0,T]$, P*-a.s. Moreover, there exists a sequence* $\{\tau_n\} \subset \mathcal{T}$ *of stationary type such that* $E|H|_{\tau_n} < \infty$ *for every* $n \in \mathbb{N}$.

Proof. Let $\tau_0 = 0$, and for $n \in \mathbb{N}$ set

$$\tau_n = \inf\left\{ t > \tau_{n-1} : \frac{L_{\tau_{n-1}} + U_{\tau_{n-1}}}{2} > U_t \quad \text{or} \quad \frac{L_{\tau_{n-1}} + U_{\tau_{n-1}}}{2} < L_t \right\} \wedge T.$$

Obviously, $\{\tau_n\}$ is nondecreasing. We shall show that it is increasing up to T. To see this, we first observe that

$$P(\tau_n = \tau_{n+1} < T) = 0, \quad n \in \mathbb{N} \cup \{0\}. \tag{3.1}$$

Indeed, suppose that $\omega \in \{\tau_n = \tau_{n+1} < T\}$. Then there exists a sequence $\{t_m\}$ such that $t_m \searrow \tau_n(\omega)$ and for every $m \in \mathbb{N}$,

$$\frac{L_{\tau_n(\omega)}(\omega) + U_{\tau_n(\omega)}(\omega)}{2} > U_{t_m}(\omega) \quad \text{or} \quad \frac{L_{\tau_n(\omega)}(\omega) + U_{\tau_n(\omega)}(\omega)}{2} < L_{t_m}(\omega).$$

Since L and U are right-continuous, this implies that

$$\frac{L_{\tau_n(\omega)}(\omega) + U_{\tau_n(\omega)}(\omega)}{2} \geqslant U_{\tau_n(\omega)}(\omega)$$

or

$$\frac{L_{\tau_n(\omega)}(\omega) + U_{\tau_n(\omega)}(\omega)}{2} \leqslant L_{\tau_n(\omega)}(\omega).$$

Hence $L_{\tau_n(\omega)}(\omega) = U_{\tau_n(\omega)}(\omega)$. Since the barriers satisfy (B1), this shows (3.1). We can now prove that $\{\tau_n\}$ is of stationary type. Suppose, on the contrary, that there is $\tau \in \mathcal{T}$ such that $\tau_n \nearrow \tau$ and $P\big(\bigcap_{n=1}^{\infty}\{\tau_n < \tau\}\big) > 0$. Then

$$P\left(\frac{L_{\tau_{n-1}} + U_{\tau_{n-1}}}{2} \to \frac{L_{\tau-} + U_{\tau-}}{2}\right) > 0.$$

This implies that $P\big(L_{\tau-} \geqslant \frac{L_{\tau-}+U_{\tau-}}{2} \geqslant U_{\tau-}\big) > 0$, and so $P(L_{\tau-} \geqslant U_{\tau-}) > 0$, which contradicts (B1). Thus $\{\tau_n\}$ is of stationary type. Set

$$H_t = \sum_{n=1}^{\infty} \frac{L_{\tau_{n-1}} + U_{\tau_{n-1}}}{2} \mathbb{1}_{[\tau_{n-1},\tau_n)}(t), \quad t \in [0,T].$$

Then $L_t \leqslant H_t \leqslant U_t$, $t \in [0,T]$, P-a.e., and $H \in \mathcal{V}$ because $\{\tau_n\}$ is of stationary type. Moreover, for each $n \in \mathbb{N}$,

$$E|H|_{\tau_n} = \sum_{k=1}^{n} E\left|\frac{U_{\tau_k} + L_{\tau_k}}{2} - \frac{U_{\tau_{k-1}} + L_{\tau_{k-1}}}{2}\right|,$$

which is finite because L, U are of class D. ∎

The following example shows that in general there is no H between barriers such that $E|H|_T$ is finite.

EXAMPLE 3.1. Let $T = 1$ and $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,1]}$ be a Brownian filtration. Let $\{B_n\}_{n\in\mathbb{N}}$ be a partition of Ω such that B_n is $\mathcal{F}_{1/4}$-measurable and $P(B_n) = Cn^{-2}$ with $C = 6\pi^{-2}$, $n \in \mathbb{N}$. Define $h : [0,1) \to \mathbb{R}$ by the formula

$$h_t = \begin{cases} \frac{1}{2}, & t \in \big[1 - \frac{1}{2n+1}, 1 - \frac{1}{2n+2}\big), \quad n \in \mathbb{N} \cup \{0\}, \\ -\frac{3}{2}, & t \in \big[1 - \frac{1}{2n}, 1 - \frac{1}{2n+1}\big), \quad n \in \mathbb{N}, \end{cases}$$

and put

$$L_t = \sum_{n=1}^{\infty} h_{t\wedge(1-1/(n+1))} \mathbb{1}_{B_n}, \quad U_t = L_t + 1, \quad t \in [0,T].$$

One can check that L, U satisfy the assumptions of Lemma 3.1. Therefore, there exists a process $H \in \mathcal{V}$ such that $L_t \leqslant H_t \leqslant U_t$, $t \in [0,T]$, P-a.s. Consider now an arbitrary process $\bar{H} \in \mathcal{V}$ such that $L_t \leqslant \bar{H}_t \leqslant U_t$, $t \in [0,T]$, P-a.s. By the construction of the barriers L and U,

$$|\bar{H}|_T \mathbb{1}_{B_n} \geqslant \sum_{t\in[0,T]} (|U_t - L_{t-}| \wedge |U_{t-} - L_t|) \mathbb{1}_{\{L_t - L_{t-} \neq 0\}}(t) \mathbb{1}_{B_n} = n\mathbb{1}_{B_n}.$$

Hence

$$E|\bar{H}|_T = \sum_{n=1}^{\infty} E|\bar{H}|_T \mathbb{1}_{B_n} \geqslant \sum_{n=1}^{\infty} nP(B_n) = \sum_{n=1}^{\infty} \frac{C}{n} = \infty.$$

Before proving our main result, we first introduce some additional notation. Assume that ξ, f satisfy (H1)–(H4), and L, U satisfy (B1) and (B2). Set

$$\underline{f}_m(t,y) = f(t,y) - m(y-U_t)^+, \quad \overline{f}_n(t,y) = f(t,y) + n(L_t - y)^+.$$

Then $\underline{f}_m, \overline{f}^n$ also satisfy (H1)–(H4), since $y \mapsto n(L_t - y)^+$ and $y \mapsto m(y-U_t)^+$ are Lipschitz continuous for $t \in [0,T]$ and L^+, U^+ are of class D. By Theorem 2.1, for each $n \in \mathbb{N}$ there exists a unique solution $(\overline{Y}^n, \overline{A}^n, \overline{M}^n)$ of the equation $\overline{RBSDE}(\xi, \overline{f}_n + dV, U)$, and for each $m \in N$ there exists a unique solution $(\underline{Y}^m, \underline{K}^m, \underline{M}^m)$ of $\underline{RBSDE}(\xi, \underline{f}_m + dV, L)$. Therefore,

$$\begin{aligned}\overline{Y}^n_t &= \xi + \int_t^T f(s, \overline{Y}^n_s)\, ds + \int_t^T dV_s \\ &\quad + \int_t^T n(L_s - \overline{Y}^n_s)^+\, ds - \int_t^T d\overline{A}^n_s - \int_t^T d\overline{M}^n_s \leqslant U_t \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\underline{Y}^m_t &= \xi + \int_t^T f(s, \underline{Y}^m_s)\, ds + \int_t^T dV_s \\ &\quad - \int_t^T m(\underline{Y}^m_s - U_s)^+\, ds + \int_t^T d\underline{K}^m_s - \int_t^T d\underline{M}^m_s \geqslant L_t \end{aligned}$$

for $t \in [0,T]$. The function $(t,y) \mapsto f(t,y) - m(y-U_t)^+ + n(L_s - y)^+$ also satisfies (H1)–(H4), so by [8], Theorem 2.7, for any $n, m \in \mathbb{N}$ there exists a solution $(Y^{n,m}, M^{n,m})$ of the BSDE

$$\begin{aligned}Y^{n,m}_t &= \xi + \int_t^T f(s, Y^{n,m}_s)\, ds + \int_t^T dV_s + \int_t^T n(L_s - Y^{n,m}_s)^+\, ds \\ &\quad - \int_t^T m(Y^{n,m}_s - U_s)^+\, ds - \int_t^T dM^{n,m}_s, \quad t \in [0,T]. \end{aligned}$$

By Theorem 2.1, for each $m \in N$ the sequence $\{Y^{n,m}\}_n$ is nondecreasing, for each $n \in \mathbb{N}$ the sequence $\{Y^{n,m}\}_m$ is nonincreasing, and

$$\underline{Y}^m_t = \sup_{n\in\mathbb{N}} Y^{n,m}_t = \lim_{n\to\infty} Y^{n,m}_t, \quad \overline{Y}^n_t = \inf_{m\in\mathbb{N}} Y^{n,m}_t = \lim_{m\to\infty} Y^{n,m}_t, \quad t \in [0,T].$$

In particular, for all $n, m \in \mathbb{N}$ we have

$$\overline{Y}^n_t \leqslant Y^{n,m}_t \leqslant \underline{Y}^m_t, \quad t \in [0,T]. \tag{3.3}$$

By Proposition 2.1 in [8], the sequence $\{\overline{Y}^n\}$ is nondecreasing, whereas the sequence $\{\underline{Y}^m\}$ is nonincreasing. Set

$$\underline{Y}_t = \inf_{m\in\mathbb{N}} \underline{Y}^m_t = \lim_{m\to\infty} \underline{Y}^m_t, \quad \overline{Y}_t = \sup_{n\in\mathbb{N}} \overline{Y}^n_t = \lim_{n\to\infty} \overline{Y}^n_t. \tag{3.4}$$

Since $\underline{Y}^m \geqslant L$ for all $m \in \mathbb{N}$ and $\overline{Y}^n \leqslant U$ for all $n \in \mathbb{N}$, we have

$$\underline{Y}_t \geqslant L_t, \quad \overline{Y}_t \leqslant U_t, \quad t \in [0,T], \quad P\text{-a.s.}$$

Also note that from (3.3) and (3.4) and the monotonicity of the sequences $\{\overline{Y}^n\}$ and $\{\underline{Y}^m\}$ it follows that

$$\overline{Y}^0 \leqslant \overline{Y}^n \leqslant \overline{Y} \leqslant \underline{Y} \leqslant \underline{Y}^m \leqslant \underline{Y}^0. \tag{3.5}$$

Since $\overline{Y}^0$ and $\underline{Y}^0$ are solutions of reflected BSDEs, they are processes of class D.

LEMMA 3.2. *Assume* (H1) *and* (H2) *are satisfied. Then for every* $r > 0$

$$t \mapsto \sup_{|y|\leqslant r} f(t,y) \in L^1(0,T).$$

Proof. By (H1), for all $y \in [-r,r]$, $t \in [0,T]$, we have

$$f(t,y) \geqslant f(t,r) - 2\mu r, \quad f(t,y) \leqslant f(t,-r) + 2\mu r.$$

Hence

$$\sup_{|y|\leqslant r} |f(t,y)| \leqslant |f(t,-r) + 2\mu r| \vee |f(t,r) - 2\mu r|.$$

It suffices to use (H2) to complete the proof. ■

LEMMA 3.3. *Let* (Y,K,A,M) *be a solution of* $RBSDE(\xi, f + dV, L, U)$ *and let* $\tau \in \mathcal{T}$. *If* $\xi \in L^1(\mathcal{F}_\tau)$, $f(t,y)\mathbb{1}_{(\tau,T]}(t) = 0$ *for all* $y \in \mathbb{R}$ *and* $t \in [0,T]$, *and*

$$V_t = V_{t\wedge\tau}, \quad L_t = L_{t\wedge\tau}, \quad U_t = U_{t\wedge\tau}, \quad t \in [0,T], \tag{3.6}$$

then

$$Y_t = Y_{t\wedge\tau}, \quad K_t = K_{t\wedge\tau}, \quad A_t = A_{t\wedge\tau}, \quad M_t = M_{t\wedge\tau}, \quad t \in [0,T]. \tag{3.7}$$

Proof. By (LU4),

$$Y_{t\wedge\tau} - Y_t = \int_{t\wedge\tau}^t dK_s - \int_{t\wedge\tau}^t dA_s - \int_{t\wedge\tau}^t dM_s. \tag{3.8}$$

Let $\{\zeta_n\}$ be a fundamental sequence for the local martingale M and let $\sigma \in \mathcal{T}$. Applying the Tanaka–Meyer formula we get

$$\begin{aligned}(Y_{(\sigma\wedge\zeta_n)\wedge\tau} - Y_{\sigma\wedge\zeta_n})^+ &\leqslant \int\limits_{(\sigma\wedge\zeta_n)\wedge\tau}^{\sigma\wedge\zeta_n} \mathbb{1}_{\{Y_{(s\wedge\tau)-} > Y_{s-}\}}\, dK_s \\ &\quad - \int\limits_{(\sigma\wedge\zeta_n)\wedge\tau}^{\sigma\wedge\zeta_n} \mathbb{1}_{\{Y_{(s\wedge\tau)-} > Y_{s-}\}}\, dA_s \\ &\quad - \int\limits_{(\sigma\wedge\zeta_n)\wedge\tau}^{\sigma\wedge\zeta_n} \mathbb{1}_{\{Y_{(s\wedge\tau)-} > Y_{s-}\}}\, dM_s \\ &\leqslant \int\limits_{(\sigma\wedge\zeta_n)\wedge\tau}^{\sigma\wedge\zeta_n} \mathbb{1}_{\{Y_\tau > Y_{s-}\}}\, dK_s - \int\limits_{(\sigma\wedge\zeta_n)\wedge\tau}^{\sigma\wedge\zeta_n} \mathbb{1}_{\{Y_\tau > Y_{s-}\}}\, dM_s.\end{aligned}$$

Taking the expectation and then letting $n \to \infty$ yields

$$E(Y_{\sigma\wedge\tau} - Y_\sigma)^+ \leqslant E \int\limits_{\sigma\wedge\tau}^{\sigma} \mathbb{1}_{\{Y_\tau > Y_{s-}\}}\, dK_s.$$

On the other hand, by (LU3) and (3.6),

$$\begin{aligned}\int\limits_{\sigma\wedge\tau}^{\sigma} \mathbb{1}_{\{Y_\tau > Y_{s-}\}}\, dK_s &= \int\limits_{\sigma\wedge\tau}^{\sigma} \mathbb{1}_{\{Y_\tau > Y_{s-}\}} \mathbb{1}_{\{Y_{s-} = L_{s-}\}}\, dK_s \\ &= \int\limits_{\sigma\wedge\tau}^{\sigma} \mathbb{1}_{\{Y_\tau > Y_{s-}\}} \mathbb{1}_{\{Y_{s-} = L_\tau\}}\, dK_s.\end{aligned}$$

Hence

$$E(Y_{\sigma\wedge\tau} - Y_\sigma)^+ \leqslant E \int\limits_{\sigma\wedge\tau}^{\sigma} \mathbb{1}_{\{Y_\tau > Y_{s-}\}} \mathbb{1}_{\{Y_{s-} = L_\tau\}}\, dK_s. \tag{3.9}$$

From now on we consider the stopping time σ defined by

$$\sigma = \inf\{t > \tau : Y_{t\wedge\tau} > Y_t\} \wedge T.$$

Observe that

$$Y_{t\wedge\tau} \mathbb{1}_{\{t<\sigma\}} \leqslant Y_t \mathbb{1}_{\{t<\sigma\}}, \quad t \in [0, T]. \tag{3.10}$$

Set

$$B_T = \Big\{ \int\limits_{\tau}^{T} \mathbb{1}_{\{Y_\tau > Y_{s-}\}} \mathbb{1}_{\{Y_{s-} = L_\tau\}}\, dK_s > 0 \Big\}.$$

Since $\mathbb{1}_{\{Y_\tau > Y_{s-}\}} \mathbb{1}_{\{Y_{s-} = L_\tau\}} \leqslant \mathbb{1}_{\{L_\tau < Y_\tau\}}$, we have

$$B_T \subset \{L_\tau < Y_\tau\}. \tag{3.11}$$

From (LU3), (3.10), (3.11) and the fact that $L_t = L_{t\wedge\tau}$ for $t \in [0,T]$ it follows that

$$\mathbb{1}_{B_T} \cdot \int_{\tau\wedge\sigma}^{\sigma} dK_s = 0. \tag{3.12}$$

By (3.9) and (3.12),

$$\begin{aligned} E(Y_\tau - Y_\sigma)^+ &\leqslant E\int_\tau^\sigma \mathbb{1}_{\{Y_\tau > Y_{s-}\}}\mathbb{1}_{\{Y_{s-}=L_\tau\}}\,dK_s \\ &= E\big(\mathbb{1}_{B_T}\int_\tau^\sigma \mathbb{1}_{\{Y_\tau > Y_{s-}\}}\mathbb{1}_{\{Y_{s-}=L_\tau\}}\,dK_s\big) = 0. \end{aligned}$$

Consequently,

$$E\big((Y_\tau - Y_\sigma)^+\mathbb{1}_{\{\sigma=T\}}\big) = 0, \qquad E\big((Y_\tau - Y_\sigma)^+\mathbb{1}_{\{\sigma<T\}}\big) = 0. \tag{3.13}$$

Suppose that $P(\sigma = T) = 1$. Then, by (3.10) and the first equality in (3.13), $(Y_{t\wedge\tau} - Y_t)^+ = 0$ P-a.s. for $t \in [0,T]$. We now prove that

$$P(\sigma < T) = 0. \tag{3.14}$$

By the second equality in (3.13),

$$P(\{Y_\tau \leqslant Y_\sigma\} \cap \{\sigma < T\}) = P(\sigma < T). \tag{3.15}$$

Observe that from the definition of σ and the fact that $L_t = L_{t\wedge\tau}$ for $t \in [0,T]$ it follows that

$$\{\sigma < T\} \subset \{L_\tau < Y_\tau\}. \tag{3.16}$$

Set

$$\zeta = \inf\left\{t > \sigma : Y_t < \frac{Y_\tau + L_\tau}{2}\right\}.$$

By the right-continuity of Y and (3.16) we have $Y_\zeta \mathbb{1}_{\{\sigma<T\}} \leqslant \frac{Y_\tau+L_\tau}{2}\mathbb{1}_{\{\sigma<T\}}$. Therefore, using (3.16), we get

$$P\left(\{Y_\zeta < Y_\tau\} \cap \{\sigma < T\}\right) = P(\sigma < T). \tag{3.17}$$

Furthermore, from (3.11), the definition of ζ and (LU3) it follows that

$$0 \leqslant \mathbb{1}_{B_T} \cdot \int_\sigma^\zeta dK_s \leqslant \mathbb{1}_{\{L_\tau < Y_\tau\}} \cdot \int_\sigma^\zeta dK_s = \mathbb{1}_{\{L_\tau < Y_\tau\}}\mathbb{1}_{\{\sigma<\zeta\}} \cdot \int_\sigma^\zeta dK_s = 0. \tag{3.18}$$

Observe that, by the definition of the set B_T,

$$E\big(\int_\tau^\zeta \mathbb{1}_{\{Y_\tau > Y_{s-}\}}\mathbb{1}_{\{Y_{s-}=L_\tau\}}\,dK_s\big) = E\big(\mathbb{1}_{B_T}\int_\tau^\zeta \mathbb{1}_{\{Y_\tau > Y_{s-}\}}\mathbb{1}_{\{Y_{s-}=L_\tau\}}\,dK_s\big).$$

By the above equality, (3.12) and (3.18),

$$\begin{aligned} E\Big(\int_\tau^\zeta \mathbb{1}_{\{Y_\tau > Y_{s-}\}} \mathbb{1}_{\{Y_{s-}=L_\tau\}}\, dK_s\Big) &= E\Big(\mathbb{1}_{B_T} \int_\tau^\sigma \mathbb{1}_{\{Y_\tau > Y_{s-}\}} \mathbb{1}_{\{Y_{s-}=L_\tau\}} dK_s\Big) \\ &\quad + E\Big(\mathbb{1}_{B_T} \int_\sigma^\zeta \mathbb{1}_{\{Y_\tau > Y_{s-}\}} \mathbb{1}_{\{Y_{s-}=L_\tau\}}\, dK_s\Big) \\ &= 0. \end{aligned}$$

This combined with (3.9) with σ replaced by ζ gives $E(Y_\tau - Y_\zeta)^+ = 0$. Consequently, $E(Y_\tau - Y_\zeta)^+ \mathbb{1}_{\{\sigma<T\}} = 0$, which together with (3.17) proves (3.14). Thus $(Y_{t\wedge\tau} - Y_t)^+ = 0$, $t \in [0,T]$, P-a.s. Applying the Tanaka–Meyer formula to the process $(Y_{t\wedge\tau} - Y_t)^-$ and using similar arguments, one can prove that $(Y_{t\wedge\tau} - Y_t)^- = 0$, $t \in [0,T]$, P-a.s. Hence

$$Y_t = Y_{t\wedge\tau}, \quad t \in [0,T]. \tag{3.19}$$

From (3.8) and (3.19) we obtain

$$0 = \int_{t\wedge\tau}^t dK_s - \int_{t\wedge\tau}^t dA_s - \int_{t\wedge\tau}^t dM_s,$$

which implies (3.7). ∎

THEOREM 3.1. *Assume that* (H1)–(H4), (B1), (B2) *are satisfied. Then there exists a unique solution* (Y,K,A,M) *of* $RBSDE(\xi, f+dV, L, U)$. *Moreover*, $Y = \overline{Y} = \underline{Y}$.

Proof. By [8], Corollary 3.2, there exists at most one solution of the equation $RBSDE(\xi, f+dV, L, U)$, so it suffices to prove the existence of a solution. To this end, first assume additionally that L, U are of class D. Then by Lemma 3.1 there exists $H \in \mathcal{V}$ such that $L_t \leqslant H_t \leqslant U_t$, $t \in [0,T]$, P-a.s. and $H_{\cdot\wedge\tau'_k} \in \mathcal{V}^1$ for some sequence $\{\tau'_k\}$ of stationary type. Set

$$\tau_k = \tau'_k \wedge \delta_k \tag{3.20}$$

and $H^{(k)} = H_{\cdot\wedge\tau_k}$, where

$$\delta_k = \inf\Big\{t \geqslant 0 : \int_0^t f(s, H_s)\, ds > k\Big\} \wedge T.$$

Observe that $H^{(k)} \in \mathcal{V}^1$ and, by Lemma 3.2, $\{\tau_k\}$ is of stationary type. The rest of the proof will be divided into five steps.

S t e p 1. We show the existence of a solution of $RBSDE(\xi, f + dV, L, U)$ on stochastic intervals $[0, \tau_k]$. Set

$$U^{(k)} = U_{\cdot\wedge\tau_k}, \quad L^{(k)} = L\mathbb{1}_{[0,\tau_k)} + (L_{\tau_k} \wedge \overline{Y}_{\tau_k})\mathbb{1}_{[\tau_k,T]},$$
$$\xi^{(k)} = \overline{Y}_{\tau_k}, \quad f^{(k)}(\cdot, y) = f(\cdot, y)\mathbb{1}_{[0,\tau_k]}, \quad V^{(k)} = V_{\cdot\wedge\tau_k},$$

where $\overline{Y}$ is defined by (3.4). By (3.5), $\xi^{(k)} \in L^1(\mathcal{F}_T)$. Also observe that $L_T^{(k)} \leqslant \xi^{(k)} \leqslant U_T^{(k)}$ and $L_t^{(k)} \leqslant H_t^{(k)} \leqslant U_t^{(k)}$, $t \in [0, T]$. Therefore, by [8], Theorem 3.3, there exists a unique solution $(Y^{(k)}, K^{(k)}, A^{(k)}, M^{(k)})$ of $RBSDE(\xi^{(k)}, f^{(k)} + dV^{(k)}, L^{(k)}, U^{(k)})$ such that

$$EK_T^{(k)} < \infty, \quad EA_T^{(k)} < \infty. \tag{3.21}$$

In particular, we have

$$Y_t^{(k)} = \xi^{(k)} + \int_t^T f^{(k)}(s, Y_s^{(k)})\, ds + \int_t^T dV_s^{(k)} + \int_t^T dK_s^{(k)} - \int_t^T dA_s^{(k)} - \int_t^T dM_s^{(k)} \tag{3.22}$$

for $t \in [0, T]$. By Lemma 3.3,

$$(Y_t^{(k)}, K_t^{(k)}, A_t^{(k)}, M_t^{(k)}) = (Y_{t\wedge\tau_k}^{(k)}, K_{t\wedge\tau_k}^{(k)}, A_{t\wedge\tau_k}^{(k)}, M_{t\wedge\tau_k}^{(k)}), \quad t \in [0, T]. \tag{3.23}$$

S t e p 2. We are going to show that for every $\tau \in \mathcal{T}$,

$$Y_\tau^{(k)} = \overline{Y}_{\tau\wedge\tau_k}. \tag{3.24}$$

By Theorem 2.1, for each $n \in \mathbb{N}$ there is a unique solution $(Y^{(k),n}, A^{(k),n}, M^{(k),n})$ of the equation $\overline{RBSDE}(\xi^{(k)}, f^{(k),n} + dV^{(k)}, U^{(k)})$ with $f^{(k),n}(t, y) = f^{(k)}(t, y) + n(L_t^{(k)} - y)^+$ and the triple $(Y^{(k),n}, A^{(k),n}, M^{(k),n})$ satisfies

$$Y_t^{(k),n} = \xi^{(k)} + \int_t^T f^{(k)}(s, Y_s^{(k),n})\, ds + \int_t^T dV_s^{(k)} + \int_t^T n(L_s^{(k)} - Y_s^{(k),n})^+\, ds - \int_t^T dA_s^{(k),n} - \int_t^T dM_s^{(k),n}, \tag{3.25}$$

and, by [8], Theorem 3.3,

$$Y^{(k),n} \nearrow Y^{(k)}. \tag{3.26}$$

Write $\tilde{Y}^n_t = Y^{(k),n}_t - \overline{Y}^n_t$, $\tilde{A}^n_t = A^{(k),n}_t - \overline{A}^n_t$, $\tilde{M}^n_t = M^{(k),n}_t - \overline{M}^n_t$. By (3.2), (3.25) and the Tanaka–Meyer formula, for all $\zeta, \tau \in \mathcal{T}$ we have

$$\begin{aligned}\tilde{Y}^{n,+}_{\tau\wedge\zeta\wedge\tau_k} \leqslant \tilde{Y}^{n,+}_{\zeta\wedge\tau_k} &+ \int_{\tau\wedge\zeta\wedge\tau_k}^{\zeta\wedge\tau_k} \mathbb{1}_{\{\tilde{Y}^n_{s-}>0\}}\big(f^{(k)}(s, Y^{(k),n}_s) - f^{(k)}(s, \overline{Y}^n_s)\big)\,ds \\ &+ \int_{\tau\wedge\zeta\wedge\tau_k}^{\zeta\wedge\tau_k} \mathbb{1}_{\{\tilde{Y}^n_{s-}>0\}} n\big((L^{(k)}_s - Y^{(k),n}_s)^+ - (L_s - \overline{Y}^n_s)^+\big)\,ds \\ &- \int_{\tau\wedge\zeta\wedge\tau_k}^{\zeta\wedge\tau_k} \mathbb{1}_{\{\tilde{Y}^n_{s-}>0\}}\,d\tilde{A}^n_s - \int_{\tau\wedge\zeta\wedge\tau_k}^{\zeta\wedge\tau_k} \mathbb{1}_{\{\tilde{Y}^{n,+}_{s-}>0\}}\,d\tilde{M}^n_s.\end{aligned}$$

Consequently, using (H1), we get

$$\begin{aligned}\tilde{Y}^{n,+}_{\tau\wedge\zeta\wedge\tau_k} \leqslant \tilde{Y}^{n,+}_{\zeta\wedge\tau_k} &+ \mu \int_{\tau\wedge\zeta\wedge\tau_k}^{\zeta\wedge\tau_k} \tilde{Y}^{n,+}_s\,ds \\ &+ \int_{\tau\wedge\zeta\wedge\tau_k}^{\zeta\wedge\tau_k} \mathbb{1}_{\{\tilde{Y}^n_{s-}>0\}} n\big((L^{(k)}_s - Y^{(k),n}_s)^+ - (L_s - \overline{Y}^n_s)^+\big)\,ds \\ &+ \int_{\tau\wedge\zeta\wedge\tau_k}^{\zeta\wedge\tau_k} \mathbb{1}_{\{\tilde{Y}^n_{s-}>0\}}\,d\overline{A}^n_s - \int_{\tau\wedge\zeta\wedge\tau_k}^{\zeta\wedge\tau_k} \mathbb{1}_{\{\tilde{Y}^n_{s-}>0\}}\,d\tilde{M}^n_s.\end{aligned} \tag{3.27}$$

Since $y \mapsto (L_s - y)^+$ is nonincreasing and $L^{(k)}\mathbb{1}_{[0,\tau_k)} = L\mathbb{1}_{[0,\tau_k)}$, we have

$$\int_{\tau\wedge\zeta\wedge\tau_k}^{\zeta\wedge\tau_k} \mathbb{1}_{\{\tilde{Y}^n_{s-}>0\}}\big((L^{(k)}_s - Y^{(k),n}_s)^+ - (L_s - \overline{Y}^n_s)^+\big)\,ds \leqslant 0. \tag{3.28}$$

Since $Y^{(k),n}_{t\wedge\tau_k} \leqslant U^{(k)}_{t\wedge\tau_k} = U^{(k)}_t$ and $\overline{Y}^n_{t\wedge\tau_k} \leqslant U_{t\wedge\tau_k} = U^{(k)}_t$, we have

$$\overline{Y}^n_{t\wedge\tau_k} \leqslant Y^{(k),n}_{t\wedge\tau_k} \vee \overline{Y}^n_{t\wedge\tau_k} \leqslant U^{(k)}_t.$$

Hence

$$\begin{aligned}\int_{\tau\wedge\zeta\wedge\tau_k}^{\zeta\wedge\tau_k} \mathbb{1}_{\{\tilde{Y}^{n,+}_{s-}>0\}}\,d\overline{A}^n_s &\leqslant \int_{\tau\wedge\zeta\wedge\tau_k}^{\zeta\wedge\tau_k} \mathbb{1}_{\{\tilde{Y}^n_{s-}>0\}} \frac{Y^{(k),n}_{s-} \vee \overline{Y}^n_{s-} - \overline{Y}^n_{s-}}{\tilde{Y}^n_{s-}}\,d\overline{A}^n_s \\ &\leqslant \liminf_{m\to\infty} m \int_{\tau\wedge\zeta\wedge\tau_k}^{\zeta\wedge\tau_k} \mathbb{1}_{\{\tilde{Y}^n_{s-}>1/m\}}(U_{s-} - \overline{Y}^n_{s-})\,d\overline{A}^n_s = 0.\end{aligned} \tag{3.29}$$

By (3.27)–(3.29),

$$\tilde{Y}^{n,+}_{\tau\wedge\zeta\wedge\tau_k} \leqslant \tilde{Y}^{n,+}_{\zeta\wedge\tau_k} + \mu \int_{\tau\wedge\zeta\wedge\tau_k}^{\zeta\wedge\tau_k} \tilde{Y}^{n,+}_s\,ds - \int_{\tau\wedge\zeta\wedge\tau_k}^{\zeta\wedge\tau_k} \mathbb{1}_{\{\tilde{Y}^n_{s-}>0\}}\,d\tilde{M}^n_s$$

for any $\tau,\zeta \in \mathcal{T}$. Let $\{\zeta_m\}$ be a fundamental sequence for the local martingale $\tilde{M}^n$. Replacing ζ by ζ_m in the above inequality and then taking the expectation, we obtain

$$E\tilde{Y}^{n,+}_{\tau\wedge\zeta_m\wedge\tau_k} \leqslant E\tilde{Y}^{n,+}_{\zeta_m\wedge\tau_k} + \mu E \int_{\tau\wedge\zeta_m\wedge\tau_k}^{\zeta_m\wedge\tau_k} \tilde{Y}^{n,+}_s \, ds.$$

The processes $Y^{(k)}, \overline{Y}^n$ are of class D as solutions of reflected BSDEs. Consequently, $\tilde{Y}^{n,+}$ is of class D. Therefore, letting $m \to \infty$ in the above inequality, we get

$$E\tilde{Y}^{n,+}_{\tau\wedge\tau_k} \leqslant E\tilde{Y}^{n,+}_{\tau_k} + \mu E \int_{\tau\wedge\tau_k}^{\tau_k} \tilde{Y}^{n,+}_s \, ds \tag{3.30}$$

for all $\tau \in \mathcal{T}$. Observe that

$$\begin{aligned} \int_{(\tau\vee t)\wedge\tau_k}^{\tau_k} \tilde{Y}^{n,+}_s \, ds &= \int_{((\tau\wedge\tau_k)\vee t)\wedge\tau_k}^{\tau_k} \tilde{Y}^{n,+}_s \, ds = \int_t^T \tilde{Y}^{n,+}_s \mathbb{1}_{[\tau\wedge\tau_k,\tau_k]}(s) \, ds \\ &\leqslant \int_t^T \tilde{Y}^{n,+}_{(\tau\vee s)\wedge\tau_k} \, ds. \end{aligned}$$

From the above inequality and (3.30) with τ replaced by $\tau \vee t$ it follows that

$$E\tilde{Y}^{n,+}_{(\tau\vee t)\wedge\tau_k} \leqslant E\tilde{Y}^{n,+}_{\tau_k} + \mu \int_t^T E\tilde{Y}^{n,+}_{(\tau\vee s)\wedge\tau_k} \, ds, \quad \tau \in \mathcal{T},\ t \in [0,T].$$

Applying Grönwall's inequality to the mapping $t \mapsto E\tilde{Y}^{n,+}_{(\tau\vee t)\wedge\tau_k}$ gives

$$E\tilde{Y}^{n,+}_{(\tau\vee t)\wedge\tau_k} \leqslant e^{\mu T} E\tilde{Y}^{n,+}_{\tau_k} \leqslant e^{\mu T} E|Y^{(k),n}_{\tau_k} - \overline{Y}^n_{\tau_k}|, \quad t \in [0,T]. \tag{3.31}$$

By (3.4), $\overline{Y}^n_{\tau_k} \nearrow \overline{Y}_{\tau_k} = \xi^{(k)}$, whereas by (3.26) and (3.23), $Y^{(k),n}_{\tau_k} \nearrow Y^{(k)}_{\tau_k} = \xi^{(k)}$. Hence, by the monotone convergence theorem,

$$E|Y^{(k),n}_{\tau_k} - \xi^{(k)}| \to 0, \qquad E|\overline{Y}^n_{\tau_k} - \xi^{(k)}| \to 0.$$

Therefore, applying Fatou's lemma and then (3.31) with $t = T$, we obtain

$$\begin{aligned} E \liminf_{n\to\infty} \tilde{Y}^{n,+}_{(\tau\wedge\tau_k)} &\leqslant \liminf_{n\to\infty} E\tilde{Y}^{n,+}_{\tau\wedge\tau_k} \\ &\leqslant \liminf_{n\to\infty} e^{\mu T}(E|Y^{(k),n}_{\tau_k} - \xi^{(k)}| + E|\overline{Y}^n_{\tau_k} - \xi^{(k)}|) = 0. \end{aligned}$$

But $\tilde{Y}^n_{\tau\wedge\tau_k} \to Y^{(k)}_{\tau\wedge\tau_k} - \overline{Y}_{\tau\wedge\tau_k} = Y^{(k)}_\tau - \overline{Y}_{\tau\wedge\tau_k}$. Hence $E(Y^{(k)}_\tau - \overline{Y}_{\tau\wedge\tau_k})^+ = 0$. In much the same way one can show that $E(Y^{(k)}_\tau - \overline{Y}_{\tau\wedge\tau_k})^- = 0$, which completes the proof of (3.24). By (3.24) and the optional cross-section theorem ([2],

p. 138-IV, (86) Theorem), the processes $Y^{(k)}$ and $\overline{Y}_{\cdot\wedge\tau_k}$ are indistinguishable. In particular, $\overline{Y}_{\cdot\wedge\tau_k}$ has *càdlàg* trajectories. By the same method we show that $Y^{(k)}$ and $\underline{Y}_{\cdot\wedge\tau_k}$ are indistinguishable.

S t e p 3. In this step we define a solution on $[0,T]$. By Step 2, for every $k\in\mathbb{N}$,

$$Y^{(k)}_{t\wedge\tau_k}=\overline{Y}_{t\wedge\tau_k}=\overline{Y}_{t\wedge\tau_k\wedge\tau_{k+1}}=Y^{(k+1)}_{t\wedge\tau_k},\quad t\in[0,T]. \tag{3.32}$$

By (3.22), (3.32) and the uniqueness of the semimartingale decomposition,

$$(Y^{(k+1)}_{t\wedge\tau_k},K^{(k+1)}_{t\wedge\tau_k},A^{(k+1)}_{t\wedge\tau_k},M^{(k+1)}_{t\wedge\tau_k})=(Y^{(k)}_{t\wedge\tau_k},K^{(k)}_{t\wedge\tau_k},A^{(k)}_{t\wedge\tau_k},M^{(k)}_{t\wedge\tau_k}),\quad t\in[0,T].$$

Therefore, we may define processes Y,K,A,M on $[0,T]$ by

$$Y_t=Y^{(k)}_t,\quad K_t=K^{(k)}_t,\quad A_t=A^{(k)}_t,\quad M_t=M^{(k)}_t,\quad t\in[0,\tau_k]. \tag{3.33}$$

By Step 2, $Y_{\tau\wedge\tau_k}=\underline{Y}_{\tau\wedge\tau_k}=\overline{Y}_{\tau\wedge\tau_k}$ for all $\tau\in\mathcal{T}$ and $k\in\mathbb{N}$, so letting $k\to\infty$ gives $Y_\tau=\overline{Y}_\tau$ for $\tau\in\mathcal{T}$. Hence, by the cross-section theorem,

$$Y=\overline{Y}.$$

The quadruple (Y,K,A,M) is a solution of $RBSDE(\xi,f+dV,L,U)$. Indeed, from (3.22), (3.33) and the stationarity of $\{\tau_k\}$ it follows that (Y,K,A,M) satisfies (LU1) and (LU4). Moreover, from the fact that $(Y^{(k)},K^{(k)},A^{(k)},M^{(k)})$ is a solution of $RBSDE(\xi^{(k)},f^{(k)}+dV^{(k)},L^{(k)},U^{(k)})$ and by (3.33) it follows that $L_{t\wedge\tau_k}\leqslant Y_{t\wedge\tau_k}\leqslant U_{t\wedge\tau_k}$, $t\in[0,T]$, P-a.s. and

$$\int_0^{\tau_k}(Y_{t-}-L_{t-})\,dK_t=\int_0^{\tau_k}(U_{t-}-Y_{t-})\,dA_t=0$$

for $k\in\mathbb{N}$. Since $\{\tau_k\}$ is of stationary type, this implies (LU2) and (LU3).

S t e p 4. Repeating the arguments from Steps 2 and 3 for $\xi^{(k)}=\underline{Y}_{\tau_k}$, we prove that $\underline{Y}=Y$, where (Y,K,A,M) is a solution of $RBSDE(\xi,f+dV,L,U)$. Therefore, by the uniqueness of solution, $Y=\overline{Y}=\underline{Y}$.

S t e p 5. We now show how to dispense with the assumption that L,U are of class D.

Let $\underset{\sim}{Y},\widetilde{Y}$ be processes appearing in Theorem 2.1. By [9], Proposition 2.1,

$$\widetilde{Y}_t\leqslant\underset{\sim}{Y}_t,\quad t\in[0,T],\quad P\text{-a.s.}$$

Let $\varepsilon>0$ and let $L^\varepsilon_t=L_t\vee(\widetilde{Y}_t-\varepsilon)$, $U^\varepsilon_t=U_t\wedge(\underset{\sim}{Y}_t+\varepsilon)$. If L,U satisfy (B1) and (B2), then also $L^\varepsilon,U^\varepsilon$ satisfy (B1) and are processes of class (D). By Steps 1–3 there exists a unique solution (Y,K,A,M) of $RBSDE(\xi,f+dV,L^\varepsilon,U^\varepsilon)$. As in the proof of Theorem 2.1, one can check that (Y,K,A,M) is also a solution of $RBSDE(\xi,f+dV,L,U)$. ∎

COROLLARY 3.1. *Assume that L, U satisfy* (B1) *and* (B2). *Then there exists a semimartingale Y of class* D *such that $L_t \leqslant Y_t \leqslant U_t$, $t \in [0,T]$, P-a.s.*

P r o o f. It is enough to consider $\xi = L_T^+ \wedge U_T$, $f \equiv 0$, $V \equiv 0$, and apply Theorem 3.1. ∎

REMARK 3.1. *Let $\{\tau_n\}$ be a sequence defined by* (3.20). *If there exists $k_0 \in \mathbb{N}$ such that*

$$P(\tau_{k_0} = T) = 1, \tag{3.34}$$

then from Step 3 *of the proof of Theorem* 3.1 *it follows that $(Y,K,A,M) = (Y^{(k_0)}, K^{(k_0)}, A^{(k_0)}, M^{(k_0)})$ is a solution of $RBSDE(\xi, f + dV, L, U)$. Furthermore, by* (3.21), *$EK_T < \infty$ and $EA_T < \infty$, and by* [9], *Lemma* 2.3, *$f(\cdot, Y) \in L^1(\mathcal{F})$. Also note that a sufficient condition for* (3.34) *to hold is the following*: *there is $H \in \mathcal{V}^1$ such that $L_t \leqslant H_t \leqslant U_t$, $t \in [0,T]$, and $t \to f(t, H_t)$ is bounded.*

The following example shows that in general EK_T and EA_T need not be finite even if $f \equiv 0$ and $V \equiv 0$.

EXAMPLE 3.2. Let $\mathbb{F}$ be a Brownian filtration and let L, U be defined as in Example 3.1. Set $\xi = (L_T + U_T)/2$ and $f \equiv 0$, $V \equiv 0$. By Theorem 3.1, there exists a unique solution (Y, K, A, M) of $RBSDE(\xi, 0, L, U)$. In particular,

$$Y_t = \xi + \int_t^T dK_s - \int_t^T dA_s - \int_t^T dM_s, \quad t \in [0,T].$$

Let $\tau_n = 1 - 1/n$. Since the filtration is Brownian, $\Delta M_{\tau_n} = 0$ P-a.s. for every $n \in \mathbb{N}$. Hence

$$\Delta Y_{\tau_n} = \Delta A_{\tau_n} - \Delta K_{\tau_n}, \quad n \in \mathbb{N}.$$

In fact, by (LU2), (LU3) and the definitions of L and U, $\Delta Y_{\tau_m} = \Delta A_{\tau_m}$ if m is even and $\Delta Y_{\tau_m} = -\Delta K_{\tau_m}$ if m is odd. Consequently, using the fact that $L \leqslant Y \leqslant U$, we infer that

$$P(\{\Delta A_{\tau_m} \geqslant 1\} \cap B_n) = Cn^{-2}, \quad 2 \leqslant m \leqslant n+1,$$

when m is even, and

$$P(\{\Delta K_{\tau_m} \geqslant 1\} \cap B_n) = Cn^{-2}, \quad 2 \leqslant m \leqslant n+1,$$

when m is odd. Hence

$$EK_T = E|K|_T = \sum_{n=1}^{\infty} E|K|_T \mathbb{1}_{B_n} \geqslant \sum_{n=2}^{\infty} \frac{n-1}{2} P(B_n) = C \sum_{n=2}^{\infty} \frac{n-1}{2n^2} = \infty$$

and

$$EA_T = E|A|_T = \sum_{n=1}^{\infty} E|A|_T \mathbb{1}_{B_n} \geqslant \sum_{n=2}^{\infty} \frac{n-1}{2} P(B_n) = C \sum_{n=2}^{\infty} \frac{n-1}{2n^2} = \infty.$$

4. DYNKIN GAMES

In this section we consider a certain stochastic game of stopping called a *Dynkin game*. For interpretation of notions which we define below (payoff function, lower and upper value of the game) we refer the reader to [1].

Let L, U be *càdlàg* processes of class D such that $L_t \leqslant U_t$, $t \in [0, T]$, P-a.s., and let f, ξ, V be as in Section 3. Also assume that conditions (H1)–(H4) are satisfied. Consider a stopping game with payoff function

$$(4.1)\qquad R_t(\sigma,\tau) = \int_t^{\sigma\wedge\tau} f(s, Y_s)\, ds + \int_t^{\sigma\wedge\tau} dV_s + \xi \mathbb{1}_{\{\sigma\wedge\tau=T\}} + L_\tau \mathbb{1}_{\{\tau<T,\tau\leqslant\sigma\}} + U_\sigma \mathbb{1}_{\{\sigma<\tau\}}, \quad \sigma,\tau \in \mathcal{T}_t,$$

where (Y, K, A, M) is a solution of $RBSDE(\xi, f+dV, L, U)$ such that $K, A \in \mathcal{V}_0^1$. By Remark 3.1, such a solution exists if (B1), (B2) and (3.34) are satisfied.

The lower value $\underline{V}$ and the upper value $\overline{V}$ of the stochastic game corresponding to R are defined by

$$\underline{V}_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_t} E\big(R_t(\sigma,\tau)|\mathcal{F}_t\big), \qquad \overline{V}_t = \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_t} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} E\big(R_t(\sigma,\tau)|\mathcal{F}_t\big).$$

We say that the game has a *value* if $\underline{V}_t = \overline{V}_t$, $t \in [0, T]$, P-a.s.

LEMMA 4.1. *Let $\{\tau_n\}$ be a sequence of stopping times such that $\tau_n \nearrow \tau$ P-a.s. and*

$$(4.2)\qquad P(\liminf_{n\to\infty}\{\tau_n = \tau\}) = 1.$$

Then for every $\sigma \in \mathcal{T}_t$, $E\big(R_t(\sigma,\tau_n)|\mathcal{F}_t\big) \to E\big(R_t(\sigma,\tau)|\mathcal{F}_t\big)$ P-a.s. as $n \to \infty$.

Proof. By (4.1) and (4.2), $R_t(\sigma,\tau_n) \to R_t(\sigma,\tau)$ P-a.s. Since V, L, U are of class D and $E|\xi| + E\int_0^T |f(t, Y_t)|dt < \infty$, we conclude from (4.1) that the family $\{R_t(\sigma,\tau_n)\}_{n\in\mathbb{N}}$ is a uniformly integrable family of random variables. Therefore, the desired convergence follows from [13], Theorem 1.3. ∎

THEOREM 4.1. *Let the assumptions of Theorem* 3.1 *hold and additionally let the relation* (3.34) *be satisfied. Then the stochastic game corresponding to the payoff function* (4.1) *has the value equal to the first component of the solution of $RBSDE(\xi, f + dV, L, U)$, i.e.*

$$(4.3)\qquad Y_t = \underline{V}_t = \overline{V}_t, \quad t \in [0, T], \quad P\text{-a.s.}$$

Proof. By [12], Lemma 5.3, to show that the game has a value it suffices to prove that for any $\varepsilon > 0$ and $t \in [0, T]$ there exist $\sigma_t^\varepsilon, \tau_t^\varepsilon \in \mathcal{T}_t$ such that for all $\sigma, \tau \in \mathcal{T}_t$,

$$(4.4)\qquad -\varepsilon + E\big(R_t(\sigma_t^\varepsilon,\tau)|\mathcal{F}_t\big) \leqslant E\big(R_t(\sigma,\tau_t^\varepsilon)|\mathcal{F}_t\big) + \varepsilon.$$

To show (4.4), we set $\sigma_t^\varepsilon = \inf\{s > t : Y_s \geqslant U_s - \varepsilon\} \wedge T$. Observe that $Y_{s-} < U_{s-}$ for $t < s \leqslant \sigma_t^\varepsilon$, and hence, by (LU3),

$$A_s \mathbb{1}_{(t,\sigma_t^\varepsilon]}(s) = A_{\sigma_t^\varepsilon} \mathbb{1}_{(t,\sigma_t^\varepsilon]}(s), \quad s \in [0,T]. \tag{4.5}$$

Clearly, for any $\tau \in \mathcal{T}_t$,

$$\{\sigma_t^\varepsilon = T\} \subset \{\tau \leqslant \sigma_t^\varepsilon\}, \quad \{\tau > \sigma_t^\varepsilon\} \subset \{\sigma_t^\varepsilon < T\}.$$

Therefore, by (4.5) it follows that on the set $\{\tau \leqslant \sigma_t^\varepsilon\}$ we have

$$\begin{aligned} R_t(\sigma_t^\varepsilon, \tau) &= \int_t^\tau f(s, Y_s)\, ds + \int_t^\tau dV_s + \xi \mathbb{1}_{\{\tau = T\}} + L_\tau \mathbb{1}_{\{\tau < T\}} \\ &\leqslant \int_t^\tau f(s, Y_s)\, ds + \int_t^\tau dV_s + \xi \mathbb{1}_{\{\tau = T\}} + Y_\tau \mathbb{1}_{\{\tau < T\}} + \int_t^\tau dK_s - \int_t^\tau dA_s \\ &\leqslant Y_t + \int_t^\tau dM_s, \end{aligned}$$

whereas on $\{\tau > \sigma_t^\varepsilon\}$ we have

$$\begin{aligned} R_t(\sigma_t^\varepsilon, \tau) &= \int_t^{\sigma_t^\varepsilon} f(s, Y_s)\, ds + \int_t^{\sigma_t^\varepsilon} dV_s + U_{\sigma_t^\varepsilon} \mathbb{1}_{\{\sigma_t^\varepsilon < \tau\}} \\ &\leqslant \int_t^{\sigma_t^\varepsilon} f(s, Y_s)\, ds + \int_t^{\sigma_t^\varepsilon} dV_s + Y_{\sigma_t^\varepsilon} + \int_t^{\sigma_t^\varepsilon} dK_s - \int_t^{\sigma_t^\varepsilon} dA_s + \varepsilon = Y_t + \int_t^{\sigma_t^\varepsilon} dM_s + \varepsilon. \end{aligned}$$

Hence

$$R_t(\sigma_t^\varepsilon, \tau) = R_t(\sigma_t^\varepsilon, \tau)\mathbb{1}_{\{\tau \leqslant \sigma_t^\varepsilon\}} + R_t(\sigma_t^\varepsilon, \tau)\mathbb{1}_{\{\tau > \sigma_t^\varepsilon\}} \leqslant Y_t + \int_t^{\sigma_t^\varepsilon \wedge \tau} dM_s + \varepsilon$$

P-a.s. Let $\{\zeta_n\}$ be a fundamental sequence for the local martingale M and let $\tau_n = \tau \wedge \zeta_n$. Then $\{\tau_n\}$ satisfies the assumptions of Lemma 4.1 and

$$E\big(R_t(\sigma_t^\varepsilon, \tau \wedge \zeta_n) | \mathcal{F}_t\big) \leqslant E\big(Y_t + \int_t^{\sigma_t^\varepsilon \wedge \tau \wedge \zeta_n} dM_s + \varepsilon \big| \mathcal{F}_t\big) = Y_t + \varepsilon.$$

Letting $n \to \infty$ and using Lemma 4.1, we obtain

$$E\big(R_t(\sigma_t^\varepsilon, \tau) | \mathcal{F}_t\big) \leqslant Y_t + \varepsilon. \tag{4.6}$$

Now, let us consider the stopping time $\tau_t^\varepsilon = \inf\{s > t : Y_s \leqslant L_s + \varepsilon\} \wedge T$. The arguments similar to those in the proof of (4.6) show that for any $\varepsilon > 0$ and $\sigma \in \mathcal{T}_t$,

$$E\big(R_t(\sigma, \tau_t^\varepsilon) | \mathcal{F}_t\big) \geqslant Y_t - \varepsilon. \tag{4.7}$$

Combining (4.6) with (4.7), we see that for any $\varepsilon > 0$,

$$-\varepsilon + E\big(R_t(\sigma_t^\varepsilon, \tau)|\mathcal{F}_t\big) \leqslant Y_t \leqslant E\big(R_t(\sigma, \tau_t^\varepsilon)|\mathcal{F}_t\big) + \varepsilon. \tag{4.8}$$

Thus (4.4) is satisfied and, in consequence, the game has a value. Moreover, from (4.8) and the definitions of $\underline{V}, \overline{V}$ it follows that $-\varepsilon + \overline{V}_t \leqslant Y_t \leqslant \underline{V}_t + \varepsilon, t \in [0, T]$, for $\varepsilon > 0$. Since we already know that the game has a value, this implies (4.3). ∎

Note that Dynkin games were studied, in different contexts, by several authors. For results related to Theorem 4.1 we refer the reader to [1], [8], [12], [14], [15] and the references given therein.

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