STOCHASTIC COMPLEX INTEGRALS ASSOCIATED WITH HOMOGENEOUS INDEPENDENTLY SCATTERED RANDOM MEASURES ON THE LINE

BY

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Abstract. Complex integrals associated with homogeneous independently scattered random measures on the line are discussed. Theorems corresponding to Cauchy's theorem and the residue theorem are given. Furthermore, the converse of Cauchy's theorem is discussed.

1. INTRODUCTION

A Lévy process is a stochastic process with stationary independent increments, stochastically continuous, starting at the origin, and having *càdlàg* paths. If the stationarity of increments is not assumed, it is called an *additive process*. Our notation and definition follow [15]. Let $\{X_t : t \geqslant 0\}$ be a Lévy process on $\mathbb{R}$. The distribution of X_1 has the Lévy–Khintchine representation of the form

$$Ee^{i\xi X_1} = \exp[C(\xi)],$$

$$C(\xi) = -2^{-1}A\xi^2 + i\gamma\xi + \int_{\mathbb{R}} \left(e^{i\xi x} - 1 - i\frac{\xi x}{1 + x^2} \right) \nu(dx),$$

where $A \geqslant 0$, $\gamma \in \mathbb{R}$, $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \min\{1, |x|^2\}\nu(dx) < \infty$. Here $C(\xi)$ is the cumulant function of the distribution of X_1. Stochastic integrals of nonrandom functions with repect to additive processes on $\mathbb{R}$ were studied by [4], [12] and [24]. Ken-iti Sato developed this theory in [16] and [17]. He has carried out various studies in [18]–[22]. These days, the connections between stochastic integral mappings and subclasses of infinitely divisible distributions are found. See [1], [2], [5], [7], [8] and [23]. Our aim is to study stochastic complex integrals based on the theory developed by Sato. Let $z = s + it \in \mathbb{C}$, where s and t are

the real and the imaginary part of z, respectively. A usual complex integral is an integral with respect to $dz = ds + idt$. Our complex integral is an integral with respect to $dz_{\mathbf{X}} = ds + idX_t$. Here we consider a process $\mathbf{X} = \{X_t : t \in \mathbb{R}\}$ on $\mathbb{R}$ defined along the construction in [13] as follows. Let $\{X_t^{(1)} : t \geqslant 0\}$ and $\{X_t^{(2)} : t \geqslant 0\}$ be independent Lévy processes such that $Ee^{i\xi X_t^{(1)}} = \exp[tC(\xi)]$ and $Ee^{i\xi X_t^{(2)}} = \exp[tC(-\xi)]$. The process $\mathbf{X}$ is defined by

$$X_t = \begin{cases} X_t^{(1)} & \text{for } t \geqslant 0, \\ X_{(-t)-}^{(2)} & \text{for } t < 0. \end{cases}$$

Throughout this paper, let $\mathbf{X} = \{X_t : t \in \mathbb{R}\}$ be the above process over an infinite time parameter set $\mathbb{R}$. We call (A, ν, γ) the *generating triplet* of $\mathbf{X}$.

Here we mention some notation and definitions: λ stands for the Lebesgue measure on $\mathbb{R}$. Let J be an interval in $\mathbb{R}$, for example, $J = \mathbb{R}$, $(-\infty, 0)$ or $[0, \infty)$. $\mathcal{B}_J^0$ is the class of bounded Borel sets in J; p-lim stands for limit in probability; ∂B stands for the boundary of B and we set $\overline{B} = B \cup \partial B$ for any set B in $\mathbb{R}^2$. By a *curve* Γ we mean a function $\Gamma : [0, 1] \to \mathbb{R}^2$ which is of class C^1. We call $\Gamma(0)$ and $\Gamma(1)$ the *beginning point* and the *end point* of Γ, respectively. By a *path* Γ we mean a sequence of curves, $\Gamma = \{\Gamma_1, \Gamma_2, \dots, \Gamma_n\}$ such that the end point of Γ_j is the beginning point of Γ_{j+1} for $j = 1, 2, \dots, n - 1$. A path Γ is called a *closed path* if the end point of Γ_n is the beginning point of Γ_1. If a closed path Γ does not intersect itself, we call it a *simple* closed path. A path Γ is called *regular* if $\Gamma_j' \neq 0$ for each j.

In this paper, let D be a bounded, connected and open set in $\mathbb{R}^2$, and suppose that $\partial D = \bigcup_{j=0}^{q} \Gamma^j$, where $0 \leqslant q < \infty$ and each Γ^j is a regular, simple closed path. In the case $q \geqslant 1$, we suppose that the region surrounded by Γ^0 includes $\bigcup_{j=1}^{q} \Gamma^j$, and that Γ^0 is parameterized counterclockwise and Γ^j, $j = 1, 2, \dots, q$, are parameterized clockwise. Let $\overline{D} \subset \widetilde{D} \subset [a, b] \times [c, d]$ for some open set $\widetilde{D}$ and some $a, b, c, d \in \mathbb{R}$. Let $f(s, t)$ be an $\mathbb{R}$-valued measurable function on $[a, b] \times [c, d]$ and of class C^1 on $\widetilde{D}$. We identify $z = s + it \in \mathbb{C}$ with $(s, t) \in \mathbb{R}^2$. Then $f(z)$ is regarded as a function of (s, t) and sometimes denoted by $f(s, t)$. Likewise, $[a, b] \times [c, d]$ is identified with $\{s + it : (s, t) \in [a, b] \times [c, d]\}$. Now we define stochastic line integrals along a simple closed path Γ. Let

$$\Delta_s^n = \{[s_{i-1}^n, s_i^n] : i = 1, 2, \dots, m_n\} \quad \text{and} \quad {}_t^n\Delta = \{[t_{j-1}^n, t_j^n] : j = 1, 2, \dots, l_n\}$$

be partitions of $[a, b]$ and $[c, d]$, respectively. Here $s_0^n = a, s_{m_n}^n = b, t_0^n = c$, and $t_{l_n}^n = d$. The set of rectangles

$$\Delta_n = \{I_{i,j}^n := [s_{i-1}^n, s_i^n] \times [t_{j-1}^n, t_j^n] : i = 1, 2, \dots, m_n, j = 1, 2, \dots, l_n\}$$

is called a *partition* of $[a, b] \times [c, d]$. Denote by $|I_{i,j}^n|$ the length of the diagonal of $I_{i,j}^n$. Then the *size* of Δ_n is defined by

$$|\Delta_n| = \max\{|I_{i,j}^n| : i = 1, 2, \dots, m_n, j = 1, 2, \dots, l_n\}.$$

Let $I_{i,j}^n \subset \widetilde{D}$. If $\partial I_{i,j}^n$ is parameterized counterclockwise, we define the line integral along $\partial I_{i,j}^n$ as follows:

$$(1.1) \qquad \int\limits_{\partial I_{i,j}^n} f(s,t)dX_t = \int\limits_{t_{j-1}^n}^{t_j^n} f(s_i^n, t)dX_t - \int\limits_{t_{j-1}^n}^{t_j^n} f(s_{i-1}^n, t)dX_t$$

$$= \int\limits_{t_{j-1}^n}^{t_j^n} \left(\int\limits_{s_{i-1}^n}^{s_i^n} \frac{\partial f}{\partial s}(s,t)ds \right) dX_t.$$

Here we interpret the values of the line integrals along $[s_{i-1}^n, s_i^n] \times \{t_j\}$ and $[s_{i-1}^n, s_i^n] \times \{t_{j-1}\}$ as zero. Hence we are able to introduce the following definition of the line integral along ∂D. The way of the definition is the first key to our analysis.

DEFINITION 1.1. Let $D_n = \bigcup_{I_{i,j}^n \subset \overline{D}} I_{i,j}^n$ and $D^n = \bigcup_{I_{i,j}^n \cap \overline{D} \neq \emptyset} I_{i,j}^n$, where $D^n \subset \widetilde{D}$. Then D_n and D^n are called the *inner partition* and the *outer partition* of $\overline{D}$ with respect to Δ_n, respectively. For each j, we set

$$D_{n,j} = \{s : (s,t) \in D_n,\ t \in (t_{j-1}^n, t_j^n)\},$$
$$D^{n,j} = \{s : (s,t) \in D^n,\ t \in (t_{j-1}^n, t_j^n)\}.$$

Let $f(s,t)$ be of class C^1 on $\widetilde{D}$. If

$$\int\limits_{\partial D_n} f(s,t)dX_t = \sum_{I_{i,j}^n \subset \overline{D}} \int\limits_{\partial I_{i,j}^n} f(s,t)dX_t = \sum_{j=1}^{l_n} \int\limits_{t_{j-1}^n}^{t_j^n} \left(\int\limits_{D_{n,j}} \frac{\partial f}{\partial s}(s,t)ds \right) dX_t$$

and

$$\int\limits_{\partial D^n} f(s,t)dX_t = \sum_{I_{i,j}^n \cap \overline{D} \neq \emptyset} \int\limits_{\partial I_{i,j}^n} f(s,t)dX_t = \sum_{j=1}^{l_n} \int\limits_{t_{j-1}^n}^{t_j^n} \left(\int\limits_{D^{n,j}} \frac{\partial f}{\partial s}(s,t)ds \right) dX_t$$

converge in probability and these limits are equal almost surely as $|\Delta_n| \to 0$ for any sequence of partitions Δ_n and if the limit does not depend on the choice of the sequence $\{\Delta_n\}$, then we call this limit the *stochastic line integral* of f along ∂D, and denote it by

$$\int\limits_{\partial D} f(s,t)dX_t.$$

REMARK 1.1. (i) *We note that* $D_n \subset \overline{D} \subset D^n \subset \widetilde{D}$. *We appropriate the path* ∂D *by the boundaries* ∂D_n *and* ∂D^n.

(ii) *If* $D_{n,j} = \emptyset$, *then we interpret the value of* $\int_{D_{n,j}} \frac{\partial f}{\partial s}(s,t)ds$ *as zero. Likewise,* $\int_{D^{n,j}} \frac{\partial f}{\partial s}(s,t)ds = 0$ *if* $D^{n,j} = \emptyset$.

In Section 2, stochastic integrals with respect to $\mathbf{X}$ are introduced and useful propositions are given without proofs. Section 3 begins with Green's theorem and Cauchy's theorem and the residue theorem are given. Furthermore, the converse of Cauchy's theorem is investigated. Proofs of the assertions in Section 3 are given in Section 4.

2. STOCHASTIC INTEGRALS

We show first the relationship of the process $\mathbf{X} = \{X_t : t \in \mathbb{R}\}$ to an independently scattered random measure (i.s.r.m.) on $\mathbb{R}$, and define stochastic integrals with respect to $\mathbf{X}$. Following [20], we define a homogeneous i.s.r.m. over $\mathbb{R}$. Also refer to [6], [12], [14], [16] and [24]. Here we note that A. Prékopa studied independently scattered random measures (not necessarily homogeneous) in the series of papers [9]–[11]. In this section, propositions are given without proofs.

DEFINITION 2.1. A family of $\mathbb{R}$-valued random variables $\{X(B) : B \in \mathcal{B}_{\mathbb{R}}^0\}$ is called an *independently scattered random measure* (i.s.r.m.) if it satisfies the following conditions:

(i) $\sum_{n=1}^{\infty} X(B_n)$ converges a.s. and equals $X\left(\sum_{n=1}^{\infty} B_n\right)$ a.s. for any sequence $B_1, B_2, \ldots$ of disjoint sets in $\mathcal{B}_{\mathbb{R}}^0$ with $\sum_{n=1}^{\infty} B_n \in \mathcal{B}_{\mathbb{R}}^0$;

(ii) $X(B_1), \ldots, X(B_n)$ are independent for any finite sequence $B_1, B_2, \ldots, B_n$ of disjoint sets in $\mathcal{B}_{\mathbb{R}}^0$;

(iii) $X(\{a\}) = 0$ a.s. for every $a \in \mathbb{R}$.

In addition, $\{X(B)\}$ is called a *homogeneous i.s.r.m.* if it satisfies

(iv) $X(B) \stackrel{d}{=} X(B + a)$ for every $B \in \mathcal{B}_{\mathbb{R}}^0$ and $a \in \mathbb{R}$.

Here, "$\stackrel{d}{=}$" means to be identically distributed.

We give a proposition corresponding to Theorem 3.2 (ii) of [16]. The proof is similar to that of Theorem 3.2 of [16].

PROPOSITION 2.1. *There is a unique homogeneous i.s.r.m.* $\{X(B) : B \in \mathcal{B}_{\mathbb{R}}^0\}$ *such that*

$$(2.1) \qquad X_t - X_s = X((s, t]) \quad \text{for } s \text{ and } t \text{ with } s < t.$$

Further, we have

$$(2.2) \qquad Ee^{i\xi X(B)} = \exp[\lambda(B)C(\xi)].$$

In particular, we have

$$(2.3) \qquad X(B) = \begin{cases} X^{(1)}(B) \text{ a.s.} & \text{for } B \in \mathcal{B}_{[0,\infty)}^0, \\ -X^{(2)}(-B) \text{ a.s.} & \text{for } B \in \mathcal{B}_{(-\infty,0)}^0. \end{cases}$$

Here, independently scattered random measures $X^{(1)}$ and $X^{(2)}$ correspond to $X_t^{(1)}$ and $X_t^{(2)}$, respectively.

REMARK 2.1. *The symbol "$-B$" means the set $\{-x : x \in B\}$. Uniqueness means that if both $\{X(B)\}$ and $\{X'(B)\}$ satisfy (2.1), then $X(B) = X'(B)$ a.s. for every $B \in \mathcal{B}_{\mathbb{R}}^0$.*

Although Rocha-Arteaga and Sato [13] directly define stochastic integrals based on $\{X_t : t \in \mathbb{R}\}$, we define those integrals based on independently scattered random measures.

DEFINITION 2.2. An $\mathbb{R}$-valued function $F(t)$ on $\mathbb{R}$ is called a *simple function* if

$$F(t) = \sum_{j=1}^{n} 1_{B_j}(t) R_j$$

for some n, where $B_1, \ldots, B_n$ are disjoint Borel sets in $\mathbb{R}$ and $R_1, \ldots, R_n \in \mathbb{R}$. If F is a simple function of this form, then we define the *integral* of F over $B \in \mathcal{B}_{\mathbb{R}}^0$ with respect to $\mathbf{X}$ as

$$\int_B F(t) dX_t = \sum_{j=1}^{n} R_j X(B \cap B_j).$$

DEFINITION 2.3. An $\mathbb{R}$-valued function $F(t)$ on $\mathbb{R}$ is said to be *locally $\mathbf{X}$-integrable* or *locally $\{X_t\}$-integrable* if it is measurable and there is a sequence of simple functions $F_n(t)$, $n = 1, 2, \ldots$, such that
(1) $F_n(t) \to F(t)$ a.e. as $n \to \infty$, and
(2) for every $B \in \mathcal{B}_{\mathbb{R}}^0$, $\int_B F_n(t) dX_t$ is convergent in probability as $n \to \infty$.
We denote by $\mathbf{L}(\mathbf{X})$ the class of locally $\mathbf{X}$-integrable functions.

Proposition 2.15 of [17] remains true even if we replace $\mathcal{B}_{[0,\infty)}^0$ with $\mathcal{B}_{\mathbb{R}}^0$:

PROPOSITION 2.2. *Let $F \in \mathbf{L}(\mathbf{X})$. If $F_n^1(t)$ and $F_n^2(t)$ are sequences satisfying (1) and (2) of Definition 2.3, then*

$$\text{p-}\lim_{n\to\infty} \int_B F_n^1(t) dX_t = \text{p-}\lim_{n\to\infty} \int_B F_n^2(t) dX_t \text{ a.s. } \text{ for each } B \in \mathcal{B}_{\mathbb{R}}^0.$$

Proposition 2.2 enables us to define stochastic integrals of locally $\mathbf{X}$-integrable functions:

DEFINITION 2.4. Let $F \in \mathbf{L}(\mathbf{X})$ and let simple functions F_n, $n = 1, 2, \ldots$, satisfy (1) and (2) of Definition 2.3. Then we define

$$\int_B F(t) dX_t = \text{p-}\lim_{n\to\infty} \int_B F_n(t) dX_t \quad \text{for } B \in \mathcal{B}_{\mathbb{R}}^0.$$

Prékopa's study gives us useful propositions, see Theorems 2.1, 2.4 and 2.5 of [10].

PROPOSITION 2.3. *If $F_1, F_2 \in \mathbf{L}(\mathbf{X})$, then, for any $a_1, a_2 \in \mathbb{R}$, $a_1F_1 + a_2F_2 \in$ $\mathbf{L}(\mathbf{X})$ and*

$$(2.4) \qquad \int_B \big(a_1 F_1(t) + a_2 F_2(t)\big) dX_t = a_1 \int_B F_1(t) dX_t + a_2 \int_B F_2(t) dX_t \quad a.s.$$

for $B \in \mathcal{B}_\mathbb{R}^0$.

PROPOSITION 2.4. *Let $F \in \mathbf{L}(\mathbf{X})$ and let $B_1, B_2 \in \mathcal{B}_\mathbb{R}^0$. If the intersection of B_1 and B_2 is empty, or if it consists of a finite number of points, then*

$$(2.5) \qquad \int_{B_1 \cup B_2} F(t) dX_t = \int_{B_1} F(t) dX_t + \int_{B_2} F(t) dX_t \quad a.s.$$

Local integrability is characterized in terms of the generating triplet (A, ν, γ) of $\mathbf{X}$, see Theorem 2.7 of [12]. Now we define

$$\varphi(u) = Au^2 + \int_\mathbb{R} \min\{1, |ux|^2\}\nu(dx)$$
$$+ \left| u\gamma + \int_\mathbb{R} \left(\frac{ux}{1 + |ux|^2} - \frac{ux}{1 + |x|^2} \right) \nu(dx) \right|$$

for $u \in \mathbb{R}$.

PROPOSITION 2.5. *Let $F(t)$ be an $\mathbb{R}$-valued measurable function on $\mathbb{R}$. Then $F \in \mathbf{L}(\mathbf{X})$ if and only if*

$$(2.6) \qquad \int_{t_0}^{t_1} \varphi\big(F(t)\big) dt < \infty \quad \text{for } -\infty < t_0 < t_1 < \infty.$$

COROLLARY 2.1. *Let $F(t)$ be an $\mathbb{R}$-valued measurable function on $\mathbb{R}$. If $F(t)$ is locally bounded, then $F \in \mathbf{L}(\mathbf{X})$.*

Lastly, we give an expression of the characteristic function of $\int_B F(t) dX_t$. See Proposition 2.6 of [12].

PROPOSITION 2.6. *If $F \in \mathbf{L}(\mathbf{X})$, then*

$$E \exp\left[i\xi \int_B F(t) dX_t \right] = \exp\left[\int_B C\big(F(t)\xi\big) dt \right]$$

for $B \in \mathcal{B}_\mathbb{R}^0$.

3. STOCHASTIC COMPLEX INTEGRALS

First, we define double integrals with repect to $dsdX_t$.

DEFINITION 3.1. Let $c \leqslant t \leqslant d$ and set $B_t = \{s : (s, t) \in B \cap \widetilde{D}\}$ for every Borel set $B \subset [a, b] \times [c, d]$. Suppose $f(s, t)$ is an $\mathbb{R}$-valued function of class C^1 on $\widetilde{D}$. Then we define

$$\iint\limits_{B} f(s,t)dsdX_t = \int\limits_{c}^{d} \left(\int\limits_{B_t} \frac{\partial f}{\partial s}(s,t)ds \right) dX_t.$$

Here, if $B_t = \emptyset$, then we interpret the value of $\int_{B_t} \frac{\partial f}{\partial s}(s,t)ds$ as zero.

As the second key of our analysis, we will show a theorem corresponding to Green's theorem. Recall that $\partial D = \bigcup_{j=0}^{q} \Gamma^j$, where Γ^0 is parameterized counterclockwise and Γ^j, $j = 1, 2, \ldots, q$, are parameterized clockwise.

THEOREM 3.1. *Suppose $f(s, t)$ is an $\mathbb{R}$-valued function of class C^1 on $\widetilde{D}$. Then*

$$(3.1) \qquad \int\limits_{\partial D} f(s,t)dX_t = \iint\limits_{\overline{D}} \frac{\partial f}{\partial s}(s,t)dsdX_t \quad a.s.$$

REMARK 3.1. *The function $\int_{(\overline{D})_t} \frac{\partial f}{\partial s}(s,t)ds$ is locally $\mathbf{X}$-integrable because it is bounded and we can use Corollary* 2.1. *Hence the right-hand side of* (3.1) *is definable.*

Let $f(z)$ be a complex-valued function for $z = s + it \in \widetilde{D}$. Then $f(z)$ can be expressed as $f(z) = u(z) + iv(z)$, where $u(z)$ and $v(z)$ are $\mathbb{R}$-valued functions.

DEFINITION 3.2. Suppose $\int_{\partial D} u(s,t)dX_t$ and $\int_{\partial D} v(s,t)dX_t$ are definable. Then we define *stochastic complex integrals* along a closed path ∂D as follows:

$$\int\limits_{\partial D} f(z)dz_{\mathbf{X}} = \int\limits_{\partial D} \big(u(s,t) + iv(s,t)\big)(ds + idX_t)$$
$$= \int\limits_{\partial D} u(s,t)ds - \int\limits_{\partial D} v(s,t)dX_t + i\big(\int\limits_{\partial D} v(s,t)ds + \int\limits_{\partial D} u(s,t)dX_t \big).$$

From Theorem 3.1 we obtain a theorem corresponding to Cauchy's theorem:

THEOREM 3.2. *Suppose $f(z)$ is holomorphic on $\widetilde{D}$. Then*

$$(3.2) \qquad \int\limits_{\partial D} f(z)dz_{\mathbf{X}} = i \iint\limits_{\overline{D}} \frac{\partial f}{\partial s}(s,t)dsd(X_t - t) \quad a.s.$$

REMARK 3.2. (i) *The process $\{X_t - t : t \in \mathbb{R}\}$ corresponds to a homogeneous i.s.r.m. $\{X(B) - \lambda(B) : B \in \mathcal{B}_{\mathbb{R}}^0\}$. See Lemma 4.2 in Section 4 below.*

(ii) *If $f(z)$ is holomorphic, then $u(s,t)$ and $v(s,t)$ are of class C^1.*

(iii) *In the case $X_t = t$, the stochastic complex integral is understood as the usual complex integral. Then the right-hand side of* (3.2) *is equal to zero, and the assertion coincides with Cauchy's theorem.*

Let $f(s,t)$ be of class C^1 on $\widetilde{D}$ excluding a finite number of points. In particular, we suppose D is simply connected. As $\int_{\partial D} f(s,t)dX_t$ is not definable in the sense of Definition 1.1, we define stochastic line integrals as follows:

DEFINITION 3.3. Let D be a bounded, simply connected, open set and let $f(s,t)$ be an $\mathbb{R}$-valued function of class C^1 on $\widetilde{D}\backslash\{a_1, a_2, \ldots, a_n\}$, where $a_j = (a_j^1, a_j^2) \in D$ for $j = 1, 2, \ldots, n$. Let

$$I_{\epsilon,\delta}(a_j) = [a_j^1 - \epsilon, a_j^1 + \epsilon] \times [a_j^2 - \delta, a_j^2 + \delta],$$

where $\partial I_{\epsilon,\delta}(a_j)$ is parameterized counterclockwise, and let

$$D_{\epsilon,\delta} = D\backslash \bigcup_{j=1}^{n} I_{\epsilon,\delta}(a_j).$$

Here we take ϵ and δ such that $I_{\epsilon,\delta}(a_j) \subset D$ and that $I_{\epsilon,\delta}(a_j)$, $j = 1, 2, \ldots, n$, are disjoint. Then we define

$$\int_{\partial D} f(z)dz_{\mathbf{X}} = \text{p-}\lim_{\delta\downarrow 0}\Big(\int_{\partial D_{\epsilon,\delta}} f(z)dz_{\mathbf{X}} + \sum_{j=1}^{n} \int_{\partial I_{\epsilon,\delta}(a_j)} f(z)dz_{\mathbf{X}} \Big)$$

and

$$\int_{\partial D} f(s,t)dX_t = \text{p-}\lim_{\delta\downarrow 0}\Big(\int_{\partial D_{\epsilon,\delta}} f(s,t)dX_t + \sum_{j=1}^{n} \int_{\partial I_{\epsilon,\delta}(a_j)} f(s,t)dX_t \Big),$$

if each convergence on the right-hand sides does not depend on ϵ. Here the integral $\int_{\partial I_{\epsilon,\delta}(a_j)} f(s,t)dX_t$ is defined in the same way as in (1.1).

REMARK 3.3. *If we regard $\partial I_{\epsilon,\delta}(a_j)$ as a part of $\partial D_{\epsilon,\delta}$, then $\partial I_{\epsilon,\delta}(a_j)$ is parameterized clockwise.*

In Theorem 3.3 and Corollary 3.1 below, we suppose D is simply connected.

THEOREM 3.3. *Let D be a bounded, simply connected, open set. Suppose $f(z)$ is holomorphic on $\widetilde{D}$ except at a finite number of isolated singular points $a_1, a_2, \ldots, a_n \in D$. Then we have the following representation*:

(3.3) $$\int_{\partial D} f(s,t)d(X_t - t) = \text{p-}\lim_{\delta\downarrow 0} \int_{\partial D_{\epsilon,\delta}} f(s,t)d(X_t - t)$$

and

$$(3.4) \qquad \int\limits_{\partial D} f(z)dz_{\mathbf{X}} = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, a_j) + i \int\limits_{\partial D} f(s,t)d(X_t - t) \quad a.s.$$

Here, $\mathrm{Res}(f, a_j)$ *stands for the residue of* $f(z)$ *at* a_j *for* $j = 1, 2, \ldots, n$.

REMARK 3.4. *We note that the limit* p-$\lim_{\delta \downarrow 0} \int_{D_{\epsilon,\delta}} f(s,t)d(X_t - t)$ *of* (3.3) *does not depend on* ϵ.

Here we mention the case where the number of isolated singular points is equal to one without proof. It is immediately derived from Theorem 3.3.

COROLLARY 3.1. *Suppose* $f(z)$ *is holomorphic on* $\widetilde{D}$. *For* $\eta \in D$, *we have*

$$(3.5) \qquad \int\limits_{\partial D} \frac{f(z)}{z - \eta} dz_{\mathbf{X}} = 2\pi i f(\eta) + i \int\limits_{\partial D} \frac{f(z)}{z - \eta} d(X_t - t) \quad a.s.$$

REMARK 3.5. *We do not state particularly that* $f(\eta)$ *has a similar representation to Cauchy's integral formula because* $f(\eta)$ *is non-random and the integrals are random, excluding the case* $X_t = t$.

At the end of this section, we investigate the converse of Cauchy's theorem. We obtain the following fact:

THEOREM 3.4. *Suppose that* $E|X_1| < \infty$ *and* $EX_1 = 1$. *Then, for any* $\mathbb{R}$-*valued function* $f(s,t)$ *of class* C^1 *on* $\widetilde{D}$,

$$(3.6) \qquad E\Big|\int\limits_{\partial D} f(z)dz_{\mathbf{X}}\Big| < \infty \quad and \quad E\Big[\int\limits_{\partial D} f(z)dz_{\mathbf{X}}\Big] = \int\limits_{\partial D} f(z)dz.$$

COROLLARY 3.2. *Suppose that* $E|X_1| < \infty$ *and* $E[X_1] = 1$. *Let* $u(s,t)$ *and* $v(s,t)$ *be* $\mathbb{R}$-*valued functions of class* C^1 *on* $\widetilde{D}$. *Then* $f(z)$ *is holomorphic on* $\widetilde{D}$ *if and only if* $E\big[\int_{\partial \triangle} f(z)dz_{\mathbf{X}}\big] = 0$ *for every closed triangle* $\triangle \subset \widetilde{D}$.

REMARK 3.6. (i) *The symbol* $\triangle$ *is used instead of* D *because the paths of stochastic integrals are triangular paths.*
(ii) *If* $X_t = t$, *then the corollary states the assertion of Morera's theorem.*

4. PROOFS

LEMMA 4.1. *Let* $F(t)$ *be an* $\mathbb{R}$-*valued function on* $\mathbb{R}$. *There are constants* K_1, K_2 *and* K_3 *such that*

$$(4.1) \qquad \big|C\big(\xi F(t)\big)\big| \leqslant K_1 \xi^2 F(t)^2 + K_2 |\xi| |F(t)| + K_3$$

for any $\xi \in \mathbb{R}$.

P r o o f. We have

$$\left|C\big(\xi F(t)\big)\right| \leqslant 2^{-1}\xi^2\big(A + \int\limits_{|x|<1} x^2\nu(dx)\big)F(t)^2$$

$$+ |\xi||F(t)|\big(|\gamma| + 2^{-1}\int\limits_{\mathbb{R}} \min\{1, x^2\}\nu(dx)\big) + 2\nu(\{|x| \geqslant 1\})).$$

Hence the lemma is true. ∎

LEMMA 4.2. (i) $\{X_t - t : t \in \mathbb{R}\}$ *corresponds to a homogeneous i.s.r.m.* $\{X(B) - \lambda(B) : B \in \mathcal{B}_{\mathbb{R}}^0\}$.

(ii) *Let* $F(t)$ *be an* $\mathbb{R}$*-valued function on* $\mathbb{R}$. *Suppose* $\int_{t_0}^{t_1} |F(t)|dt < \infty$ *for any* $t_0, t_1 \in \mathbb{R}$ *with* $t_0 < t_1$. *If* $F \in \mathbf{L}(\mathbf{X})$, *then* $F(t)$ *is locally* $\{X_t - t\}$*-integrable and*

(4.2) $$\int\limits_B F(t)dX_t - \int\limits_B F(t)dt = \int\limits_B F(t)d(X_t - t) \quad a.s.$$

for any $B \in \mathcal{B}_{\mathbb{R}}^0$.

P r o o f. (i) Let M be a homogeneous i.s.r.m. corresponding to $\{X_t - t\}$. Then we have

$$M\big((s,t]\big) = (X_t - t) - (X_s - s) = X\big((s,t]\big) - \lambda\big((s,t]\big)$$

for s and t with $s < t$. Notice that $\{X(B) - \lambda(B)\}$ is a homogeneous i.s.r.m. Hence it follows from Proposition 2.1 that

$$M(B) = X(B) - \lambda(B) \text{ a.s.} \quad \text{for every } B \in \mathcal{B}_{\mathbb{R}}^0.$$

(ii) It is obvious from Proposition 2.5 that $F(t)$ is locally $\{X_t - t\}$-integrable. If $F(t)$ is a simple function, then (4.2) holds. Hence (4.2) is true. ∎

LEMMA 4.3. *Let* $f(s,t)$ *be an* $\mathbb{R}$*-valued function of class* C^1 *on* $\widetilde{D}$. *For* D_n *and* D^n *of Definition* 1.1, *we have*

(4.3) $$\int\limits_{\partial D_n} f(s,t)dX_t = \int\limits_c^d \left(\int\limits_{(D_n)_t} \frac{\partial f}{\partial s}(s,t)ds\right)dX_t \quad a.s.,$$

(4.4) $$\int\limits_{\partial D^n} f(s,t)dX_t = \int\limits_c^d \left(\int\limits_{(D^n)_t} \frac{\partial f}{\partial s}(s,t)ds\right)dX_t \quad a.s.$$

P r o o f. Recall that $D_{n,j} = \{s : (s,t) \in D_n, \ t_{j-1}^n < t < t_j^n\}$. From the definition we see that

$$\int\limits_{\partial D_n} f(s,t)dX_t = \sum_{j=1}^{l_n} \int\limits_{t_{j-1}^n}^{t_j^n} \left(\int\limits_{D_{n,j}} \frac{\partial f}{\partial s}(s,t)ds\right)dX_t$$

$$= \sum_{j=1}^{l_n} \int\limits_{t_{j-1}^n}^{t_j^n} \left(\int\limits_{(D_n)_t} \frac{\partial f}{\partial s}(s,t)ds\right)dX_t = \int\limits_c^d \left(\int\limits_{(D_n)_t} \frac{\partial f}{\partial s}(s,t)ds\right)dX_t \quad a.s.$$

Here we used Proposition 2.4 in the last equality. In the same way as above, we can prove that (4.4) is true. The lemma has been proved. ∎

We are now ready to prove Theorem 3.1.

P r o o f o f T h e o r e m 3.1. As $\int_{D_t} \frac{\partial f}{\partial s}(s,t)ds$ is locally bounded, it follows from Corollary 2.1 that it is $\mathbf{X}$-integrable. Denote by D_n and D^n the inner partition and the outer partition of $\overline{D}$ with respect to Δ_n (see Definition 1.1). We infer from Propositions 2.3 and 2.6 that

$$(4.5) \quad E \exp\left[i\xi\left(\iint_{\overline{D}} \frac{\partial f}{\partial s}(s,t)dsdX_t - \int_{\partial D_n} f(s,t)dX_t\right)\right]$$

$$= E \exp\left[i\xi\int_c^d \left(\int_{\overline{D}_t\backslash(D_n)_t} \frac{\partial f}{\partial s}(s,t)ds\right)dX_t\right]$$

$$= \exp\left[\int_c^d C\left(\xi\int_{\overline{D}_t\backslash(D_n)_t} \frac{\partial f}{\partial s}(s,t)ds\right)dt\right].$$

Let $\epsilon > 0$ and set $E_n := \left\{t \in [c,d] : \lambda\left(\overline{D}_t\backslash(D_n)_t\right) > \epsilon\right\}$. Now we split the above integral in two:

$$\int_c^d C\left(\xi\int_{\overline{D}_t\backslash(D_n)_t} \frac{\partial f}{\partial s}(s,t)ds\right)dt = \int_{E_n} C\left(\xi\int_{\overline{D}_t\backslash(D_n)_t} \frac{\partial f}{\partial s}(s,t)ds\right)dt$$

$$+ \int_{[c,d]\backslash E_n} C\left(\xi\int_{\overline{D}_t\backslash(D_n)_t} \frac{\partial f}{\partial s}(s,t)ds\right)dt$$

$$\equiv I_1 + I_2.$$

Let

$$M := \sup_{(s,t)\in\overline{D}} \left|\frac{\partial f}{\partial s}(s,t)\right|.$$

Here it follows from Lemma 4.1 that

$$\left|C\left(\xi\int_{\overline{D}_t\backslash(D_n)_t} \frac{\partial f}{\partial s}(s,t)ds\right)\right| \leqslant K_1\xi^2\left(\int_{\overline{D}_t\backslash(D_n)_t} \frac{\partial f}{\partial s}(s,t)ds\right)^2$$

$$+ K_2|\xi|\left|\int_{\overline{D}_t\backslash(D_n)_t} \frac{\partial f}{\partial s}(s,t)ds\right| + K_3$$

$$\leqslant K_1\xi^2\left(M(b-a)\right)^2 + K_2|\xi|M(b-a) + K_3.$$

Notice that $\lim_{n\to\infty} \lambda(E_n) = 0$ because

$$\lim_{n\to\infty} \int_c^d \lambda\big(\overline{D}_t \backslash (D_n)_t\big) dt = 0.$$

This implies that

$$\lim_{n\to\infty} |I_1| \leqslant \Big(K_1 \xi^2 \big(M(b-a)\big)^2 + K_2|\xi|M(b-a) + K_3\Big) \lim_{n\to\infty} \lambda(E_n) = 0.$$

Furthermore, we see that

$$\left| \int_{\overline{D}_t \backslash (D_n)_t} \frac{\partial f}{\partial s}(s,t) ds \right| \leqslant M\lambda\big(\overline{D}_t \backslash (D_n)_t\big) \leqslant M\epsilon$$

on $[c,d] \backslash E_n$. Let $\delta > 0$. As $C(\xi)$ is continuous, it follows that

$$|I_2| \leqslant \int_{[c,d] \backslash E_n} \left| C\Big(\xi \int_{\overline{D}_t \backslash (D_n)_t} \frac{\partial f}{\partial s}(s,t) ds\Big) \right| dt < \delta\lambda([c,d] \backslash E_n) \leqslant \delta(d-c)$$

for sufficiently small $\epsilon > 0$ and sufficiently large n. This implies $\lim_{n\to\infty} I_2 = 0$. Hence we obtain

$$\iint_{\overline{D}} \frac{\partial f}{\partial s}(s,t) ds dX_t - \int_{\partial D_n} f(s,t) dX_t \to 0 \quad \text{in probability}$$

because (4.5) goes to one as $n \to \infty$. Furthermore, we have

$$E \exp\left[i\xi\Big(\int_{\partial D^n} f(s,t) dX_t - \iint_{\overline{D}} \frac{\partial f}{\partial s}(s,t) ds dX_t \Big) \right]$$

$$= \exp\left[\int_c^d C\Big(\xi \int_{(D^n)_t \backslash \overline{D}_t} \frac{\partial f}{\partial s}(s,t) ds\Big) dt \right].$$

In the same way as above, we arrive at the conclusion that

$$\int_{\partial D^n} f(s,t) dX_t - \iint_{\overline{D}} \frac{\partial f}{\partial s}(s,t) ds dX_t \to 0 \quad \text{in probability}.$$

Hence the theorem has been proved. ■

We are now ready to prove Theorem 3.2. In the proofs of Theorems 3.2 and 3.3 and Corollary 3.2, we use the representation $f(z) = u(z) + iv(z)$. Here $u(z)$ and $v(z)$ are the real part and the imaginary part of $f(z)$, respectively.

P r o o f o f T h e o r e m 3.2. From Theorem 3.1 it follows that almost surely

$$
\begin{aligned}
\int\limits_{\partial D} f(z) dz_{\mathbf{X}} &= \int\limits_{\partial D} u(s,t) ds - \int\limits_{\partial D} v(s,t) dX_t \\
&\quad + i\Big(\int\limits_{\partial D} v(s,t) ds + \int\limits_{\partial D} u(s,t) dX_t \Big) \\
&= -\iint\limits_{\overline{D}} \frac{\partial u}{\partial t}(s,t) dt ds - \iint\limits_{\overline{D}} \frac{\partial v}{\partial s}(s,t) ds dX_t \\
&\quad + i\Big(-\iint\limits_{\overline{D}} \frac{\partial v}{\partial t}(s,t) dt ds + \iint\limits_{\overline{D}} \frac{\partial u}{\partial s}(s,t) ds dX_t \Big) \\
&\equiv I.
\end{aligned}
$$

Here we also used the usual Green's theorem. The Cauchy–Riemann equations tell us that

$$
\frac{\partial u}{\partial s}(s,t) = \frac{\partial v}{\partial t}(s,t) \quad \text{and} \quad \frac{\partial u}{\partial t}(s,t) = -\frac{\partial v}{\partial s}(s,t).
$$

These mean that

$$
\begin{aligned}
I &= -\iint\limits_{\overline{D}} \frac{\partial u}{\partial t}(s,t) ds dt - i \iint\limits_{\overline{D}} \frac{\partial v}{\partial t}(s,t) ds dt \\
&\quad + i\Big(i \iint\limits_{\overline{D}} \frac{\partial v}{\partial s}(s,t) ds dX_t + \iint\limits_{\overline{D}} \frac{\partial u}{\partial s}(s,t) ds dX_t \Big) \\
&= -i \iint\limits_{\overline{D}} \frac{\partial f}{\partial s}(s,t) ds dt + i \iint\limits_{\overline{D}} \frac{\partial f}{\partial s}(s,t) ds dX_t \\
&= i \iint\limits_{\overline{D}} \frac{\partial f}{\partial s}(s,t) ds d(X_t - t) \quad \text{a.s.}
\end{aligned}
$$

The last equality is due to Lemma 4.2. The theorem has been proved. ∎

We are now ready to prove Theorem 3.3.

P r o o f o f T h e o r e m 3.3. Let $a_j = (a_j^1, a_j^2) \in \mathbb{R}^2$ for $1 \leqslant j \leqslant n$. Arrange $\{a_j^2 : j = 1, 2, \ldots, n\}$ in ascending order. They are denoted by

$$
\beta_1 < \beta_2 < \ldots < \beta_l \quad (l \leqslant n).
$$

Furthermore, for each β_k, we arrange $\{a_j^1 : a_j^2 = \beta_k, 1 \leqslant j \leqslant n\}$ in ascending order. They are denoted by

$$
\alpha_1^k < \alpha_2^k < \ldots < \alpha_{m_k}^k.
$$

Here $\sum_{k=1}^{l} m_k = n$. We take a sequence $\{\delta_n\}$ satisfying $\delta_n \downarrow 0$ as $n \to \infty$. Let $\widetilde{X}_t = X_t - t$ and denote by $\widetilde{C}(z)$ the cumulant function of the distribution of $X_1 - 1$. From Lemma 4.2 and Cauchy's theorem we see that

$$\int\limits_{\partial D_{\epsilon,\delta_n}} f(z)dz_{\mathbf{X}} = \int\limits_{\partial D_{\epsilon,\delta_n}} f(z)dz + i \int\limits_{\partial D_{\epsilon,\delta_n}} f(s,t)d\widetilde{X}_t$$

$$= i \int\limits_{\partial D_{\epsilon,\delta_n}} f(s,t)d\widetilde{X}_t \quad \text{a.s.}$$

We examine the convergence of $\int_{\partial D_{\epsilon,\delta_n}} f(s,t)d\widetilde{X}_t$. Let $n > m$. From Theorem 3.1 and Proposition 2.3 we infer that almost surely

$$\int\limits_{\partial D_{\epsilon,\delta_n}} u(s,t)d\widetilde{X}_t - \int\limits_{\partial D_{\epsilon,\delta_m}} u(s,t)d\widetilde{X}_t$$

$$= \int_c^d \left(\int\limits_{(D_{\epsilon,\delta_n})_t} \frac{\partial u}{\partial s}(s,t)ds \right)d\widetilde{X}_t - \int_c^d \left(\int\limits_{(D_{\epsilon,\delta_m})_t} \frac{\partial u}{\partial s}(s,t)ds \right)d\widetilde{X}_t$$

$$= \sum_{k=1}^{l} \sum_{j=1}^{m_k} \int_c^d \left(\int\limits_{J_{k,j}(t)} \frac{\partial u}{\partial s}(s,t)ds \right)d\widetilde{X}_t \equiv I,$$

where

$$J_{k,j}(t) = \begin{cases} [\alpha_j^k - \epsilon, \alpha_j^k + \epsilon] & \text{if } t \in [\beta_k - \delta_m, \beta_k - \delta_n] \cup [\beta_k + \delta_n, \beta_k + \delta_m], \\ \emptyset & \text{otherwise.} \end{cases}$$

Furthermore, we have

$$E \exp\left[i\xi \int_c^d \left(\int\limits_{J_{k,j}(t)} \frac{\partial u}{\partial s}(s,t)ds \right)d\widetilde{X}_t \right]$$

$$= \exp\left[\left(\int\limits_{\beta_k - \delta_m}^{\beta_k - \delta_n} + \int\limits_{\beta_k + \delta_n}^{\beta_k + \delta_m} \right) \widetilde{C}\big(\xi\{u(\alpha_j^k + \epsilon, t) - u(\alpha_j^k - \epsilon, t)\}\big)dt \right].$$

Notice that $u(\alpha_j^k + \epsilon, t)$ and $u(\alpha_j^k - \epsilon, t)$ are continuous in t and thereby bounded on $[\beta_k - \delta_m, \beta_k - \delta_n] \cup [\beta_k + \delta_n, \beta_k + \delta_m]$. By Lemma 4.1, this implies that

$$\int_c^d \left(\int\limits_{J_{k,j}(t)} \frac{\partial u}{\partial s}(s,t)ds \right)d\widetilde{X}_t \to 0 \quad \text{in probability}$$

as $n, m \to \infty$. Hence p-$\lim_{n,m\to\infty} I = 0$. Even if we replace $u(s,t)$ with $v(s,t)$, I goes to zero in probability. Hence $\int_{\partial D_{\epsilon,\delta}} f(s,t)d\widetilde{X}_t$ converges in probability as $\delta \downarrow 0$.

Next, we show that this limit does not depend on ϵ. Let $\epsilon' > \epsilon > 0$. Then we have almost surely

$$\int\limits_{\partial D_{\epsilon,\delta}} u(s,t)d\widetilde{X}_t - \int\limits_{\partial D_{\epsilon',\delta}} u(s,t)d\widetilde{X}_t = \iint\limits_{D_{\epsilon,\delta}} \frac{\partial u}{\partial s}(s,t)dsd\widetilde{X}_t - \iint\limits_{D_{\epsilon',\delta}} \frac{\partial u}{\partial s}(s,t)dsd\widetilde{X}_t$$

$$= \sum_{k=1}^{l}\sum_{j=1}^{m_k}\int_c^d \left(\int\limits_{L_{k,j}(t)} \frac{\partial u}{\partial s}(s,t)ds\right)d\widetilde{X}_t,$$

where

$$L_{k,j}(t) = \begin{cases} [\alpha_j^k - \epsilon', \alpha_j^k - \epsilon] \cup [\alpha_j^k + \epsilon, \alpha_j^k + \epsilon'] & \text{if } t \in [\beta_k - \delta, \beta_k + \delta], \\ \emptyset & \text{otherwise.} \end{cases}$$

As we see that

$$E\exp\left[i\xi\int_c^d \left(\int\limits_{L_{k,j}(t)} \frac{\partial u}{\partial s}(s,t)ds\right)d\widetilde{X}_t\right]$$

$$= \exp\left[\int\limits_{\beta_k-\delta}^{\beta_k+\delta}\widetilde{C}\big(\xi\{u(\alpha_j^k+\epsilon',t)-u(\alpha_j^k+\epsilon,t)+u(\alpha_j^k-\epsilon,t)-u(\alpha_j^k-\epsilon',t)\}\big)dt\right],$$

it follows that

$$\text{p-}\lim_{\delta\downarrow 0}\int_c^d \left(\int\limits_{L_{k,j}(t)} \frac{\partial u}{\partial s}(s,t)ds\right)d\widetilde{X}_t = 0.$$

This implies that

$$\int\limits_{\partial D_{\epsilon,\delta}} u(s,t)d\widetilde{X}_t - \int\limits_{\partial D_{\epsilon',\delta}} u(s,t)d\widetilde{X}_t \to 0 \quad \text{in probability}$$

as $\delta \downarrow 0$. This convergence also holds for v in place of u. Hence we conclude that $\text{p-}\lim_{\delta\downarrow 0}\int_{\partial D_{\epsilon,\delta}} f(s,t)d\widetilde{X}_t$ does not depend on ϵ.

The rest is to examine $\int_{\partial I_{\epsilon,\delta}(a_j)} f(s,t)d\widetilde{X}_t$:

$$E\exp\left[i\xi\int\limits_{\partial I_{\epsilon,\delta}(a_j)} u(s,t)d\widetilde{X}_t\right] = E\exp\left[i\xi\iint\limits_{I_{\epsilon,\delta}(a_j)} \frac{\partial u}{\partial s}(s,t)dsd\widetilde{X}_t\right]$$

$$= E\exp\left[i\xi\int\limits_{a_j^2-\delta}^{a_j^2+\delta}\{u(a_j^1+\epsilon,t)-u(a_j^1-\epsilon,t)\}d\widetilde{X}_t\right]$$

$$= \exp\left[\int\limits_{a_j^2-\delta}^{a_j^2+\delta}\widetilde{C}\big(\xi\{u(a_j^1+\epsilon,t)-u(a_j^1-\epsilon,t)\}\big)dt\right].$$

This converges to one as $\delta \downarrow 0$. Hence p-$\lim_{\delta \downarrow 0} \int_{\partial I_{\epsilon,\delta}(a_j)} u(s,t) d\widetilde{X}_t = 0$. This limit holds for v in place of u and thereby

$$\int_{\partial D} f(s,t) d\widetilde{X}_t = \text{p-}\lim_{\delta \downarrow 0} \int_{\partial D_{\epsilon,\delta}} f(s,t) d\widetilde{X}_t \quad \text{a.s.}$$

Hence we infer from Theorems 3.1 and 3.2 and the usual residue formula that almost surely

$$\int_{\partial D} f(z) dz_{\mathbf{X}} = \text{p-}\lim_{\delta \downarrow 0} \Big(i \int_{\partial D_{\epsilon,\delta}} f(s,t) d\widetilde{X}_t + \sum_{j=1}^{n} \int_{\partial I_{\epsilon,\delta}(a_j)} f(z) dz \Big)$$

$$= i \int_{\partial D} f(s,t) d\widetilde{X}_t + 2\pi i \sum_{j=1}^{n} \text{Res}(f, a_j).$$

The theorem has been proved. ∎

We are now ready to prove Theorem 3.4.

P r o o f   o f   T h e o r e m   3.4. We have

$$\Big| \int_{\partial D} f(z) dz_{\mathbf{X}} \Big| \leqslant \Big| \int_{\partial D} f(s,t) ds \Big| + \Big| \int_{\partial D} f(s,t) dX_t \Big|.$$

Let

$$M := \max \Bigg\{ \sup_{(s,t) \in \overline{D}} \Big| \frac{\partial f}{\partial s}(s,t) \Big|, \ \sup_{(s,t) \in \overline{D}} \Big| \frac{\partial f}{\partial t}(s,t) \Big| \Bigg\}.$$

Firstly, we see that

$$\Big| \int_{\partial D} f(s,t) ds \Big| \leqslant \iint_{\overline{D}} \Big| \frac{\partial f}{\partial t}(s,t) \Big| dt ds \leqslant M \iint_{\overline{D}} dt ds < \infty.$$

By Proposition 2.6 we have

$$(4.6) \qquad E \exp \Big[i\xi \int_{\partial D} f(s,t) dX_t \Big] = \exp \Big[\int_c^d C \Big(\xi \int_{D_t} \frac{\partial f}{\partial s}(s,t) ds \Big) dt \Big]$$

and its Lévy measure is given by

$$\nu_0(B) = \int_c^d dt \int_{\mathbb{R}} 1_B \Big(x \int_{D_t} \frac{\partial f}{\partial s}(s,t) ds \Big) \nu(dx) \quad \text{for any Borel set } B.$$

By Theorem 25.3 of [15], $E|X_1| < \infty$ if and only if $\int_{|x|>1} |x| \nu(dx) < \infty$. Now we have

$$\int_{|x|>1} |x| \nu_0(dx) = \int_c^d dt \int_{|A(x,t)|>1} \Big| x \int_{D_t} \frac{\partial f}{\partial s}(s,t) ds \Big| \nu(dx)$$

$$\leqslant M \iint_{\overline{D}} ds dt \int_{|x|>(M(b-a))^{-1}} |x| \nu(dx) < \infty, \quad \text{where } A(x,t) = x \int_{D_t} \frac{\partial f}{\partial s}(s,t) ds.$$

From Theorem 25.3 of [15] again it follows that $E\left|\int_{\partial D} f(s,t)dX_t\right| < \infty$, and thereby $E\left|\int_{\partial D} f(z)dz_{\mathbf{X}}\right| < \infty$.

Differentiating both sides of (4.6) in ξ and letting $\xi = 0$, we obtain the following mean:

$$E\Big[\int_{\partial D} f(s,t)dX_t\Big] = \int_c^d \int_{D_t} \frac{\partial f}{\partial s}(s,t)dsdt \cdot EX_1 = \int_{\partial D} f(s,t)dt.$$

This yields

$$E\Big[\int_{\partial D} f(z)dz_{\mathbf{X}}\Big] = \int_{\partial D} f(s,t)ds + iE\Big[\int_{\partial D} f(s,t)dX_t\Big]$$
$$= \int_{\partial D} f(s,t)ds + i\int_{\partial D} f(s,t)dt = \int_{\partial D} f(z)dz.$$

The theorem has been proved. $\blacksquare$

We immediately obtain the corollary of Theorem 3.4.

P r o o f   o f   C o r o l l a r y   3.2. From Theorem 3.4 we see that

$$E\Big[\int_{\partial \triangle} f(z)dz_{\mathbf{X}}\Big] = E\Big[\int_{\partial \triangle} u(z)dz_{\mathbf{X}}\Big] + iE\Big[\int_{\partial \triangle} v(z)dz_{\mathbf{X}}\Big]$$
$$= \int_{\partial \triangle} u(z)dz + i\int_{\partial \triangle} v(z)dz = \int_{\partial \triangle} f(z)dz.$$

The usual Cauchy theorem and Morera's theorem tell us that the assertion is true. $\blacksquare$

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